Essays in Price Competition and Statistical Applications

Submitted by

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Declaration

I, Ramakanta Patra, hereby declare that Part I of this thesis, comprising Chapters 1, 2, and 3 are entirely my own research work. Part II is comprised of collaborative research. Chapter 4 was conducted in collaboration with Professor David Heyes from the Physics department at Royal Holloway. In this paper, my independent contribution is three-fold: Apart from developing the idea with my coauthor (1) I establish a relationship between Pearson's rank correlation coefficient and the beta-coefficient in the ordinary least squares (OLS) regression. I use statistical logic to discuss the gain in identifying the direction of causality when we use OLS coefficient to establish a relationship as opposed to Pearson's coefficient (2) I ran simulations using the software "DynamO" to generate data on temperature from square-well collisions in a media with constant density in different magnitudes and different number of collisions (3) I produced simple one-variable OLS regression results with temperature as the dependent variable and number of collisions as the independent variable, interpreted and discussed the results and generated diagrams representing the fit of OLS. For this I used STATA software.

Signed .... (Ramakanta Patra)

Date:

*To my daughter, **Mrunmayee Kannagi Patra**, who brought a world of hope and strength to me and my family.*

Abstract

This thesis is to address three questions on price competition and one question in statistical application in physics that has a possible application in economics. The aim of the first paper is to investigate the equilibrium in a dynamic Bertrand duopoly where firms do not know the cost of other firms and firms face an avoidable sunk cost when they decide to enter the market. In this model, firms are allowed to monitor rival's entry decision before making their pricing decision. Firms are also allowed to communicate with each other via announcements before they make the entry decision. I show that there exists two classes of Pure-Strategy Bayesian-Nash equilibria in this game. In one class of equilibrium only the low cost firms enter, and in the other class of equilibrium only one firm enters while the other stays out irrespective of their types. This is a new existence result and the paper provides full characterization of the *Perfect Bayesian Equilibria (PBE)*. Communication among firms is just 'cheap talk' and has no effect on the set of equilibria in this game.

One-shot price competition among identical firms facing avoidable fixed cost generally leads to a permanent inefficiency when costs are unknown. This stems from the fact that the market is not served with positive probabilities. However, in reality, firms interact repeatedly. My second paper shows that market inefficiency in the one-shot game can be restored with infinitely repeated interaction among competing firms. I demonstrate that with infinite interaction of firms in a Bertrand setting, competing firms can self-impose collusive conduct via communication. I provide a characterization of the *Perfect Public Equilibrium (PPE)* where firms collude and as a part of this equilibrium firms employ asymmetric penal codes. In this game, pre-play communication has a positive value which is absent in the one-shot game. The results indicate that the presence of an avoidable fixed cost in this setting makes it easier for firms to collude.

In the third paper, I consider a Bertrand duopoly where one firm's cost is publicly known and the other firm's cost is private information. In this paper, we provide a full characterization of the [**?**] of this game under equal market sharing rule. We point out that one-sided cost uncertainty and bounded known cost type is sufficient to guarantee

the existence of the *PSBNE*.

Finally, in the fourth paper we use basic techniques from econometrics and statistics, in particular Ordinary Least Square regression and Pearson's rank correlation method, to study second order fluctuations along the fluid side of the melting line of the Lennard-Jones (LJ). We use Molecular Dynamic computer simulation to generate data on the cross correlation between the configurational part of the pressure and potential energy the repulsive and attractive parts of the potential energy. By using the statistical techniques we notice a qualitative change along the melting line.

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Chapter 1

Introduction

A big part of this thesis analyzes Bertrand Competition when firms do not know each other's cost in different settings. In the second part, we analyze a statistical application to predict second order fluctuations of temperature along the Lennard-Jones melting line using different ensembles. The first part of my thesis has three chapters. The first chapter is motivated by Baye and Kovenock (2008) and several others that establish that, under equal market sharing rule, the existence of pure strategy equilibrium in a standard full information Bertrand game with avoidable fixed cost is difficult. Saporiti et al (2010) show that an equilibrium with pure strategies is possible when cost functions of firms are not sub-additive. However, when marginal costs are unknown and due to the fundamental discontinuity of the game, Sharkey et al (1993) and Spulber (1993) indicate that the existence of a pure strategy equilibrium is not possible. In my paper, I show that when firms are allowed to monitor each others entry decision in the market, pure strategy equilibrium exists. The paper investigates the equilibrium in a dynamic Bertrand duopoly where firms do not know the cost of other firms and firms face an avoidable sunk cost when they decide to enter the market. In this model, firms are allowed to monitor rival's entry decision before making their pricing decision. Firms are also allowed to communicate with each other via announcements before they make the entry decision. I show that there exists two classes of Pure-Strategy Bayesian-Nash equilibria in this game. In one class of equilibrium (symmetric) only the low cost firms enter, and in the other class of equilibrium (asymmetric) only one firm enters while the other (inefficient firms) stay out irrespective of their cost types. This is a new existence result and the paper provides full characterization of the Perfect Bayesian Equilibria. The paper also explores the effect of pre-play communication in the game to see whether this brings cooperation among firms, however, the results indicate that such communication is cheap talk and has no effect on the set of equilibria. In the second paper, I research the possibility of collusion among price competing firms that try to obtain private information about each other by observing a third party public information (such as media publication, accounting report or the like). Athey and Bagwell (2001, 2008) have shown that collusion among infinitely Bertrand competing firms with asymmetric and unknown marginal costs is possible under proper inter-temporal market sharing agreements between firms. They suggest schemes to implement the first-best outcome that supports a *Perfect Public Equilibrium (PPE)*. In my paper, I assume a similar set-up, but also assume that firms pay an avoidable fixed cost of entry in the period they decide to participate in the market. This set-up is widely present across industries such as the airlines industry where firms have to renew their terminal lease agreements every period before competing on price. In line with Athey and Bagwell (2001, 2008), I maintain that in each period firms receive an *iid* cost shock and are allowed to communicate before making their entry decisions, but to be more consistent with the wider reality present in todays economy, I do not allow for explicit market sharing agreements. As a result, and unlike Athey and Bagwell (2008), the stage game in this set-up has permanent inefficiencies (Patra, 2015) where either the market is not served with positive probability or the entering firm earns a negative profit with positive probability. But with infinite interaction, the collusive equilibrium (a *PPE*) presented in my paper develops a strategy to restore market efficiency where the market is always served and the entering firms receives its share of the monopoly profit. Allowing for communication among firms facilitates a self-enforcing collusive agreement among competing firms and I study the value of this communication both from collusion and efficiency perspective. I provide a characterization of the *PPE* and show that there exists a discount factor strictly less than one for which this equilibrium exists. The conclusion that emerges then is that the presence of the avoidable fixed cost makes it easier for the firms to collude, and market efficiency is achieved in this *PPE* in the sense that, among the firms who enter, only lowest cost firms produce.

In my third paper, I observe that the classical outcome of competitive profit when firms with symmetric constant marginal costs compete one-shot *a la* Bertrand while having perfect information about each other has been difficult to replicate in different informational and asymmetric cost conditions. Spulber (1995) shows that, in a standard Bertrand game with parameterized asymmetric costs, all but the highest cost firm expect positive profit when costs are drawn from a continuous distribution. In a recent paper, Routledge (2010) shows that in a classical model of Bertrand competition with homogeneous goods and constant marginal costs, only a mixed strategy Nash-equilibrium exists when there is discrete cost uncertainty. In my paper, I show that, under equal market sharing rule (which is the assumption maintained in the previous two cases and several others), there exists a set of Pure Strategy Bayesian-Nash Equilibria (PSBNE) in a Bertrand duopoly where one firm's cost is known and the other firm's cost is a draw from a commonly known probability distribution on a support of two discrete costs. I provide a full characterization of the equilibrium and point out that one-sided cost uncertainty and bounded known cost types are sufficient to guarantee the existence of *PSBNE*.

Finally, in my fourth paper in the second part of the thesis we study statistical fluctuations and correlations between thermodynamic properties along the fluid side of the melting line of the *Lennard-Jones (LJ)*. Using the physical properties of *Molecular Dynamics (MD)* computer simulation we generate data between the configurational part of the pressure and potential energy, and the repulsive and attractive parts of the potential energy. We compute the coefficients of *Ordinary Least Squares (OLS)* regression and the Pearson coefficient and other statistical measures. The cross correlation between Scatter plots show that at constant temperature the *Weeks-Chandler-Andersen (WCA)* decomposition of the Lennard-Jones repulsive and attractive potential energy components show a qualitative change along the melting line. At low temperature the two components are correlated, while they are anticorrelated in the high temperature limit. There is an intermediate temperature range in which the two potential energy components are effectively decorrelated. The various trends along the melting line were found to be weakly dependent on the force field used to generate the distribution of states, namely, the LJ potential, inverse power potential with exponent 12, and the repulsive term in the WCA decomposition of the LJ potential.

Part I

Bertrand Competition with Unknown Costs

Chapter 2

Introduction

2.1 Introduction

The aim of this paper is to fill a gap in the literature on equilibrium characterization in Bertrand games where firms do not know each other's marginal cost and firms face an avoidable fixed cost[24] of entry. Fixed costs are important considerations in economics in general and strategic games in particular, due largely to the technical fact that such a cost renders a firm's decision to participate in the market an endogenous choice. Market examples of avoidable fixed cost would include periodic renewal of license, participation fee in auctions, expenses conducting a market survey before assuming business etc.. Firms incur these expenses before they compete with each other with respect to any strategic variable. As such, firms make their choice to enter the market by comparing the entry fee with their expected profit based on the nature of competition that is going to ensue once they enter the market. What is interesting here is that we allow firms to observe rival's entry decision before they make a decision on their price. Observation of such decision by firms is more close to reality as compared to the prior literature where much focus has been on firms making both entry and pricing decisions simultaneously[1]. The avoidable

[1]See Binmore[3]

fixed cost considered here is an exogenous sunk cost[2] which has no effect on the marginal cost of the firms. We show the existence of classes of Pure Strategy Perfect Bayesian-Nash Equilibrium (PSBNE) in this game. In one class of equilibrium only the lowest cost firms decide to enter the market and supply. In the other class of equilibrium, one firm enters the market and and the other firm stays out irrespective of their types. We provide a full characterization of this equilibrium along with a numerical illustration of the game.

2.2 Literature Review

Francois Bertrand (1822-1900) [2], as a review response to the quantity competing oligopoly model proposed by Cournot in 1883, modeled competition among firms using price as their strategic variable. Bertrand's observation that such a competition might lead to indefinite undercutting of prices among oligopolists was falsified later with the advent of game theoretic tools in economics. It was shown that with a small number of identical firms producing homogeneous goods and having perfect information about the market and rivals, Bertrand competition would yield competitive outcome in equilibrium, leaving 'zero' profit for the oligopolists. However, due to the existence of the fundamental discontinuity in the profit functions of Bertrand competing firms with homogeneous product, many existence results involve tedious characterization of the equilibrium. As such, existence results in Bertrand games are still an active area of research. Fixed cost, on the other hand, is an important economic variable for firms. A firm may not want to undertake production if its expected returns, given the strategic situation of the firm, are not high enough to cover the fixed costs. Thus, given the nature of the fixed cost[3] and timing of the game, the structure of equilibrium in the pricing game changes. For example Baye and Kovenock (2008) [1] showed that with a fully avoidable fixed cost and constant marginal cost of firms there does not exist any pure or mixed strategy Nash-equilibrium in the full information Bertrand game. Kreps and Scheinkman (1983)[5] (KS hereafter), showed that in a two

[2]see Sutton[24]

[3]endogenous or exogenous

stage game where firms decide on capacities (a fixed cost) in the first stage and compete in price in the second stage, the unique Nash equilibrium is equivalent to the Cournot outcome. However, in the KS game the capacity level decided at the first stage has a cost reducing effect in the second stage. Spulber (1995)[5] looked into a basic one-shot game of price competition with unknown costs. He has shown that, with asymmetric costs and other regularity assumptions, all but the highest cost firm expect positive profit when costs are unknown. He has also shown that when firms are operating in a contestable market and when there are unknown fixed cost of entry, the resulting equilibrium is that all firms except the highest cost firm enters, since all but the firm with the highest fixed cost would have positive expected profit of entry.

Saporiti and Coloma (2010) [10] work out equilibrium situations where price competing firms face, among others, avoidable fixed costs in a perfect information game. Under equal market sharing rule [4], they show that, with an avoidable fixed cost and the variable cost function of firms not being sub-additive when they produce market supply at the lowest price, there always exists a Bertrand equilibrium in pure strategies[5].

This paper is organized in the following order. First, we present the basic set-up of the model. Next, we present the equilibrium analysis of the stage game. Finally, in the last section, we present a numerical example to illustrate the model.

2.3 Basic Set-Up

In this game $\mathcal{G}$, two *ex ante* identical firms, 1 and 2, compete one-shot in a standard Bertrand model with homogeneous goods. The inverse demand function $D(p)$ satisfies regularity conditions $D'(p) < 0 < D(p)$. Let $p_i \in \mathbb{R}^+$ be the price chosen by firm $i \in \{1, 2\}$. Firms in this game face an exogenous fixed cost [6], F, when they decide to participate in

[4] firms agree to split market shares equally when they charge the same price in the market

[5] need not be symmetric

[6] See Sutton (Chapter 1) for a detailed discussion [24]

the market. Such fixed costs may be viewed as renewal of licenses, leases etc. that businesses normally incur every year before they make production and pricing decisions for the subsequent year. Thus, effectively, firms make a decision to 'enter' the market or 'not enter' based on their comparison between the expected pay-off from the ensuing competition after they decide to enter and the fixed cost they will need to pay when they enter.

Accordingly, we assume that every firm has a total cost function represented by

$$C_i(q) = \begin{cases} c_i q + F, & \text{if enter} \\ \\ 0, & \text{otherwise} \end{cases}$$

where c_i is the realized marginal cost of firm i. c_i is an IID random variable that is equal to c_L with probability η_L and c_H with probability $1 - \eta_L$. We assume $\eta_L < \frac{1}{2}$. This assumption ensures that there is a higher probability that a firm will realize high cost and thus represents a more interesting scenario, since this is a situation firms would have a stronger motivation to cooperate than the reverse situation[7]. We call a firm to be of type L (*or* H) if it faces a marginal cost c_L (*or* c_H). For convenience we will use c_L(*or* c_H) and L(*or* H) interchangeably. The state space of types is represented here by $\Omega = \{L, H\} \times \{L, H\}$ where $\Omega^i = (L, H)$ and the types are realized from a common prior which is common knowledge to all.

The timing of the game is as follows: (1) firms observe their type, L or H (2) firm i makes a decision to enter the market or not based on the realization of its own type,

[7] In fact, we only need an asymmetric probability weight for the types in order to characterize our *PSBNE*. So, WLOG, we have assumed $\eta_L < \frac{1}{2}$

$c_i \in \{c_L, c_H\}$, (3) Firms monitor each other's entry decision and subsequently compete in price. We denote the space of entry decision for firm i as Υ^i where $\Upsilon^i = \{E, N\} \equiv \{E = enter, N = not\ \ enter\}$ and Υ to be the state space of entry decisions pertaining to the available types defined by $\Upsilon = \Upsilon^i \times \Upsilon^{-i} = (E, N) \times (E, N)$. The entry decision function is $e^i(c_i) : \Omega^i \to \Upsilon^i$ and we denote the realized entry decision of firm i by $\upsilon^i \in \Upsilon^i$ where $\upsilon^i = e^i(c_i)$. Denote $\boldsymbol{\upsilon} = (\upsilon^i, \upsilon^{-i})$ where $\boldsymbol{\upsilon} \in \Upsilon$. For future notational use we define $\boldsymbol{\Upsilon} = (\Upsilon^i, \Upsilon^{-i})$. As is evident, in this set up, if a firm enters it incurs a fixed cost of F. If a firm doesn't enter, its profit is identically equal to '0'.

This game can be represented in the following diagram.

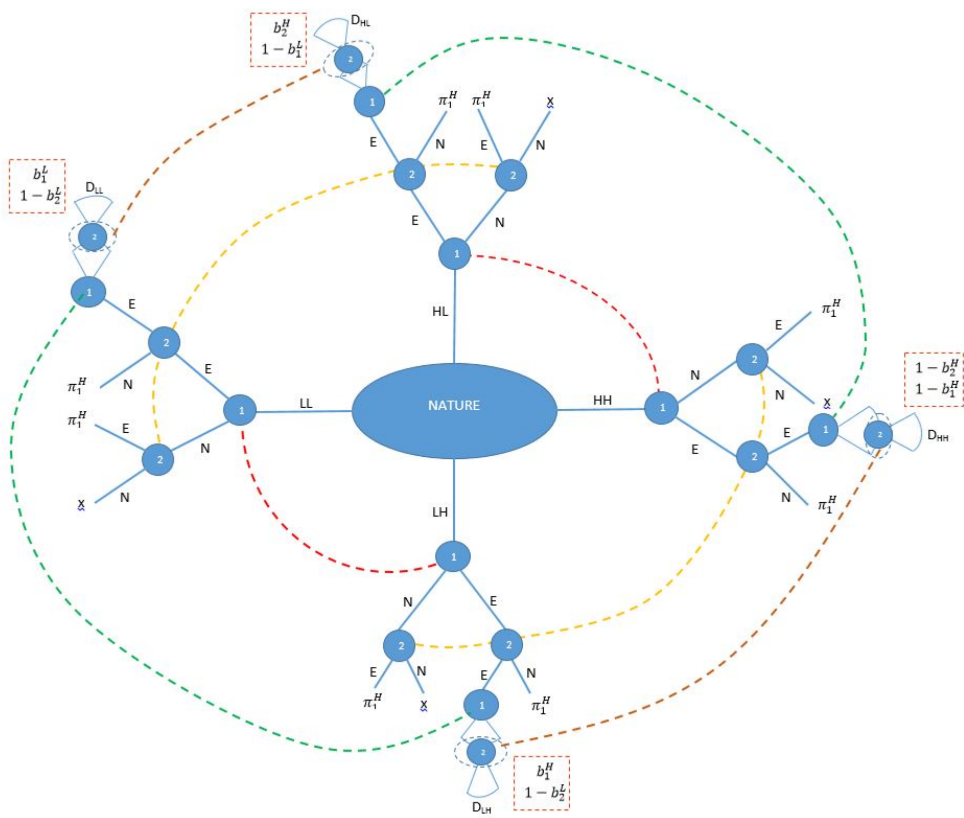

Figure 2.1: Game $\mathcal{G}$

(x): no pay-off for any firm; (π_i^j): monopoly pay-off for firm $i \in \{1, 2\}$ when its type $j \in \{H, L\}$; Nature : 'nature' is choosing type of firm $i \in \{1, 2\}$; (D_{st}): profit from duopoly price competition where firm 1 is type type $s \in \{L, H\}$ and firm 2 is type $t \in \{L, H\}$; b_x^y: belief of player x that the rival is of type y

We now state our assumptions of the model.

Assumption 1. *There exists a* $\hat{p} < \infty$ *such that* $D(\hat{p}) = 0$.

Assumption 2. *There exists a price,* $\overline{p}$ *such that* $D_i(\overline{p})\overline{p} - c_H D_i(\overline{p}) - F \geq 0, i \in \{1, 2\}$.

Let's denote the optimal monopoly price, quantity and profit depending upon its cost type by p^j, q^j, Π^j for types $j \in \{L, H\}$. We also define the operating profit of each firm by $\Pi^{j-} = \Pi^j + F, j \in \{L, H\}$.

Assumption 3. $(1-\eta_L)\Pi^{H^-} < F < (1-\eta_L)\Pi^{L^-}$, $\Pi^{H^-} > (1-\eta_L)(c_H - c_L)q(c_H) > F$. Assumption 2 guarantees that there is sufficient demand in the market for a firm to operate at profit even if all the firms find that they have the highest possible marginal cost. The first part of Assumption 3 ensures that the fixed entry cost is below the expected operating monopoly profit of a low cost firm and above the expected operating monopoly profit of a high cost firm. The second part of Assumption 3 implies that the expected profit of a low cost firm when it undercuts a type H rival's marginal cost still earns a high enough profit to enter the market without incurring losses. To make economic sense I have assumed that this profit is lower than the operating monopoly profit of the high cost firm.

2.4 Equilibrium Analysis of the Game

We solve for the *Perfect Bayesian Equilibrium* in this game where we will consider both *symmetric* and *asymmetric* equilibria. We define the strategy structure of this game as follows. We retain the entry strategy of firm i to be $e^i(c_i) : \Omega^i \rightarrow \Upsilon^i$ since entry is dependent on type of the firm only which is the respective firm's private information. Then the subsequent pricing strategy for firm i can be defined as $\rho^i(c_i, \boldsymbol{\upsilon}) = \Omega^i \times \Upsilon^{-i} \to \mathbb{R}^+$ since a firm observes rival's entry decision before making its pricing decision. We define an entry policy vector in this game by $\boldsymbol{e}(\boldsymbol{c}) \equiv (e^1(c_1), e^2(c_2))$ where $\boldsymbol{c} \equiv (c_1, c_2)$ is the vector of realized cost types of the duopolists. Finally, the pricing strategy profile can be represented by a vector $\boldsymbol{\rho}(\boldsymbol{c}, \boldsymbol{\upsilon}) = (\rho^1(c_1, \upsilon^2), \rho^2(c_2, \upsilon^1))$.

We denote an equilibrium pricing strategy profile by $\varrho^* = (p^{1*}, p^{2*}) \in \boldsymbol{p}^*$ where both p^{1*}, p^{2*} are a six-tuple consisting of the entry strategy of the form $e^{j*} = (\nu_L^j, \nu_H^j) \equiv$ *(firm j's entry decision when it realizes type L, firm j's entry decision when it realizes type H)* when $(\nu_L^j, \nu_H^j) \in \Upsilon,\ j \in \{1, 2\}$, and pricing strategy of the form $p^{j*}_{\theta,\gamma}$ where $(\theta, \gamma) \in \{L, H\} \times \{E, N\} \in \Omega^j \times \Upsilon^{-j}$ and $p^{i*}_{\theta,\gamma} \equiv$ *(price when type L and the other enters, price when type L and the other does not enter, price when type H and the other enters, price when type H and the other does not enter).* Therefore, we denote an entry strategy profile in this game by $(\nu_L^1, \nu_H^1), (\nu_L^2, \nu_H^2)$ and the resulting full pricing strategy profile by

$$((\nu_L^1, \nu_H^1), (\nu_L^2, \nu_H^2), (p_{LE}^1, p_{LN}^1, p_{HE}^1, p_{HN}^1), (p_{LE}^2, p_{LN}^2, p_{HE}^2, p_{HN}^2))$$

where $\boldsymbol{p}^* \in \mathbf{R}^{2+|\Omega^i \times \Upsilon^{-i}|+|\Omega^{-i} \times \Upsilon^i|}$. Note that a firm does not make any pricing decision when it does not enter in this game. Now, let's assume p^L, p^H to be optimal monopoly prices for type L and type H respectively when they face demand $D(p)$. From the conditions above we can deduce that when a rival, say firm 2, does not enter then, observing this, firm 1 sets its price at the monopoly price with respect to its realized cost. As such we can fix the prices in such cases at their respective monopoly price. So, the updated entry and pricing strategy profile will be

$$((\nu_L^1, \nu_H^1), (\nu_L^2, \nu_H^2), (p_{LE}^1, p_{LN}^1 = p^L, p_{HE}^1, p_{HN}^1 = p^H), (p_{LE}^2, p_{LN}^2 = p^L, p_{HE}^2, p_{HN}^2 = p^H))$$

Now we define the beliefs in this game at different nodes. Observe that the beliefs of players after 'Nature' has drawn the types (call it first-stage beliefs) are trivially defined, because all of the corresponding informations sets are reached with positive probability. For example, after the nature has selected that firm 1 is of type L, the belief of firm 1 that firm 2 is type L with probability η_L i, which is derived from the given prior. Other such beliefs are analogously defined with consistency since they are trivial as discussed in the above example. We will not be using additional notation to explicitly define those beliefs into the model. We will, however, introduce notations to define beliefs during the price competition stage of the game. Let $b_{yz}^{i\mathcal{J}}, i \in \{1, 2\}, \mathcal{J} \in \{L, H\}$ be the belief of type $\mathcal{J}$ (already assigned by nature) player i that player $\neg i \equiv -i \in \{1, 2\}$ is of type $y \in \{L, H\}$ when player $-i$ has taken action $z \in \{E, N\}$. Thus, the system of beliefs are given by a sixteen-tuple $\boldsymbol{\mu} \equiv \{b_{yz}^{i\mathcal{J}} | \, i, -i \in \{1, 2\}; y, \mathcal{J} \in \Omega; z \in \{E, N\}\}$. However, notice that the complexity of representation can be drastically reduced since (1) belief of any firm i's about the rival $-i$'s type is not dependent on the realization of firm i's own type, since the cost draws are IID; (2) if firm i believes that rival is of type L with probability $u \in [0, 1]$ then he also believes that the rivals of type H with probability $1 - u$; and (3) the belief that firm i forms about firm $-i$ when firm $-i$ has not entered does not play an active part in the description of the *PBE*. Thus we denote that firm i believes that the probability that firm $-i$ is of type L when firm $-i$ has entered is $b^i, i \in \{1, 2\}$ which are the only beliefs we will describe in the strategy profiles.

Proposition 2.4.1. *This game $\mathcal{G}$ possesses two classes of Perfect Bayesian Equilibria (PBE) corresponding to the entry and pricing strategy profiles as follows:*

1. *Symmetric Equilibrium: In this equilibrium a type L firm always enters and a type H firm stays out of the market. Notationally,* $((E,N),(E,N),(p^1_{LE},p^1_{HE},p^1_{LN}=p^L,p^1_{HN}=p^H),(p^2_{LE},p^2_{HE},p^2_{LN}=p^L,p^2_{HN}=p^H))$

2. *Asymmetric Equilibrium: In this equilibrium, only one firm enters irrespective of its type. Notationally,* $((E,E),(N,N),(p^1_{LE},p^1_{HE},p^1_{LN}=p^L,p^1_{HN}=p^H)),(p^2_{LE},p^2_{HE},$ $p^2_{LN}=p^L,p^2_{HN}=p^H))$[8]

3. *No other entry strategy profile in this game forms a part of any pure strategy equilibrium in this game, $\mathcal{G}$.*

We proceed with the proof of this theorem in a case-by-case basis and we define the beliefs as a part of the proof.

2.4.0.1 Symmetric Equilibrium: Entry Strategy (EN,EN)

The strategy profile $((E,N),(E,N),(p^1_{LE},p^1_{HE},p^1_{LN}=p^L,p^1_{HN}=p^H)$**,** $(p^2_{LE},p^2_{HE},p^2_{LN}=p^L,p^2_{HN}=p^H))$ **forms a part of a *PBE* of the game, $\mathcal{G}$.**

Proof: First, by consistency, the beliefs of players in any *PBE* with this strategy profile are given by $(b^1=1,b^2=1)$. Given these beliefs, the best response function of firm 2,

[8] Of course as a corollary, an equilibrium strategy profile can be analogously defined as $((N,N),(E,E),(p^1_{LE},p^1_{HE},p^1_{LN}=p^L,p^1_{HN}=p^H),(p^2_{LE},p^2_{HE},p^2_{LN}=p^L,p^2_{HN}=p^H))).See\,case 2.4.0.11.$

from the optimality of p^2_{LE} :

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{LE} = p^1_{LE} - \epsilon, & \text{if } p^1_{LE} > c_L \\ p^2_{LE} \geq p^1_{LE}, & \text{if } p^1_{LE} = c_L \\ p^2_{LE} > p^1_{LE}, & \text{if } p^1_{LE} < c_L \end{cases}$$

The best response function of firm 1, from the optimality of p^1_{LE} :

$$BR_1(p^2_{LE}) = \begin{cases} p^1_{LE} = p^2_{LE} - \epsilon > c_L, & \text{if } p^2_{LE} > c_L \\ p^1_{LE} \geq p^2_{LE} & \text{if } p^2_{LE} = c_L \\ p^1_{LE} > p^2_{LE}, & \text{if } p^2_{LE} < c_L \end{cases}$$

As we can observe, for all price levels in the set $[0, c_L)$, type L firms' best responses are to set price above each other. But $\nexists$ a maximum in the set $[0, c_L)$ implying $\nexists$ a fixed point in the best response correspondences of both firms when they are type L and the assumed entry strategy profile is $((E, N), (E, N))$. In the similar setting, for prices in the set (c_L, ∞), the best responses are firms undercutting each other. Since $\nexists$ a minimum in this set of prices there is no fixed point correspondence of undercutting best responses. Finally, we verify that the best response correspondences are satisfied at the pricing strategy profile $\{p^1_{LE} = p^2_{LE} = c_L, p^1_{HE} = p^2_{HE} = c_H, p^1_{LN} = p^2_{LN} = p^L, p^1_{HN} = p^2_{HN} = p^H\}$ [9]. Given this equilibrium pricing profile, the associated expected profit of a type H firm, say firm

[9] Note that if firm $i \in \{1, 2\}$ enters, any price set by the firm i above c_H will be undercut by rival $j \neq i$ irrespective of j's type. Additionally, any price lower than c_H for a type H firm is dominated by c_H, since the firm runs the risk of being over bid by its rival when the rival is of type H and is forced to supply the whole market. So a type H firm in this case never gets any more than '0' expected operating profit when the rival enters.

1, is '0' when it stays out, and $(1-\eta_L)\Pi^{H^-} - F$ when it enters. This is because, if firm 1, being type H, observes that firm 2 has not entered, makes monopoly profit $\Pi^{H^-} - F$ and makes a profit of '0-F' if it observes firm 2 enter . Since every firm's belief is (EN, EN) and since $prob(c_H) = 1-\eta_L$, firm 1 believes that the firm 2 would not enter with probability $1-\eta_L$, in which case firm 1 would expect to make type H monopoly profit. Similarly, for type L firm, when the firm enters the game, it earns an expected profit of $(1-\eta_L)\Pi^{L^-} - F$ and '0' if it stays out. Since $(1-\eta_L)\Pi^{H^-} < F < (1-\eta_L)\Pi^{L^-}$, its obviuos that a type H firm would not expect to make any positive return when it enters where as a type L firm would always expect to make a positive profit when it enters, enforcing the belief of each other. ■

2.4.0.2 Asymmetric Equilibrium: Entry Strategy ((E,E),(N,N))

The strategy profile $((E,E),(N,N),(p^1_{LE} = c_L, p^1_{HE} < c_H, p^1_{LN} = p^L, p^1_{HN} = p^H))$, $(p^2_{LE}, p^2_{HE}, p^2_{LN} = p^L, p^2_{HN} = p^H))$ **forms a part of a *PSBNE* of the game, $\mathcal{G}$.**

Proof: In order to analyze if this strategy profile is an equilibrium, first we consider the pricing strategy[10] of firm 2 given the prices that firm 1 sets. We then assume that the entry strategy $((E,E),(N,N))$ forms a part of the equilibrium of the game. Then we analyze the game to see if firm 2 could profitably deviate given the strategy of firm 1 in order to disprove our assumption. Note here that since firm 1 believes that firm 2 is not going to enter irrespective of its type, if firm 2 actually enters, any belief that firm 1 forms about the type of firm 2 is going to be Bayes-consistent. Thus, the belief of players in this section is given by $(b^1 = \mathcal{P}, b^2 = \eta_L)$ for any $\mathcal{P} \in [0,1]$.

[10] See appendix for the pricing strategy of firm 2

Observe that if a type L firm 2 does not find it profitable to enter then the equilibrium in the pricing game must be such that firm 2, irrespective of its type, does not get an expected profit above F. We will now show by contradiction that this implies that type H firm 1 must be pricing below c_H when both firms enter. Assume that a type H firm 1 sets $p^1_{HE} \geq c_H$. But then, a type L firm 2 can set $P^2_{LE} = c_H - \epsilon$ which would guarantee an expected profit of at least $(1 - \eta_L)(c_H - c_L)D(c_H)$ and this will be enough to cover the cost of entry F for firm 2 (by assumption 3). This means that a type L firm 2 will find it profitable to enter which is a contradiction that firm 2 will always choose to stay out as a part of its entry strategy. Now we have established that for entry strategy profile $((E, E), (N, N))$ to be in equilibrium it must be true that a type H firm 1 is pricing strictly below c_H. If a type H firm 2 prices strictly above p^1_{HE} it will lose the market to firm 1 and make '0' profit if firm 1 is type H. On the other hand, if firm 2 prices weakly below $p^1_{HE}(< c_H)$ then firm 2 will gain market, but it will make negative profit in expectation, since, by assumption, firm 2 knows that firm 1 is type H with probability $1 - \eta_L > 0$. Thus, firm 2 will find it more profitable to price strictly above p^1_{HE}. Given this, suppose that when firm 2 enters, firm 1 assigns a positive probability weight to firm 2 being type H. Since we have shown that p^1_{HE} is strictly below c_H in equilibrium, and since a type H firm 2 is going to price above p^1_{HE}, a type H firm 1 will receive negative expected pay-off in this case. In such a case, a type H firm 1 will find it more profitable to price strictly above p^2_{HE}, which contradicts our assumption that firms are in equilibrium. So in the equilibrium with entry strategy profile $((E, E), (N, N))$, firm 1 must have no probability weight on firm 2 being type H. So now we fix that it is a consistent belief of firm 1 that if firm 2 has entered then firm 2 is type L. Now, we inspect the pricing strategy of a type L

firm 1. Given that firm 1 believes that an entering firm 2 is type L, a type L firm 1 would price at $p^1_{LE} = p^2_{LE} - \epsilon$ if p^2_{LE} is above c_L and $p^1_{LE} > p^2_{LE}$ if p^2_{LE} is below c_L. Now lets fix that both p^1_{LE} and p^1_{HE} are above c_L. Now, given that firm 2 knows the probability distribution of types of firm 1 whose strategy is to enter the market irrespective of its realized type and given that firm 2 knows that a type H firm would price strictly below c_H, a type L firm 2 will set price $p^2_{LE} = p^1_{HE} - \epsilon > c_L$ if $p^1_{HE} \geq c_L + \frac{q(p^1_{LE})(p^1_{LE} - c_L)}{(1-\eta_L)q(p^1_{HE})} = \mathcal{K}$, otherwise firm 2 will set a price equal to $p^1_{LE} - \epsilon > c_L$. Then a type L firm 1 knows that the entering firm 2 (which firm 1 believes to be type L only) would set the price at $p^2_{LE} = p^1_{HE} - \epsilon$. As such, a type L firm 1 would set a price below p^2_{LE} since by doing such firm 1 will gain the market and make a profit greater than 'zero', which is aprofitable deviation. Arguing similarly, if $p^1_{HE} < \mathcal{K} < c_H$ a type L firm 2 will set its price at $p^1_{LE} - \epsilon$. In such a case, a type L firm 1 would undercut firm 2's price, which is, again, a deviation from the equilibrium. Now, suppose that a type L firm 1 sets a price above p^1_{HE}. Then a type L entering firm 2 would undercut p^1_{LE} by ϵ in this case. Given this, a type L firm 1 will undercut the price set by firm 2 in this case, which is a profitable deviation. Finally, for any price that a type L firm 1 sets strictly below c_L, a type L frim 2 will best respond with setting a price above it. In such a case the type L firm 1 can profitably deviate by setting a price above firm 2's price and strictly below c_L. So far, we have exhausted all candidates for equilibrium in the support of prices. The only surviving candidate is a type L firm 1 pricing at c_L. If a type L firm 1 prices at c_L then an entering type L firm 2 would set a price at c_L . In such a case a type L firm 1 does not have any profitable deviation. Thus $p^1_{LE} = c_L$ is an equilibrium condition. Finally, we inspect the equilibrium pricing strategy of a type H firm 2. At this point, firm 2 knows the probability distribution of types of firm 1, belief of firm 1 (entering firm 2 is type L), a type H firm 1 would price strictly below c_H and a type L firm 1 would price at c_L. Given this information, a type H firm 2 would have to price below $p^1_{HE} < c_H$ to gain positive market share in expectation. But in such a case $p^2_{HE} < p^1_{HE} < c_H$ and thus firm 2 will earn a negative profit. So a profitable deviation for type H firm 2 will be to charge a price strictly above p^1_{HE}, which is also the equilibrium pricing strategy for a type H firm 2. The deductions above define the complete set of conditions for an equilibrium to hold where $((E, E), (N, N))$ forms a part of the equilibrium strategy profile.

Note that in the*asymmetric equilibrium* involving the entry strategy profile ((E,E),(N,N)) (or ((N,N),(E,E))) a type H entering firm prices below its marginal cost, c_H, which is a weakly dominated strategy. This is an unattractive feature of this equilibrium as compared to the the symmetric equilibrium.

2.4.0.3 No other entry strategy profile in this game forms a part of any equilibrium in this game, $\mathcal{G}$.

We present the proof of this section in a case-by-case basis in the appendix 3.

2.5 Conclusion

In this paper we have shown that when firms make an endogenous entry in a standard Bertrand duopoly with discrete cost uncertainty, there exists two classes of PSBNE, one with the entry strategy profile $((E, N), (E, N))$ and the other with the entry strategy profile $((E, E), (N, N))$. Baye and Kovenock [1] and Spulber[5] have shown previously, under symmetric market sharing and full information, there is no-equilibrium existence in games with price competition and entry, either in pure or in mixed strategy. In a related work Routledge (2010)[4] has shown that in a classical Bertrand model with two-sided symmetric uncertainty and discrete costs, there does exist only mixed strategy Bayesian-Nash equilibria of the game. Thus our result adds an existence of PSBNE result in a price competition game with cost uncertainty and entry. Our existence result depends largely on the fact that firms can observe each other's entry decision before they compete in price. However, there are some bleak features of the classes of equilibria we find in this game. With the equilibrium following from the entry strategy profile (EN, EN), the market will not be served with positive probability, i.e., when both firms draw high cost simultaneously. In the second equilibrium following from the entry strategy profile $((E, E), (N, N))$ (or $((N, N), (E, E))$) a type H entering firm will be charging a price below its marginal cost, c_H, in equilibrium. Therefore, one of the future research directions would be to consider possibilities that would help improve equilibrium outcome of this price game in different settings. In particular, it will be interesting to consider extending this model to a repeated play of this game and explore the equilibrium, since such consideration is close to real market situations.

2.6 Appendix 1

The pricing strategy of firm 2,the not entering firm, when the entry strategy is $((E, E), (N, N))$, is as follows:

1. $Case\ \ 1 : p^1_{LE} \leq p^1_{HE}\ <\ c_L$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ > \ p^1_{HE} \\ \\ p^2_{HE} \ > \ p^1_{HE} \end{cases}$$

2. $Case \ \ 2 : p^1_{LE} \leq p^1_{HE} \ = \ c_L$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \geq p^1_{HE} \\ \\ p^2_{HE} \ > \ p^1_{HE} \end{cases}$$

3. $Case \ \ 3 : p^1_{LE} \ < \ c_L \ < \ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{HE} - \epsilon, \\ \\ p^2_{HE} \ > \ p^1_{HE} \end{cases}$$

4. $Case \ \ 4 : c_L \ = \ p^1_{LE} \ < \ p^1_{HE} \ < \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{HE} - \epsilon \\ \\ p^2_{HE} \geq p^1_{HE}, \end{cases}$$

5. $Case \ \ 5 : c_L \ < \ p^1_{LE} \ = \ p^1_{HE} \ < \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{LE} - \epsilon \\ \\ p^2_{HE} \ > \ p^1_{HE} \end{cases}$$

6. $Case \ \ 6 : c_L \ < \ p^1_{LE} \ = \ p^1_{HE} \ < \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A : p^2_{LE} \ = \ p^1_{HE} - \epsilon & \text{if } p^1_{HE} \geq c_L + \frac{q(p^1_{LE})(p^1_{LE} - c_L)}{(1-\eta_L)q(p^1_{HE})} \\ B : p^2_{LE} \ > \ p^1_{HE} & \text{if } otherwise \\ p^2_{HE} \geq c_H \end{cases}$$

7. $Case \quad 7 : c_L \ < \ p^1_{LE} \ < \ p^1_{HE} \ = \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A : p^2_{LE} \ = \ p^1_{LE} - \epsilon & \text{if } p^1_{LE} \geq c_L + (1 - \eta_L) \frac{q(c_H)(c_H - c_L)}{q(p^1_{LE})} \\ B : p^2_{LE} \ > \ p^1_{HE} & \text{if } otherwise \\ p^2_{HE} \geq c_H \end{cases}$$

8. $Case \quad 8 : c_L \ < \ p^1_{LE} \ = \ p^1_{HE} \ = \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{LE} - \epsilon \\ p^2_{HE} \geq p^1_{HE} \end{cases}$$

9. $Case \quad 9 : c_L \ < \ p^1_{LE} \ < \ c_H \ < \ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A : p^2_{LE} \ = \ p^1_{LE} - \epsilon & \text{if } p^1_{LE} \geq c_L + (1 - \eta_L) \frac{q(p^1_{HE})(p^1_{HE} - c_L)}{q(p^1_{LE})} \\ B : p^2_{LE} \ = \ p^1_{HE} - \epsilon & \text{if } otherwise \\ p^2_{HE} \ = \ p^1_{HE} - \epsilon \end{cases}$$

10. $Case\ \ 10 : c_H\ = p_{LE}^1\ <\ p_{HE}^1$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} A : p_{LE}^2\ = p_{LE}^1 - \epsilon & \text{if } p_{HE}^1 \geq c_L + \frac{1}{(1-\eta_L)} \frac{q(c_H)(c_H - c_L)}{q(p_{HE}^1)} \\ B : p_{LE}^2\ = p_{HE}^1 - \epsilon & \text{if } otherwise \\ p_{HE}^2\ = p_{HE}^1 - \epsilon \end{cases}$$

11. $Case\ \ 11 : c_L\ = p_{LE}^1\ <\ p_{HE}^1\ = c_H$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2\ = p_{HE}^1 - \epsilon \\ p_{HE}^2 \geq p_{HE}^1 \end{cases}$$

12. $Case\ \ 12 : c_L\ = p_{LE}^1\ <\ c_H\ <\ p_{HE}^1$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2\ = p_{HE}^1 - \epsilon \\ p_{HE}^2\ = p_{HE}^1 - \epsilon \end{cases}$$

13. $Case\ \ 13 : p_{LE}^1\ <\ c_L\ <\ c_H\ <\ p_{HE}^1$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2\ = p_{HE}^1 - \epsilon \\ p_{HE}^2\ = p_{HE}^1 - \epsilon \end{cases}$$

14. $Case\ \ 14 : c_L\ <\ c_H\ <\ p_{LE}^1\ <\ p_{HE}^1$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A: p^2_{LE} \ = \ p^1_{LE} - \epsilon & \text{if } p^1_{LE} \geq c_L + (1-\eta_L)\frac{q(p^1_{HE})(p^1_{HE}-c_L)}{q(p^1_{LE})} \\ B: p^2_{LE} \ = \ p^1_{HE} - \epsilon & \text{if } otherwise \\ C: p^2_{HE} \ = \ p^1_{LE} - \epsilon & \text{if } p^1_{HE} \leq c_H + \frac{1}{(1-\eta_L)}\frac{q(p^1_{LE})(p^1_{LE}-c_H)}{q(p^1_{HE})} \\ D: p^2_{HE} \ = \ p^1_{HE} - \epsilon & \text{if } otherwise \end{cases}$$

15. $Case \ \ 15 : c_L \ < \ c_H \ < \ p^1_{LE} \ = \ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{LE} - \epsilon \\ p^2_{HE} \ = \ p^1_{HE} - \epsilon \end{cases}$$

2.7 Appendix 2

		$p^1_{LE} \leq p^1_{HE} < c_L$	$p^1_{LE} \leq p^1_{HE} = c_L$	$p^1_{LE} < c_L < p^1_{HE}$	$c_L = p^1_{LE} < p^1_{HE} < c_H$	$c_L < p^1_{LE} = p^1_{HE} < c_H$	$c_L < p^1_{LE} < p^1_{HE} = c_H$ *CaseA & B*	$p^1_{LE} = p^1_{HE} = c_H$	$c_L < p^1_{LE} < c_H < p^1_{HE}$ *Case A & B*	$c_H = p^1_{LE} < p^1_{HE}$ *CaseA & B*	$c_L = p^1_{LE} < p^1_{HE} = c_H$	$c_L = p^1_{LE} < c_H < p^1_{HE}$	$p^1_{LE} < c_L < c_H < p^1_{HE}$	$c_L < c_H < p^1_{LE} < p^1_{HE}$ *CasesA, B, C & D*	$c_L < c_H < p^1_{LE} = p^1_{HE}$
	BR2 / BR1	$p^2_{LE} > p^1_{HE}$ $p^2_{HE} > p^1_{HE}$	$p^2_{LE} \geq p^1_{HE}$ $p^2_{HE} > p^1_{HE}$	$p^2_{LE} = p^1_{HE} - \varepsilon$ $p^2_{HE} > p^1_{HE}$	$p^2_{LE} = p^1_{HE} - \varepsilon$ $p^2_{HE} > p^1_{HE}$	$p^2_{LE} = p^1_{LE} - \varepsilon$ $p^2_{HE} > p^1_{HE}$	$p^2_{LE} = p^1_{LE} - \varepsilon(A)$ $p^2_{LE} = p^1_{HE} - \varepsilon(B)$ $p^2_{HE} \geq p^1_{HE}$	$p^2_{LE} = p^1_{LE} - \varepsilon$ $p^2_{HE} > p^1_{HE}$	$p^2_{LE} = p^1_{LE} - \varepsilon(A)$ $p^2_{LE} = p^1_{HE} - \varepsilon(B)$ $p^2_{HE} = p^1_{HE} - \varepsilon$	$p^2_{LE} = p^1_{LE} - \varepsilon(A)$ $p^2_{LE} = p^1_{HE} - \varepsilon(B)$ $p^2_{HE} = p^1_{HE} - \varepsilon$	$p^2_{LE} = p^1_{HE} - \varepsilon$ $p^2_{LE} \geq p^1_{HE}$	$p^2_{LE} = p^1_{HE} - \varepsilon$ $p^2_{LE} = p^1_{HE} - \varepsilon$	$p^2_{LE} = p^1_{HE} - \varepsilon$ $p^2_{LE} = p^1_{HE} - \varepsilon$	$p^2_{LE} = p^1_{LE} - \varepsilon(A)$ $p^2_{LE} = p^1_{HE} - \varepsilon(B)$ $p^2_{HE} = p^1_{LE} - \varepsilon(C)$ $p^2_{HE} = p^1_{HE} - \varepsilon(D)$	$p^2_{LE} = p^1_{HE} - \varepsilon$ $p^2_{LE} = p^1_{HE} - \varepsilon$
$p^2_{LE} \leq p^2_{HE} < c_L$	$p^1_{LE} > p^2_{HE}$ $p^1_{HE} > p^2_{HE}$	Cyclic. No fixed Point.	x	x	x	x	x	x	x	x	x	x	x	x	x
$p^2_{LE} \leq p^2_{HE} = c_L$	$p^1_{LE} \geq p^2_{HE}$ $p^1_{HE} > p^2_{HE}$	x	x	x	Y, but cyclic.	x	Y, but cyclic.	x	x	x	x	x	x	x	x

$p^2_{LE} < c_L < p^2_{HE}$	$p^1_{LE} = p^2_{HE} - \varepsilon$ $p^1_{HE} > p^2_{HE}$	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$c_L = p^2_{LE} < p^2_{HE} < c_H$	$p^1_{LE} = p^2_{HE} - \varepsilon$ $p^1_{HE} > p^2_{HE}$	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$c_L < p^2_{LE} = p^2_{HE} < c_H$	$p^1_{LE} = p^2_{LE} - \varepsilon$ $p^1_{HE} > p^2_{HE}$	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$c_L < p^2_{LE} < p^2_{HE} = c_H$ $Case A \ \& \ B$	$p^1_{LE} = p^2_{LE} - \varepsilon(A)$ $p^1_{LE} = p^2_{HE} - \varepsilon(B)$ $p^1_{HE} \geq p^2_{HE}$	x	x	x	x	x	Y but cyclic.	x	Y, but cyclic.	x	x	x	x	x	x
$p^2_{LE} = p^2_{HE} = c_H$	$p^2_{LE} = p^1_{LE} - \varepsilon$ $p^2_{HE} \geq p^1_{HE}$	x	x	x	x	x	Y but cyclic.	x	Y, but cyclic.	x	x	x	x	x	x

$c_L < p^2_{LE} < c_H < p^2_{HE}$ *CaseA & B*	$p^1_{LE} = p^2_{LE} - \varepsilon(A)$ $p^1_{LE} = p^2_{HE} - \varepsilon(B)$ $p^1_{HE} = p^2_{HE} - \varepsilon$	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$c_H = p^2_{LE} < p^2_{HE}$ *CaseA & B*	$p^1_{LE} = p^2_{LE} - \varepsilon(A)$ $p^1_{LE} = p^2_{HE} - \varepsilon(B)$ $p^1_{HE} = p^2_{HE} - \varepsilon$	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$c_L = p^2_{LE} < p^2_{HE} = c_H$	$p^1_{LE} = p^2_{HE} - \varepsilon$ $p^1_{LE} \geq p^2_{HE}$	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$c_L = p^2_{LE} < c_H < p^2_{HE}$	$p^1_{LE} = p^2_{HE} - \varepsilon$ $p^1_{LE} \geq p^2_{HE} - \varepsilon$	x	x	x	x	x	x	x	x	x	x	x	x	Y, but cyclic.	Y, but cyclic.
$p^2_{LE} < c_L < c_H < p^2_{HE}$	$p^1_{LE} = p^2_{HE} - \varepsilon$ $p^1_{LE} \geq p^2_{HE} - \varepsilon$	x	x	x	x	x	x	x	x	x	x	x	x	Y, but cyclic.	Y, but cyclic.

$c_L < c_H < p^1_{LE} < p^1_{HE}$	$p^1_{LE} = p^2_{LE} - \varepsilon (A)$ $p^1_{LE} = p^2_{HE} - \varepsilon (B)$ $p^1_{HE} = p^1_{LE} - \varepsilon (C)$ $p^2_{HE} = p^1_{HE} - \varepsilon (D)$	x	x	x	x	x	x	x	x	x	x	x	x	Y, but cyclic.	Y, but cyclic.
$c_L < c_H < p^1_{LE} = p^1_{HE}$	$p^1_{LE} = p^2_{HE} - \varepsilon$ $p^1_{LE} = p^2_{HE} - \varepsilon$	x	x	x	x	x	x	x	x	x	x	x	x	Y, but cyclic.	Y, but cycic.

x: not a compatible best response correspondence.

y: a compatible best response correspondence

For the indicated conditions like A, B etc., please refer to the paper.

As is established here there does not exist any equilibrium pricing strategy for the entry strategy profile (EE, EE).

2.8 Appendix 3

2.8.0.4 Entry Strategy ((E,N),(E,E))

The strategy profile $((E, N), (E, E), (p^1_{LE}, p^1_{HE}, p^1_{LN}, p^1_{HN})), (p^2_{LE}, p^2_{HE}, p^2_{LN}, p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$.

Proof: First, by consistency, the belief of players in this section is given by $(b^1 = \eta_L, b^2 = 1)$. The best response function of firm 2 from the optimality of p^2_{LE} can be deduced as follows: ;

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{LE} = p^1_{LE} - \epsilon, & \text{if } p^1_{LE} > c_L \\ p^2_{LE} \geq p^1_{LE}, & \text{if } p^1_{LE} = c_L \\ p^2_{LE} > p^1_{LE}, & \text{if } p^1_{LE} < c_L \end{cases}$$

and from the optimality of p^2_{HE};

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{HE} = p^1_{L} - \epsilon, & \text{if } p^1_{LE} > c_H \\ p^2_{HE} \geq p^1_{LE}, & \text{if } p^1_{LE} = c_H \\ p^2_{HE} > p^1_{LE}, & \text{if } p^1_{LE} < c_H \end{cases}$$

Now we discuss the best responses of firm 1 from the optimality of p^1_L in a case-by-case basis. We will inspect firm 1 trying to set prices in the following intervals, which will exhaust cases of all possible prices:

1. $Case\ \ 1 : p^1_{LE} < c_L$

In this case, firm 2 would best respond with

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{LE} > p^1_{LE}, & \text{if } p^1_{LE} < c_L \\ p^2_{HE} > p^1_{LE}, & \text{if } p^1_{LE} < c_L \end{cases}$$

Given this best response and since the interval $[0, c_L)$ is open upwards, there does not exist an optimal strategy for firm 1. So $\nexists$ an equilibrium pricing strategy in this interval.

2. $Case \;\; 2 : p^1_{LE} = c_L$

In this case, firm 2 would best respond with

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{LE} \geq c_L, & \text{if } p^1_{LE} = c_L \\ p^2_{HE} > c_L, & \text{if } p^1_{LE} = c_L \end{cases}$$

But then, immediately, firm 1's best response would be to charge $p^2_{HE} - \epsilon > c_L$. So $\nexists$ an equilibrium pricing strategy in at c_L.

3. $Case \;\; 3 : c_L < p^1_{LE} < c_H$

In this case, firm 2 would best respond with

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{LE} \geq p^1_{LE} - \epsilon > c_L, & \text{if } c_L < p^1_{LE} < c_H \\ p^2_{HE} > p^1_{LE} > c_L, & \text{if } c_L < p^1_{LE} < c_H \end{cases}$$

But then, immediately, firm 1's best response would be to charge $p^2_{LE} - \epsilon > c_L$. So $\nexists$ an equilibrium pricing strategy in the interval $c_L < p^1_{LE} < c_H$.

4. $Case\ \ 4 : p^1_{LE}\ =\ c_H$

   In this case, firm 2 would best respond with

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{LE}\ =\ p^1_{LE} - \epsilon\ >\ c_L, & \text{if } p^1_{LE}\ =\ c_H \\ p^2_{HE} \geq c_H, & \text{if } p^1_{LE}\ =\ c_H \end{cases}$$

   But then, immediately, firm 1's best response would be to charge $p^2_{LE} - \epsilon\ >\ c_L$. So $\nexists$ an equilibrium pricing strategy when $p^1_{LE}\ =\ c_H$.

5. $Case\ \ 5 : p^1_{LE}\ >\ c_H$

   In this case, firm 2 would best respond with

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{LE}\ =\ p^1_{LE} - \epsilon\ >\ c_L, & \text{if } p^1_{LE}\ >\ c_H \\ p^2_{HE}\ =\ p^1_{LE} - \epsilon, & \text{if } p^1_{LE}\ >\ c_H \end{cases}$$

   But then, immediately, firm 1's best response would be to charge $p^2_{LE} - \epsilon\ >\ c_L$. This is periodic. So $\nexists$ an equilibrium pricing strategy when $p^1_{LE}\ >\ c_H$.

This completes the proof that there does not exists an optimal pricing strategy in the strategy profile outlined in this case. Consequently, there does not exist and equilibrium strategy profile in this case. ■

2.8.0.5 Entry Strategy ((E,N),(N,E))

The strategy profile, $((E, N), (N, E), (p^1_{LE}, p^1_{HE}, p^1_{LN}, p^1_{HN})), (p^2_{LE}, p^2_{HE}, p^2_{LN}, p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. Proof: First, the belief of players in this section is given by $(b^1\ =\ 0, b^2\ =\ 1)$. From the optimality of p^2_{HE} we get the best response correspondence as follows:

$$BR_2(p^1_{LE}) = \begin{cases} p^2_{HE} \; = p^1_{LE} - \epsilon, & \text{if } p^1_{LE} \; > \; c_H \\ p^2_{HE} \geq p^1_{LE}, & \text{if } p^1_{LE} \; = \; c_H \\ p^2_{HE} \; > \; p^1_{LE}, & \text{if } p^1_{LE} \; < \; c_H \end{cases}$$

From the optimality of p^1_{LE}

$$BR_1(p^2_{HE}) = \begin{cases} p^1_{LE} \; = p^2_{HE} - \epsilon, & \text{if } p^2_{HE} \; > \; c_L \\ p^1_{LE} \geq p^2_{HE}, & \text{if } p^2_{HE} \; = \; c_L \\ p^1_{LE} \; > \; p^2_{HE}, & \text{if } p^2_{HE} \; < \; c_L \end{cases}$$

With the best responses defined, we now inspect for Nash Equilibrium in different intervals of the price space, $[0, \infty)$.

1. Case 1 ($< c_L$): The best responses for firms 1 and 2 in this region respectively are $p^1_{LE} > p^2_{HE}$ and $p^2_{HE} > p^1_{LE}$, which is cyclic. So we do not have a NE in this region.

2. Case 2 ($= c_L$): The best responses for firms 1 and 2 in this region respectively are $p^1_{LE} \geq p^2_{HE}$ and $p^2_{HE} > p^1_{LE}$, which is cyclic. So we do not have a NE in this region.

3. Case 3 (c_L, c_H): The best responses for firms 1 and 2 in this region respectively are $p^1_{LE} = p^2_{HE} - \epsilon$ and $p^2_{HE} > p^1_{LE}$, which is cyclic. So we do not have a NE in this region.

4. Case 4 $(= c_H)$: The best responses for firms 1 and 2 in this region respectively are

$p^1_{LE} = p^2_{HE} - \epsilon$ and $p^2_{HE} \geq p^1_{LE}$, which is cyclic. So we do not have a NE in this region.

5. Case 5 ($> c_H$): The best responses for firms 1 and 2 in this region respectively are $p^1_{LE} = p^2_{HE} - \epsilon$ and $p^2_{HE} = p^1_{LE} - \epsilon$, which is cyclic. So we do not have a NE in this region.

Now we have exhausted all cases and we establish that we do not have an equilibrium pricing strategy in the case of the entry strategy profile $((E,N),(N,E))$. Thus the entry strategy profile $((E,N),(N,E))$ cannot form a part of a Nash Equilibrium strategy profile.

2.8.0.6 Entry Strategy ((E,N),(N,N))

The strategy profile $((E,N),(N,N),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. Proof: First, the belief of players in this section is given by $(b^1 = \mathcal{P}, b^2 = 1)$ for any $\mathcal{P} \in [0,1]$. Now, suppose the above strategy profile forms an equilibrium of the game. Now consider the following deviation. Firm 2 plays 'enter' and charges $p^1_{LE} = c_L$ and $p^1_{LN} = p^L$. Apparently the expected profit from such deviation is $(1-\eta_L)\Pi^{L^-} - F$ which is positive. So such a deviation is credible. Q.E.D.

2.8.0.7 Entry Strategy ((E,E),(E,E))

The strategy profile $((E,E),(E,E),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. Proof: First, the belief of players in this section is given by $(b^1 = \eta_L, b^2 = \eta_L)$. WLOG, we first describe the best responses for firm 2 from the optimality of its type dependent pricing strategy.

1. $Case \;\; 1 : p_{LE}^1 \leq p_{HE}^1 \; < \; c_L$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2 \; > \; p_{HE}^1 \\ p_{HE}^2 \; > \; p_{HE}^1 \end{cases}$$

2. $Case \;\; 2 : p_{LE}^1 \leq p_{HE}^1 \; = \; c_L$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2 \geq p_{HE}^1 \\ p_{HE}^2 \; > \; p_{HE}^1 \end{cases}$$

3. $Case \;\; 3 : p_{LE}^1 \; < \; c_L \; < \; p_{HE}^1$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2 \; = \; p_{HE}^1 - \epsilon, \\ p_{HE}^2 \; > \; p_{HE}^1 \end{cases}$$

4. $Case \;\; 4 : c_L \; = \; p_{LE}^1 \; < \; p_{HE}^1 \; < \; c_H$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2 \; = \; p_{HE}^1 - \epsilon \\ p_{HE}^2 \geq p_{HE}^1, \end{cases}$$

5. $Case \;\; 5 : c_L \; < \; p_{LE}^1 \; = \; p_{HE}^1 \; < \; c_H$

$$BR_2(p_{LE}^1, p_{HE}^1) = \begin{cases} p_{LE}^2 \; = \; p_{LE}^1 - \epsilon \\ p_{HE}^2 \; > \; p_{HE}^1 \end{cases}$$

6. $Case\ \ 6 : c_L \ < \ p^1_{LE} \ = \ p^1_{HE} \ < \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A : p^2_{LE} \ = \ p^1_{HE} - \epsilon & \text{if } p^1_{HE} \geq c_L + \frac{q(p^1_{LE})(p^1_{LE} - c_L)}{(1-\eta_L)q(p^1_{HE})} \\ B : p^2_{LE} \ > \ p^1_{HE} & \text{if } otherwise \\ p^2_{HE} \geq c_H \end{cases}$$

7. $Case\ \ 7 : c_L \ < \ p^1_{LE} \ < \ p^1_{HE} \ = \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A : p^2_{LE} \ = \ p^1_{LE} - \epsilon & \text{if } p^1_{LE} \geq c_L + (1-\eta_L)\frac{q(c_H)(c_H - c_L)}{q(p^1_{LE})} \\ B : p^2_{LE} \ > \ p^1_{HE} & \text{if } otherwise \\ p^2_{HE} \geq c_H \end{cases}$$

8. $Case\ \ 8 : c_L \ < \ p^1_{LE} \ = \ p^1_{HE} \ = \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{LE} - \epsilon \\ p^2_{HE} \geq p^1_{HE} \end{cases}$$

9. $Case \ \ 9: c_L \ < \ p^1_{LE} \ < \ c_H \ < \ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A: p^2_{LE} \ = \ p^1_{LE} - \epsilon & \text{if } p^1_{LE} \geq c_L + (1-\eta_L)\frac{q(p^1_{HE})(p^1_{HE}-c_L)}{q(p^1_{LE})} \\ B: p^2_{LE} \ = \ p^1_{HE} - \epsilon & \text{if } otherwise \\ p^2_{HE} \ = \ p^1_{HE} - \epsilon \end{cases}$$

10. $Case \ \ 10: c_H \ = \ p^1_{LE} \ < \ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A: p^2_{LE} \ = \ p^1_{LE} - \epsilon & \text{if } p^1_{HE} \geq c_L + \frac{1}{(1-\eta_L)}\frac{q(c_H)(c_H-c_L)}{q(p^1_{HE})} \\ B: p^2_{LE} \ = \ p^1_{HE} - \epsilon & \text{if } otherwise \\ p^2_{HE} \ = \ p^1_{HE} - \epsilon \end{cases}$$

11. $Case \ \ 11: c_L \ = \ p^1_{LE} \ < \ p^1_{HE} \ = \ c_H$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{HE} - \epsilon \\ p^2_{HE} \geq p^1_{HE} \end{cases}$$

12. $Case \ \ 12: c_L \ = \ p^1_{LE} \ < \ c_H \ < \ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE} \ = \ p^1_{HE} - \epsilon \\ p^2_{HE} \ = \ p^1_{HE} - \epsilon \end{cases}$$

13. $Case\ \ 13 : p^1_{LE}\ <\ c_L\ <\ c_H\ <\ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE}\ = p^1_{HE} - \epsilon \\ \\ p^2_{HE}\ = p^1_{HE} - \epsilon \end{cases}$$

14. $Case\ \ 14 : c_L\ <\ c_H\ <\ p^1_{LE}\ <\ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} A : p^2_{LE}\ = p^1_{LE} - \epsilon & \text{if } p^1_{LE} \geq c_L + (1-\eta_L)\frac{q(p^1_{HE})(p^1_{HE}-c_L)}{q(p^1_{LE})} \\ \\ B : p^2_{LE}\ = p^1_{HE} - \epsilon & \text{if } otherwise \\ \\ C : p^2_{HE}\ = p^1_{LE} - \epsilon & \text{if } p^1_{HE} \leq c_H + \frac{1}{(1-\eta_L)}\frac{q(p^1_{LE})(p^1_{LE}-c_H)}{q(p^1_{HE})} \\ \\ D : p^2_{HE}\ = p^1_{HE} - \epsilon & \text{if } otherwise \end{cases}$$

15. $Case\ \ 15 : c_L\ <\ c_H\ <\ p^1_{LE}\ =\ p^1_{HE}$

$$BR_2(p^1_{LE}, p^1_{HE}) = \begin{cases} p^2_{LE}\ = p^1_{LE} - \epsilon \\ \\ p^2_{HE}\ = p^1_{HE} - \epsilon \end{cases}$$

Since the strategy profile is symmetric, the best responses of player 1 will also be symmetric to that of player 2. We then inspect for a NE of pricing strategy using the table provided. We conclude from the table that there does not exist an equilibrium pricing strategy for the subgame with entry strategy profile $((E, E), (E, E))$. Therefore, we can also conclude that the strategy profile $((E, E), (E, E))$ cannot form part of a NE strategy profile for the game we are analyzing. Hence, we do not have any equilibrium pay-off for players in this game.

2.8.0.8 Entry Strategy ((E,E),(N,E))

The strategy profile $((E,E),(N,E),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. Proof: First, the belief of players in this section is given by $(b^1 = 0, b^2 = \eta_L)$. Now, lets consider this equilibrium profile. Then $\exists$ a price p^2_{HE} for which the profit of firm 2 is greater than '0'. Now consider the following deviation by firm 2. Play 'E' when type L and set $p^2_{LE} = p^2_{HE}$. Its obvious that this is a profitable deviation for firm 2. Q.E.D.

2.8.0.9 Entry Strategy ((N,E),(N,E))

The strategy profile $((N,E),(N,E),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. Proof: First, the belief of players in this section is given by$(b^1 = 0, b^2 = 0)$. Assume the above strategy profile is an equilibrium of this game. Now consider the following deviation by player 1. Play 'Enter' when type L and charge $p^1_{LE} = p^1_{HE}$. It is apparent that this is a profitable deviation for firm 1. Q.E.D.

2.8.0.10 Entry Strategy ((N,E),(N,N))

The strategy profile $((N,E),(N,N),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$.

Proof: First, the belief of players in this section is given by $(b^1 = \mathcal{P}, b^2 = 0)$ for any $\mathcal{P} \in (0,1)$. Now, assume the above strategy profile to be an equilibrium strategy profile. Now consider the following deviation: firm 1 enters when low cost and charges price $p^1_{LE} = p^1_{HE}$ and $p^1_{LN} = p^L$. If firm 1 makes positive profit playing 'enter' in the equilibrium being type H, then it must be true that it will make positive profit by playing 'enter' when it is type L. So this deviation is credible. Q.E.D.

2.8.0.11 Entry Strategy ((N,N),(NN))

The strategy profile $((N,N),(N,N),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$.

Proof: First, the belief of players in this section is given by $(b^1 = \mathcal{P}, b^2 = \mathcal{P})$ for any $\mathcal{P} \in (0,1)$. WLOG, consider any equilibrium price profile $(p^1_{LE},p^1_{HE},(p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},$ $p^2_{LN},p^2_{HN})$. Since every player's strategy is to not enter irrespective of its type, consider the following deviation by player 1; enter when type L and charge $p^1_{LE} = p^L$. The resulting profit in such a case is Π^{L^-} which is greater than F. Thus this deviation is profitable. Q.E.D.

2.8.0.12 Entry Strategies ((N,N),(E,E)), ((E,E),(E,N)),((N,E),(E,N)), ((N,N),(E,N)), ((N,E),(E,E)), ((N,N),(N,E))

The strategy profile $((N,N),(E,E),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ forms a part of a *PSBNE* of the game, $\mathcal{G}$. This case is symmetric to case 2. The strategy profile $((E,E),(E,N),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. This case is symmetric to case 3. The strategy profile $((N,E),(E,N),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. This case is symmetric to case 4. The strategy profile $((N,N),(E,N),(p^1_{LE},p^1_{HE},p^1_{LN},p^1_{HN}),(p^2_{LE},p^2_{HE},p^2_{LN},p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$.This case is symmetric to case 5. The strategy profile $((N, E), (E, E), (p^1_{LE}, p^1_{HE}, p^1_{LN}, p^1_{HN}), (p^2_{LE}, p^2_{HE}, p^2_{LN}, p^2_{HN}))$ does not form a part of a *PS-BNE* of the game, $\mathcal{G}$. This proof is symmetric to case 7. The strategy profile $((N, N), (N, E),$ $(p^1_{LE}, p^1_{HE}, p^1_{LN}, p^1_{HN}), (p^2_{LE}, p^2_{HE}, p^2_{LN}, p^2_{HN}))$ does not form a part of a *PSBNE* of the game, $\mathcal{G}$. This case is symmetric to case 9. Q.E.D.

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Chapter 3

Collusion with Private Information and Fixed Costs

3.1 Introduction

The wide presence of oligopolistic market structure and resulting strategic competition among participating firms is one of the major reasons for the continued interest in this particular field. Price being a major strategic variable in market competition has warranted due attention under different market situations and with different market constraints. Since profit of identical firms competing *a la* Bertrand falls to 'zero' in a single play of the game, a large body of literature is suggestive that firms in such a market tend to contemplate collusion both in one-shot and repeated interactions on finite and infinite horizons. Collusion has important anti-trust implications. A number of anti-trust cases all over the world and across industries [1], and more recently, an upsurge of a number of theoretical and empirical papers exploring the welfare effects of those collusive practices[18], provide a strong evidence of the anti-welfare effects of collusion. possibility of collusion as an equilibrium outcome[2] has sparked interest among economists. Current development of game theoretic tools like Nash Equilibrium and its application to games with different information structures have proven to be very handy to identify the incentive conditions for oligopolists to collude, which is an important benchmark in the theory of oligopoly equilibrium [3]. A large class of models, which have evolved from Green and Porter (1984)[12], Abreu, Pearce, and Stacchetti (1986)[1](hence forth APS1) and Fudenberg, Levin, Maskin(1994)[10](hence forth FLM) analyze situations where oligopolists face a natural barrier to collusion when the strategic behavior of firms is imperfectly observable. For example, in input markets firms negotiate their prices individually with their suppliers and any information about this negotiation is mostly private information that competitors do not observe [3][4]. Even in cases where firm's strategic decisions may be publicly observable, firms might not have sufficient information about each other. In such cases, it is not always clear if there can exist a separating equilibrium where firms may truthfully reveal their private information and collude efficiently leading to a high profit outcome. However, recent literature establishes that firms can rely on the realizations of a public signal like price, published report of firms' accounts etc. [19] as a part of their strategy to enforce collusive equilibrium when they cannot monitor each others actions directly. FLM and APS1 developed the related equilibrium concept which they defined as a strategy profile where every firm's strategy is dependent only on publicly observable outcomes (called public history) and for each time period and that period public history

[1] For example the Alcoa Case, the Lysine Cartel, the Air Tours case and Airlines in EU, to name a few

[2] of course in repeated setting

[3] Conditions for the incentive to collude dates back to Chamberlin, i.e., the kinked demand curve solution where collusion is being enforced by the threat of retaliation

[4] It might be interesting to see if suppliers can be incentivised to reveal truthful information to firms about each other's cost, but then it could not be truthfully verified.

the strategy profile induces a Nash-equilibrium from that point on. Such equilibrium is called *Perfect Public Equilibrium* (henceforth *PPE*). But, imperfections in interpreting the public signal do exist due to a number of exogenous factors like level of law enforcement etc. . These imperfections often pose problems in the characterization of the *(PPE)*[5] by directly affecting the incentives among firms to collude. Several earlier papers have looked into the issue of collusion among firms involving unknown costs in both repeated and dynamic settings over both finite and infinite horizons. A subset of them discuss various aspects of public monitoring and analyze its role in enforcing collusion. We discuss some of these papers later in the literature review section.
In this paper, we propose two fundamental changes to the game that Athey and Bagwell have analyzed in their 2001, 2004 (with Chris Sanchirico) and 2008 papers. First, we assume a downward-sloping demand curve instead of a unit demand by consumers for a very good reason that downward sloping demand is a better representation of the real world. Secondly, we introduce an avoidable fixed cost into the firms' cost structure that induces firms make a decision to participate in the market before going on to competing in prices. This makes firms participation in this game endogenous and the stage game changes to a dynamic game, a major departure from Athey and Bagwell's set-up. We explore, with the changed conditions, the possibility of collusion among firms. We also analyze if it is harder to collude in this set-up compared to a situation when there is no avoidable fixed cost. We also analyze if productive efficiency (only the low cost firm producing in the market) can be achieved and if it can sustain in the new circumstances. Finally, we explore if allowing for pre-play communication facilitates collusion.The pre-play communication is introduced to exchange information (make announcements) as a part of the commitment mechanism in the collusive strategy of the players. In particular, players announce their cost types before playing the entry and pricing strategies in the repeated game which they try to verify later via a public signal mechanism. Without the pre-play communication, players will have no chance to reveal their types and thus the collusive agreement to attain the first-best outcome could not be facilitated[6] Unlike Ziv (1993)[26], we assume no monetary transfers between players.

3.2 Literature Review

Spulber (1995)[5] looked into a basic one-shot game of price competition with unknown costs. He has shown that, with assymetric costs and other regularity assumptions, all but

[5]Note that a *PPE* is an analogous concept as SPNE where the strategy of players is dependent on the observation of a public signal and thus the public history. It is a Nash equilibrium that induces a Nash Equilibrium in a repeated game from any time t onwards.

[6]For a discussion see Chakrabarti (2010) [7]. We outline the details of this communication in the model.

the highest cost firm expect positive profit when costs are unknown.

Compte (1988), Matsushima and Kandori (1998)[14], Matsushima (2001)[20], Cole and Kocherlakota (2001) [8] and Kennan (2001)[15] have looked into different aspects of collusive behavior in dynamic Bertrand games when there is private information. This private information is generated by a random cost shock that firms receive every period and as such states are privately observed by some of the players but not by all. Hanazono and Yang (2007)[13] analyze collusive behavior when the firms receive private signals by independently and identically distributed (IID) demand shocks affecting the demand side of the market primarily. In a similar setting, Gerlach (2009)[11] designed a stochastic market sharing rule to substitute for pre-play communication that is dependent on the state of demand, in order for collusion to sustain when demand is fluctuating arbitrarily. For example, in his model, partial communication in high demand states are sufficient in order for firms to achieve the best and full communication collusive outcome since communication eliminates the possibility of opportunistic price cuts when demand is fluctuating.

In a series of papers, Athey and Bagwell (2001)[3] (2008)[5] and Sanchirico (2004)[4], very rigorously discuss the possibility of collusion when firms cannot observe each others costs. These papers use the market signal correlated with firms actions to characterize jointly profit maximizing collusive equilibrium when firms are competing *a la* Bertrand and when firms have private information about their own costs. In the 2001 paper, competing firms [7] receive an IID shock about their cost types at the beginning of each period which is private information. They characterize the *Perfect Public Equilibrium (PPE)* where productive efficiency is achieved only when the high cost firm is willing to give up its market share. Collusive equilibrium becomes most profitable in this situation when the high cost firms are promised higher market share in future in order to implement efficiency in the present period.

In the 2008 paper, existing firms in the market are assumed to play an infinite-horizon version of the Bertrand price-setting game in which the prices are perfectly observed, firms receive a private cost shock every period, but firms' type remains persistent over time. It is evident that this change in setting makes it a dynamic game. In this game cost shocks are independent across firms, but within a firm cost shocks follow a first-order Markov process. The paper shows that the firms can collude at the monopoly price by agreeing on appropriate splits of the market share. The high-cost firm will be willing to give up market share because it expects higher profit in the future. This result, however, as opposed to the 2001 paper, does not depend crucially on the condition that a high-cost firm today could receive a technology "shock" in the future that would make it a low-cost firm.

Athey and Bagwell (2008)'s main result indicates that if the distribution of costs is

[7] They use discrete types i.e. low-type and high-type

log-concave and the firms are sufficiently patient, then price rigidity is supported in the optimal collusive equilibrium, i.e., firms set the same price and share the market equally, regardless of their respective costs. In their model, productive efficiency can be achieved under some circumstances, but such equilibria are not optimal. It should, however, be noted that the firms play a Bertrand price-setting game in each period and (this could be a key factor that drives their results) the demand side is given by a unit mass of identical consumers with a fixed reservation price (assumed to be strictly above the highest possible marginal cost). This indeed makes the market sharing rule much more tractable than in a case where a downward sloping demand schedule is assumed, since the firms know that the reserve price is the optimal collusive price irrespective of the privately observed costs of the firms. In particular when the demand is given by the usual downward sloping demand curve, the optimal collusive price depends on the realized costs of the firms. In these three papers by Athey and Bagwell and Athey, Bagwell and Sanchirico, the authors have used dynamic programming squared technique proposed by APS1, Abreu, Pearce and Staccetti (1990)[2](henceforth APS2) and Fudenberg, Levin and Maskin (1994) [10]. These techniques provide a very efficient approach to modeling *PPE* by focusing on the pay-offs rather than the strategy itself. However, they are useful when there exists a unique stage game equilibrium in pure strategies which the firms can unequivocally bank upon for a punishment strategy.

A later paper by Chakrabarti (2010)[7] models a similar situation in the case of Cournot competing firms. However, he does not explicitly make use of the techniques that Athey and Bagwell use and his conclusions are significantly different from Athey and Bagwell (2008). He shows that, with signaling, the firms will play the strictly separating Bayesian Nash equilibrium in period 1 and produce the optimal incentive compatible collusive quantity vector from period 2 onwards. But with communication, the first period play of strictly separating Bayesian Nash equilibrium in period 1 is no more optimal. He concludes that the separating equilibria with communication yields larger pay-off among the two equilibria he was considering.

Our work is well timed and placed in the literature in the sense that many of the existence results that we use in our analysis in one-shot Bertrand games are still an active area of research. This paper is organized in the following order. First, we present the basic set-up of the model where we discuss the results of the stage game. Next, we present the repeated game with imperfect public monitoring where we characterize a *PPE* that results in efficient production (low cost firm producing). In the both the sections above we consider that firms get an IID cost shock every period. Finally, in the last section, we present a numerical example to illustrate the model.

3.3 Basic Set-Up

We assume that two *ex ante* identical firms, 1 and 2, compete repeatedly in a standard Bertrand model with homogeneous goods in periods $t = 0, 1, 2, \dots$. The inverse demand function $D(p)$ satisfies regularity conditions $D'(p) < 0 < D(p)$. Let $p_{i,t} \in \mathbb{R}^+$ be the price chosen by firm $i \in \{1, 2\}$ in period t. Firms discount future with rate $\delta \in (0, 1)$ and we use $-i$ to represent convention.

We assume that every firm has a total cost function represented by

$$C_{i,t}(q) = \begin{cases} c_{i,t}q + F, & \text{if enter} \\ 0, & \text{otherwise} \end{cases}$$

where F is an exogenous fixed cost [8] that a firm pays to participate in the market and $c_{i,t}$ is its realized marginal cost in period t. Such fixed costs may be viewed as renewal of licenses, leases etc. that businesses normally incur every year before they make pricing and production decisions for the subsequent year. We assume that the marginal cost of firms $c_{i,t}$ is a random variable with support $\{c_L, c_H\}$, $c_H > c_L > 0$ and $prob(c_L) = \eta_L < \frac{1}{2}$. The reason we assign this asymmetric probability to the cost types is to rid our model of any mixed strategy equilibria and effectively use the *PSBNE* from the stage game equilibria for our modeling purposes[9]. We call a firm to be of type L (*or* H) if it faces a marginal cost c_L (*or* c_H). For convenience we will use c_L(*or* c_H) and L(*or* H) interchangeably. The state space of types is represented here by $\Omega = \{L, H\} \times \{L, H\}$ where $\Omega^i = (L, H)$ are realized from a common prior. For exposition, we assume that prob $(c_L) = \eta_L \in (0, \frac{1}{2})$, which is common knowledge. Since we are also interested in examining the effect of pre-play communication we allow for firms to announce their cost type before they engage in price competition. Notice that every firm will have to announce its type here and not announcing its type is not an option. Every firm i announces its type $a^i \in A \equiv \{L, H\}$ where A is same across all firms. Thus the announcement space is $\boldsymbol{A} = A^2$.

The game follows a schedule every time period t in the following manner: (1) firms observe their type, L or H (2) firms communicate with each other or engage in "Cheap Talk" and make announcements $a^i = \psi^i(c_i)$ where $\psi^i(c_i)$ is firm i's announcement function (we allow for such pre-play communication in order to examine if it facilitates collusion, an important anti-trust issue) (3) firms make a decision to enter the market or not based on their own announcement, a^i, the announcement of other firms, $a^{-i} = \boldsymbol{\psi}^{-i}$ where $a^{-i} \in A$ and the realization of their own type, c_i. Denote $\boldsymbol{a} = (a^i, a^{-i})$ as the announcement vector of all firms where $\boldsymbol{a} \in \boldsymbol{A}$ and $\boldsymbol{A}$ is the space of announcements of all firms. We denote

[8] See Sutton (Chapter 1) for a detailed discussion [24]

[9] WLOG we have assumed $\eta_L < \frac{1}{2}$

the space of entry decision for firm i as Υ^i where $\Upsilon^i = \{E, N\} \equiv \{E = enter, N = not\ \ enter\}$. The entry decision function is $e^i(c_i, \boldsymbol{a}) : \Omega^i \times A \times A \to \Upsilon^i$ and we denote the entry decision of firm i by $\upsilon^i \in \Upsilon^i$ where $\upsilon^i = e^i(c_i, \boldsymbol{a})$. Denote $\boldsymbol{\upsilon} = (\upsilon^i, \upsilon^{-i})$ where $\boldsymbol{\upsilon} \in \boldsymbol{\Upsilon} = (\Upsilon^i, \Upsilon^{-i})$ and $\Upsilon = \Upsilon^i \times \Upsilon^{-i} = (E, N) \times (E, N)$. Υ is the state space of entry decisions pertaining to the available types. As is evident, in this set up, if a firm enters it incurs a fixed cost of F. If a firm doesn't enter, its profit is identically equal to '0'. (4) Firms monitor each other's entry decision and subsequently compete in price.

We describe the public monitoring mechanism that we will be using in our analysis. Abreu, Pearce and Staccetti (1990) and Fudenberg, Levine and Maskin (1994) [10] have done pioneering work in developing very powerful techniques to model repeated games with imperfect public information using dynamic programming. A public monitoring mechanism is essentially a signal that the players receive by observing a strategic variable in the market. For example, Green and Porter identify price as the public monitoring mechanism that firms use to monitor collusion while competing in Cournot. Another example of such a monitoring is Team Production where players choose between high effort or low effort. The probability of success of the project depends on the sum of efforts of both players, but only the joint outcome is observed publicly. In a perfect public monitoring firms can infer the correct actions from the public signal they observe which is also equivalent to saying that firms can observe each other actions correctly. On the other hand, in imperfect public monitoring, firms receive a noisy signal and the distribution of the signal depends on the actions that firms take. We assume that the support of the distribution of the public signal is constant across the set of all action profiles. This definition of imperfect public monitoring can be directly extended to repeated games and dynamic games. Mailath and Samuelson [19] represent the idea of public monitoring via a public correlation device. Such device could capture the idea of a range of public events that firms might use as a coordination mechanism which in essence captures the idea of a public monitoring mechanism.

We now state our assumptions of the model.

Assumption 4. *There exists a* $\hat{p}$ *such that* $D(\hat{p}) = 0$ *and* $\hat{p} < \infty$.

Assumption 5. *There exists a price,* $\overline{p}$ *such that* $D_i(\overline{p})\overline{p} - c_H D_i(\overline{p}) - F \geq 0$.

Let's denote the optimal monopoly price, quantity and profit depending upon its cost type by $\Pi^L(\Pi^H), p^L(p^H), q^L(q^H)$ for types $L(H)$. We also define $\Pi^{L^-}(\Pi^{H^-}) = \Pi^L(\Pi^H) + F$.

Assumption 6. $(1 - \eta_L)\Pi^{L^-} \leq \Pi^{H^-}$, $\Pi^{H^-} > F$.

Assumption 5 guarantees that there is sufficient demand in the market for a firm to operate at profit even if all the firms find that they have the highest possible marginal cost. Assumption 6 specifies that the expected operating monopoly profit when a low cost firm enters is less than the operating monopoly profit of a high cost firm. This assumption allows us to reduce the multiplicity of equilibria using symmetric distribution on one side of the inequality.

3.4 The Stage Game

We now describe the one-shot play of the stage game which forms the background for our subsequent repeated extensive form game with incomplete information. We solve for the Nash-Equilibriums in the stage game using the strategic trade-offs that the firms face when making their entry decisions followed by their pricing decisions. First, we define the strategy structure of the stage game as follows. Since announcements are "Cheap Talk" in a one-shot game, we denote the entry strategy of firm i by $e^i(c_i) : \Omega^i \to \Upsilon^i$ since entry is dependent on type of the firm only . Then the subsequent pricing strategy for firm i can be defined as $\rho^i(c_i, \boldsymbol{\upsilon}) = \Omega^i \times \Upsilon^i \times \Upsilon^{-i} \to \mathbb{R}^+$ since a firm observes the entry decision of all firms before making its pricing decision. We define an entry strategy profile in this game as a vector $\boldsymbol{e}(\boldsymbol{c}) \equiv (e^1(c_1), e^2(c_2))$ where $\boldsymbol{c} \equiv (c_1, c_2)$ is the vector of realized cost types of the duopolists. Finally, the pricing strategy profile can be represented by a vector $\boldsymbol{\rho}(\boldsymbol{c}, \boldsymbol{\upsilon}) = (\rho^1(c_1, \boldsymbol{\upsilon}), \rho^2(c_2, \boldsymbol{\upsilon}))$.

We denote the equilibrium pricing strategy profile by $p* = (p^{1*}, p^{2*}) \in \boldsymbol{p}^*$ where both p^{1*}, p^{2*} are a six-tuple consisting of the entry strategy of the form $e^{j*} = (\nu^j_L, \nu^j_H) \equiv$ *(firm j's entry decision when it realizes type L, firm j's entry decision when it realizes type H)* when $(\nu^j_L, \nu^j_H) \in \Upsilon$, $j \in \{1, 2\}$, and pricing strategy of the form $p^{j*}_{\theta,\gamma}$ where $(\theta, \gamma) \in \{L, H\} \times \{E, N\} \in \Omega^j \times \Upsilon^{-j}$ and $p^{i*}_{\theta,\gamma} \equiv$ *(price when type L and the other enters, price when type L and the other does not enter, price when type H and the other enters, price when type H and the other does not enter).* Thus a full pricing strategy profile for the one-shot stage game can be represented as follows.

$$((\nu^1_L, \nu^1_H), (\nu^2_L, \nu^2_H), (p^1_{LE}, p^1_{LN}, p^1_{HE}, p^1_{HN}), (p^2_{LE}, p^2_{LN}, p^2_{HE}, p^2_{HN}))$$

where $\boldsymbol{p}^* \in \mathbf{R}^{2+|\Omega^i \times \Upsilon^{-i}|+|\Omega^{-i} \times \Upsilon^i|}$. Let's assume p^L, p^H to be optimal monopoly prices for type L and type H respectively. From the conditions above we can deduce that when the rival, say firm 2, does not enter, then firm 1 sets its price at the monopoly price with respect to its realized cost. Thus we will only be solving for the situations where both firms enter and decide on the pricing strategy. Also, note that the entry strategy profiles $((N, E), (N, E))$ and $((N, N), (N, N))$ will only be considered as part of off-equilibrium strategy profile of the subsequent repeated game since the former entry strategy profile is counter intuitive on incentive compatibility grounds and the latter leads to no production in the market making it uninteresting from our modeling perspective.

Now we can state the following proposition.

Proposition 3.4.1. *The stage game where Bertrand duopolists face symmetric discrete cost uncertainty and an avoidable fixed cost, possesses two classes of Pure Strategy Perfect Bayesian equilibria (PBE) corresponding to the entry and pricing strategy profiles as follows:*

- *Symmetric Equilibrium: In this equilibrium a type L firm always enters and a type H firm stays out of the market. Notationally,* $((E,N),(E,N),(p^1_{LE},p^1_{HE},p^1_{LN}=p^L,p^1_{HN}=p^H),(p^2_{LE},p^2_{HE},p^2_{LN}=p^L,p^2_{HN}=p^H))$
- *Asymmetric Equilibrium: In this equilibrium, only one firm enters irrespective of its type. Notationally,* $((E,E),(N,N),(p^1_{LE},p^1_{HE},p^1_{LN}=p^L,p^1_{HN}=p^H)),(p^2_{LE},p^2_{HE},p^2_{LN}=p^L,$ $p^2_{HN}=p^H))$
- *No other entry strategy profile in this game forms a part of any pure strategy equilibrium in this game,* $\mathcal{G}$.

Proof: *See Patra, 2015[21].*

From this proposition, it is obvious that the symmetric equilibrium strategy profile $((E,N),(E,N),(p^1_{LE}=c_L,p^1_{HE}>c_L,p^1_{LN}=p^L,p^1_{HN}=p^H),(p^2_{LE}=c_L,p^2_{HE}>c_L,p^2_{LN}=p^L,p^2_{HN}=p^H))$ constitutes a natural equilibrium of the game. This is the most important result from our repeated game perspective. So $p^*((E,N),(E,N)) = ((p^1_{LE}=c_L,p^1_{HE}>c_L,p^1_{LN}=p^L,p^1_{HN}=p^H))$. The asymmetric equilibrium strategy profile is an artificial one where a type H firm prices below c_H when the rival firm has entered. We use this result to motivate the penal code in our colussive equilibrium strategy in the repated game that we analyze below.

Notice that, unlike Athey and Bagwell (2001[3] and 2008[4]), the stage game here is an extensive form game and, in equilibrium, it possesses permanent inefficiencies where either the market is not served with positive probability or the firm which always enters earns a negative profit with positive probability.

Now we proceed to the infinitely repeated game[10] . Our primary objective in this section is to investigate production efficiency and collusive behavior among firms by allowing for pre-game communication.

[10]another interpretation is that firms believe that the game will be played in the subsequent period with positive probability

3.5 The Repeated Game

In this section we define and analyze the repeated game. We also provide a theorem to show the existence of a *PPE* that is production efficient.

3.5.1 Recursive Representation of the Stage Game

In the papers by Athey and Bagwell and Sanchirico that we have discussed earlier there existed a unique Bertrand-Nash equilibrium (without any side payment possibilities) in the stage game where $p = \mathbb{E}[\min_i\{mc_i\}]$ where $\mathbb{E}$ is the Kolmogoroff Expectation Operator. However, we have a multiplicity of equilibria (See the section on the stage game)in the stage game we are considering due to the endogeniety of participation of firms. In this repeated setting we include announcements, which did not explicitly appear in the one-shot stage game due to the classic "Cheap Talk" argument. As we have discussed earlier, we analyze whether announcements may facilitate collusion, an important concern for anti-trust. Now we go on to define the firm strategies in the stage game of the repeated game in the following manner. We represent the space of policies of a firm by

$$S^i = \{\alpha^i|\alpha^i : \Omega^i \to A^i\} \times \{e^i : \Omega^i \times A^i \times A^{-i} \to \Upsilon^i\} \times \{\rho^i : \Omega^i \times A^i \times A^{-i} \times \Upsilon^i \times \Upsilon^{-i} \to \mathbb{R}\}.$$

A typical policy for a firm i when its realized cost type is c_i, firm j's announcement a^j and firm j's entry decision e^j is denoted by $s^i(c_i, a^i, a^j, \upsilon^i, \upsilon^j) = (\alpha^i(c_i), e^i(c_i, a^i, a^j),$ $\rho^i(c_i, a^i, a^j, \upsilon^i, \upsilon^j))$. Now, let's assume $\boldsymbol{c} = (c_i, c_{-i})$ where $c_i = c_j, c_{-i} = c_k$ in state $(j, k) \in \Omega$, $\boldsymbol{a} = (a^i, a^{-i})$, $\boldsymbol{e} = (e^i, e^{-i})$ and $\boldsymbol{\upsilon} = (\upsilon^i, \upsilon^{-i})$ where υ^i and e^i represents the announcement and entry decision of firm i. We also define several vectors here for future use.

$$\boldsymbol{\alpha}(\boldsymbol{c}) \equiv (\alpha^i(c_i), \alpha^{-i}(c_{-i}))$$

$$\boldsymbol{e}(\boldsymbol{c}, \boldsymbol{a}) \equiv (e^i(c_i, \boldsymbol{a}), e^{-i}(c_{-i}, \boldsymbol{a}))$$

$$\boldsymbol{\rho}(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{\upsilon}) \equiv (\rho^i(c_i, \boldsymbol{a}, \boldsymbol{\upsilon}), \rho^{-i}(c_{-i}, \boldsymbol{a}, \boldsymbol{\upsilon}))$$

$$\boldsymbol{s}(\boldsymbol{c}, \boldsymbol{a}) \equiv (s^i(c_i, \alpha^i(c_i), e^i(c_i, \boldsymbol{a}), s^{-i}(c_{-i}, \alpha^{-i}(c_{-i}), e^{-i}(c_i, \boldsymbol{a})))$$

The policy vector $\boldsymbol{s}(c)$ mentioned above determines the path of the game, i.e., it determines announcements as well as the entry and price responses to these announcements. Thus we can write down the stage game payoffs conditional upon the realization of types as $\pi^i(\boldsymbol{s}) = \mathbb{E}_{c_i \in \Omega^i}[\pi^i(\mathbf{s}(\mathbf{c}), c_i)]$. Since we are dealing with a duopoly here, i and $-i$ can be safely replaced by 1 and 2. We will be using this interchangeably for the rest of the game without further declaration when we do so.

Since we will be exploring the possibility of collusion among the duopolists during the repeated play of the game we make a stop here to declare some of the quantities that form a part of the expected return of firms every period they collude. We denote $\pi^{L^-} = \frac{\Pi^{L^-}}{2}$ to represent the profit of a single firm when the duopolists realize cost c_L enter the market announcing type L, and collude at price p^L to split the monopoly profit equally. The quantity $\pi^{H^-} = \frac{\Pi^{H^-}}{2}$ is defined analogously for type H. As indicated earlier, we discount these payoffs by a factor $\delta \in (0, 1)$ per time period when we compute the present value of the returns.

As we have mentioned earlier, firms do not have a direct monitoring mechanism in this game and they use public signals to monitor the actions of their rival. Since a firm's pay off function is dependent on the history of publicly observable outcomes like announcements, entry decisions and prices and not on any of their own private history, the solution concept of a *PPE* seems applicable for this purpose[11]. Thus, before we go on to state the main theorem, we would like to introduce some definitions and solution concepts for the type of game we are considering. The solution concept we will primarily be using here is Perfect Public Equilibrium where firms condition their strategy on realization of public signals.

3.5.1.1 Perfect Public Monitoring

We deal with perfect public monitoring as a special case of Imperfect Public Monitoring that we define below.

3.5.1.2 Imperfect Public Monitoring

The concept of Imperfect Public Monitoring is very important in our analysis. Since monitoring equilibrium action of players' in this game is of primary importance, and since there is no proper mechanism to monitor the type and policy function of the firms, players in our game turn to a public signal in order to try inferring the strategy that the rival actually played. We call this imperfect public monitoring. In repeated games with Imperfect Public Monitoring (hence forth IPM), players information is a stochastic public signal and the distribution of the public signal is dependent on the strategy profile chosen by the firms. Let Y be the space for public signals and is finite, and the probability that a public signal $y \in Y$ is generated following a strategy profile s is $\mu(y|s)$. For a game with perfect monitoring $Y = A$.

Also, like Athey and Bagwell ('01, '04), we will not be analyzing mixed strategy profiles in this game[12] In our case even though the strategy profile is history dependent, since we

[11]the concept of sequential equilibrium by Kreps and Wilson[16] does not exclusive condition on the public history

[12]This is purely for analytical purposes and to avoid cumbersome measurability details arising from a

are assuming IID cost realizations over time t, some of the actions are not correlated over time. To be precise, in this repeated game we try to detect if a firm has deviated from a proposed equilibrium strategy and we use realizations of public signal to detect such deviation[13]. Now, suppose $s^i = (\alpha^i, e^i, \rho^i)$ is an equilibrium strategy. A firm can deviate from the equilibrium by choosing a deviant announcement, $a^i \neq \alpha^i(c_i)$, a deviant entry decision, $v^i \neq e^i(c_i)$, a price deviation $p^i \neq \rho^i(c_i, \boldsymbol{a}, \boldsymbol{v})$ or any combination of these deviations. All these deviations can be represented by alternative policy function formulations, $\tilde{s}^i \neq s^i$. Formally, we distinguish them into *on-schedule* and *off-schedule* deviations. An *on-schedule* deviation is a deviation that is not observable to the players during the equilibrium path of play. In the specific game that we are considering such a deviation will come from what we call a *mimicking* deviation where a type H firm will pretend to be type L, and share the market with type L. In terms of policy function, this refers to a situation when a type H firm adopts the policy that the equilibrium specifies for type L. Formally, $\tilde{s}^i$ specifies that cost type c_j mimics type $\hat{c}_j \neq c_j$, i.e., $\tilde{\alpha}^i(c_j) = \alpha^i(\hat{c}_j)$, $\tilde{e}^j(c_j, \tilde{\alpha}^i(c_j)) = e^j(\hat{c}_j, \alpha^i(\hat{c}_j))$ and $\forall \boldsymbol{a}$ and $\forall \boldsymbol{v}$, $\tilde{\rho}^i(c_j, \boldsymbol{a}, \boldsymbol{v}) = \rho^i(\hat{c}_j, \boldsymbol{a}, \boldsymbol{v})$. Since a low type firm has natural disincentives to mimic the high type in our setting, we will only be considering cases where a high type firm mimics a low type. Thus, we will also have to enforce the incentive compatibility for equilibrium announcement, price and entry decisions in order to compute our continuation values. An *off-schedule deviation* , on the other hand, is observable to the firms. They are defined as actions or a series of actions that no cost type should adopt in equilibrium. For example, in our model, decisions such as under-cutting the equilibrium price, entering the market when a firm has announced type H are *off-schedule* deviations. As will be discussed later, this refers to the deviation as "off-the-equilibrium-path" deviations. The current literature prescribes harsh punishments for such a deviation in order to make it incentive compatible for firms to play the collusive equilibrium[14].

As has been indicated from the discussion above, since *on-schedule* deviations are not observable, firms use a public signal (a signal that is generated by a mechanism/device that deduces the cost of the rival firm by observing the rival firm's published accounting profit, its announcement, the demand function and an assumed tolerance for statistical errors and compares the result to the announcement of the rival firm) for detection of deviation and this is public information. In a simple sense this means that if a firm has played a strategy that is not conforming with its true type then the machine will detect such a difference of play with probability less than 1 (note that if such a play can be detected with probability

strategy profile being a mapping of the history (actions taken until that point) at a time period $t$and the public realizations at the time period t to the current strategy profile.

[13]In the standard literature notion of a Public Correlation Device is employed in such situations

[14]In this case the literature uses mechanism design approach to deduce the participation constraints

1, its the case of perfect information). However, if the rival has played the strategy that conforms with its true cost type, then the mechanism will always detect fair play. That is to say, the public signal mechanism can assign probability weights conditioning upon an element from the state space of on-schedule deviating moves in any period $\Xi = (D, ND) \times (D, ND) = (\text{Deviation}, \text{No deviation}) \times (\text{Deviation}, \text{No deviation})$ where for every state $(a, b) \in \Xi$,'a' is firm i's strategy and 'b' is firm $-i$'s strategy and (Deviation, No Deviation)$\equiv$ (playing $\tilde{s}^i(c_H, ., .)$, playing $s^i(c_L, ., .)$). We denote the public signal in the similar manner as the state space of deviations . $\boldsymbol{\mu}^i$ represents the probability distribution that firm i's deviation will be detected given the action tuple of all firms in the market. This probability assignment is symmetric and is public information.

$$\boldsymbol{\mu}^i(D|\underline{a}) = \begin{cases} \eta_C & if \;\; a_i = D \\ 0 & otherwise \end{cases}$$

where $\eta_c \in (0, 1)$, $\underline{a} = (a_i, a_{-i}) \in \Xi$ and $a_i \in (D, ND)$. In the case of *off-schedule* deviations $\eta_C = 1$ always. In our game, all firms have access to the public signal.[15]

The most interesting thing with imperfect monitoring is that a firm might use this to its advantage in order to deviate from a collusive equilibrium. However, note that in the particular signaling mechanism that we have introduced earlier does not have a full support, i.e., firms only initiate punishment when the rival in fact has deviated. In a situation where the rival has not deviated the signaling mechanism will detect it with probability 1 and such no punishment will be initiated. In a technical note, this signaling mechanism helps us move away from the premises of APS1 and APS2 since it does not satisfy the full support assumption of the signaling function. This plays a key role when we design the collusive equilibrium.

Now we go on to describe the strategy of the repeated game. Clearly, we are dealing with a game where continuation strategy of firms after a stage game is a function of public monitoring and it does not condition on any private actions that firms take during the stage game. Note that after the realization of the respective pay-offs in every stage game, that stage game ends. Then, the firms observe the public signal before moving on to the subsequent stage game. At the beginning of the subsequent stage game firms realize their new type which is independent of the type they realized in the previous stage game. Then firms play their stage game strategy as a mapping from their realized type and announcement of their rival and proceed with subsequent strategies according to the time line of the stage game set out earlier. Specifically, firms condition their strategy on the history of realized announcements, entry decisions, price decisions and public signals and

[15] See appendix C for a helpful discussion.

not on their private history of types or policy schedules. Such strategies are called *Public Strategies*. We provide the following definition for *Public Strategy*.

Definition 3.5.1. *Public Strategy:* Let $h^t = (h_0, h_1, \cdots, h_{t-1})$ be the t period public history of firm i, i.e., a sequence of all publicly observed signals in the past. Every h^t is a 4-vector tuple, $\{\boldsymbol{a}^{t-1}, \boldsymbol{v}^{t-1}, \boldsymbol{p}^{t-1}, \boldsymbol{y}^{t-1}\} \in \mathbb{R}^{|\boldsymbol{a}^{t-1}|+|\boldsymbol{v}^{t-1}|+|\boldsymbol{p}^{t-1}|+|\boldsymbol{y}^{t-1}|}$ of realized announcements, entry decisions, price decisions up to time period t and public signals and every h_t is a 4-tuple $\{a^t, v^t, p^t, y^t\}$. Let the space of all public signals at period t as $Y^t \subset \boldsymbol{Y}$ and $\boldsymbol{y} \in \boldsymbol{Y}$.

Now, let H^t be the space of public histories h^t at time t. We define the strategy of firm i in period t as $\sigma_i^t : H^t \to S^i$ where S^i has the same meaning as before. We also denote σ^t as period t strategy profile, $\boldsymbol{\sigma}$ as a sequence of strategy profiles over $t = 1, \cdots, \infty$ and $\sigma_i \in \Sigma_i$, $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}$ and $\sigma = (\sigma_i, \cdots, \sigma_I)$. Finally, given history h^t, we define per period stage game payoff of firm i as $\hat{\pi}^i(\sigma^t(h^t))$ and the expected payoff of the game as $\mathbb{E}(\sum_{t=1}^{\infty} \delta^{t-1}\hat{\pi}^i(\sigma^t(h^t)))$ where h^1 is a null set.

Having defined the strategy (which is a *public strategy*) in this game, we now go on to define the equilibrium concept associated with such strategy along the lines of Fudenberg, Levin and Maskin (1994)[10](henceforth FLM) and APS2 in order to analyze the game. The rationale for using this kind of equilibrium is the dependence of the strategy on public realizations only. Since strategies are not necessarily public in the standard concept of pure strategy Sequential Equilibrium proposed by Kreps and Wilson(1982)[16] and since the full support assumption does not hold in our case (due to the fact that imperfect detection mechanism that firms use assigns a positive probability to an unfavorable signal only if (but not necessarily if) a deviation has occurred), we use *Perfect Public Equilibrium (PPE)* as the equilibrium concept which we define below.

Definition 3.5.2. *Perfect Public Equilibrium: A strategy profile* $\sigma^* = (\sigma_i, ..., \sigma_I)$ *is a* Perfect Public Equilibrium (PPE) if,

1. σ_i^* is a *Public Strategy* for all i
2. For each date t and history h^t the strategies are a Nash Equilibrium from that point on. Formally,
$$V_i(\sigma^*|_{h^t}) \geq V_i(\sigma_i, \sigma_{-i}^*|_{h^t}), \quad \forall \sigma_i \in \Sigma_i, \forall i.$$

Note that, unlike FLM and APS2 the use of *PPE* in our game is greatly simplified compared to due to the nature of the monitoring mechanism we have discussed earlier. This is because the monitoring mechanism we have is a composite function of public signals generated by the actions taken by the firms, the application of the folk theorem provided by FLM becomes much simpler. As an aside, we divert away from the symmetric structure of the response mechanism of the penal code by the punished firm and the punishing firm that APS2 prescribes. Now we state the main result of our paper in the following theorem.

Theorem 3.5.1. *Given that firms are sufficiently patient, there exists a Perfect Public Equilibrium (PPE) in an infinitely repeated Bertrand Duopoly facing avoidable fixed cost with imperfect public monitoring for a fixed cost $F \leq (1-\eta_L)\Pi^{H^-}$. This equilibrium is production efficient, i.e., in equilibrium only lowest cost firms enter and produce*[16] *in the market.*

Proof: We provide our proof of this theorem by construction. First, we define the equilibrium strategy in two parts; (1) A collusive agreement, (2) A penal code that enforces the collusive agreement. Then we use standard tools from game theory to show that the strategy outlined in part(1) constitutes such an equilibrium by the characterization of the *PPE.*

Collusive Agreement: The collusive agreement in our Bertrand Duopoly game is outlined as follows.
(i) If both firms announce type L, both enter and set price p^L and split the monopoly profit equally. Each firm gets $\pi^{L^-} - F$ in this situation.
(ii) If one firm announces type L, only type L enters the market and sets price equal to p^L. In this case, this firm gets the whole market share and gets profit equal to $\Pi^{L^-} - F$.
(iii) If both firms announce type H, then both enter, set price equal to p^H, and split the monopoly profits equally. In this case each firm earns $\pi^{H^-} - F$.
Notice that the collusive agreement aims to support productive efficiency of the firms while optimizing the per period payoff of every firm in the repeated game[17].

In order to enforce the above collusive agreement we need to design a credible penal code. But before we do so, we take a moment here to discuss possible deviations and monitoring mechanism in this game. As we have indicated earlier, firms in our game depend on realization of public signals to detect deviation. This means, after firms have played their set of actions in our stage game, a public signal is generated to exhibit if a firm has deviated from the collusive agreement in that game or not, and this signal becomes publicly visible such that all players notice if a particular firm has deviated or not. As has been discussed earlier, we make a distinction between *on-schedule* and *off-schedule* deviations here. As we have indicated in section 5.1.2 firm is said to deviate

[16] Note that the lowest cost firm in this case is with respect to the firms' announcement and as such it does not have to be of type L. For example, if both firms declare type H, then the lowest cost firm is type H

[17] see appendix E for further discussion on this.

off-schedule if it chooses an action or a combination of actions in the stage game that is not specified for any cost realization. For example, in our game, price under-cutting by a firm is an example of such deviation. As we have indicated, in our monitoring mechanism, an *off-schedule* deviation can be detected with probability 'one'. This means that, with probability 'one', the public signal will visibly indicate that a firm has deviated when it has deviated *off-schedule*. The standard literature prescribes the worst available punishment to deter this kind of deviation. On the other hand, an *on-schedule* deviation occurs when a firm chooses an action or a combination of actions that are not prescribed for its own type, but for some other existing type. In our game a *mimicking* deviation is an example of such deviation. The actions are thus "on-the-equilibrium-path". The standard way to prevent this kind of deviation is to play the worst possible Nash equilibrium as a credible threat. But, such a Nash Equilibrium does not exist in our game for the level of fixed cost F we are considering. However, since the specific public monitoring mechanism available in our game can detect such *on-schedule* deviation with probability $\eta_C \in (0, 1)$ via a public signal, we could also use the worst available punishment here just as we do for an under-cutting to the lowest possible cost c_L when the deviant enters as a threat to deter such *on-schedule* deviation. Therefore, under proper mechanism, we will be using this worst available pricing as threat against all kinds of deviation.

Now we prescribe the *penal code* to explain the punishment mechanism and we will subsequently derive the incentive compatibility constraints.

Penal Code:

(a) If the public signal positively detects an *on-schedule* or *off-schedule* deviation, the strategy of the punishing firm and the deviating firm are as follows.

(i) Punishing firm: Enters the market every subsequent period irrespective of its cost type. If the deviating firm has entered, punishing firm sets price equal to c_L. Else, the punishing firm charges monopoly price with respect to its type. Note that the punishing firm would like to floor the price to the lowest possible marginal cost in the market in order to ensure that the rival faces a loss for sure if it enters the market, irrespective of its cost type.

(ii) Deviating firm: Stays out of the market for every subsequent period. If it enters during the punishment phase then it sets its price equal to its marginal cost which is apparently its dominant strategy irrespective of its type.

(b)If no deviation is detected:

Every firm plays according to the initial collusive agreement.

In order to support this penal code we deduce the incentive compatibility constraints in the following section.

Incentive Compatibility Constraints:

As mentioned earlier, we use standard tools from the literature to characterize the con-

tinuation value function and the present-value expected return to derive the incentive compatibility constraints. Even though we are using the solution principle of dynamic programming squared via characterizing a Bellman operator as a function of an endogenous Bellman operator, we have not made an explicit exposition of the technique here. This is due to the very reason that the public monitoring mechanism is deterministic in our program. But this model can be extended to a non-deterministic public monitoring set-up where we replace our public signaling mechanism by a mechanism with simple public correlation with minor changes in the notation and introducing some additional definitions as well as operators. Such modeling will condition the actions on the *PPE* pay-offs and not on public monitoring directly which is not the case here. So we proceed as follows.
We define:

$$\overline{V} = \left[\eta_L^2 \pi^L + \eta_L(1-\eta_L)\Pi^L + (1-\eta_L)\eta_L.0 + (1-\eta_L)^2\pi^H\right]$$

where $\pi^L = \pi^{L^-} - F$ and $\pi^H = \pi^{H^-} - F$. Notice here that $\overline{V}$ is the expected stage game pay-off of a firm in our repeated game when they are playing the game according to the collusive agreement. Now we can write down the incentive compatibility constraints as follows.

IC-1: (Under-cutting being type L): $(1-\delta)\Pi^L \leq \overline{V}$.

As we have discussed earlier, an under-cutting deviation occurs when both firms enter after both declaring type L and one sets a price equal to $p^L - \epsilon, \epsilon > 0$ to win the whole market. Since this is an *off-schedule* deviation this will be detected with probability 'one'. As a result, according to the prescribed penal code, the deviating firm will be earning no profit after such a deviation.
IC-2: (Under-cutting being type H): $(1-\delta)\Pi^H \leq \overline{V}$. This constraint is analogous to IC-1, but for type H duopolists.
IC-3: (Mimicking): $(1-\delta)\pi^c + \delta(1-\eta_C)\overline{V} \leq \overline{V}$ where $\pi^c = (p^L - c_H)\frac{q^L}{2} - F$ is the profit of the *mimicking* firm. We note here from a quick inspection that all incentive compatibility constraints including IC-2 are satisfied when IC-1 and IC-3 hold. For future use, let's denote $\pi^{c^-} = (p^L - c_H)\frac{q^L}{2}$.
Solving the constraints yields us

$$\frac{\Pi^L - \overline{V}}{\Pi^L} \leq \delta \leq 1.$$

As is evident the left limit of the δ, it is a decreasing function of the expected value from playing collusion $\overline{V}$ and an increasing function of the lost cost monopoly profit Π^L. This shows that collusion becomes easier when the ratio between $\overline{V}$ and Π^L are increasing since firms do not have to be very patient to play the punishment. This shows that for the above value of δ, the incentive compatibility constraints hold in order for the designed penal code to be credible which in turn supports the aforesaid collusive equilibrium. ■

3.6 Numerical Example

For a simple exposition of the model, consider a linear inverse demand function of the market given by $Q = A - P$ and we assume the market organization, time-line of the game as described in the previous sections. As before, we assume that every firm pays identical fixed cost F to enter the market. Firms realize cost types with the following distribution; $\mathbb{P}(c_L) = 1/3$ and $\mathbb{P}(c_H) = 2/3$, where $\mathbb{P}$ is the probability operator. Then, the profits, following the description of the game above and maintaining the terms as defined in the game, can be calculated to be:

$$\Pi^{L^-} = \frac{(A-c_L)^2}{4}; \quad \pi^{L^-} = \frac{\Pi^{L^-}}{2} = \frac{(A-c_L)^2}{8}; \quad \pi^L = \pi^{L^-} - F = \frac{(A-c_L)^2}{8} - F$$

$$\Pi^{H^-} = \frac{(A-c_H)^2}{4}; \quad \pi^{H^-} = \frac{\Pi^{H^-}}{2} = \frac{(A-c_H)^2}{8}; \quad \pi^H = \pi^{H^-} - F = \frac{(A-c_H)^2}{8} - F$$

and

$$\pi^{c^-} = \frac{(A-c_L)(A+c_L-2c_H)}{8};$$

where $p^L = \frac{A+c_L}{2}, q^L = \frac{A-c_L}{2}$.

Now for the equilibrium we are considering, $((E,N),(E,N))$, we will need to satisfy the conditions $\frac{2}{3}\Pi^{L^-} - F > 0$; $\frac{2}{3}\Pi^{H^-} - F < 0$ (see appendix D for a helpful discussion), for the Nash equilibrium in the one-shot game to hold. Note that we showed previously, that *PPE* holds automatically when F falls in the following bounds that we obtain by combining the two conditions above.

$$\frac{2}{3}\Big(\frac{A-c_L}{2}\Big)^2 > F > \frac{2}{3}\Big(\frac{A-c_H}{2}\Big)^2.$$

As we have discussed in the theorem 5.1, we also explore the possibility of collusion when

$$\frac{2}{3}\Big(\frac{A-c_H}{2}\Big)^2 > F.$$

Solving from the IC constraints as defined in the characterization of *PPE*, it is immediate that the *PPE* holds when δ is in the following bounds.

$$\frac{13}{18} - \frac{2}{9}\left(1 + \frac{c_L - c_H}{A - c_L}\right)^2 \leq \delta \leq 1.$$

The following diagram represents this relationship.

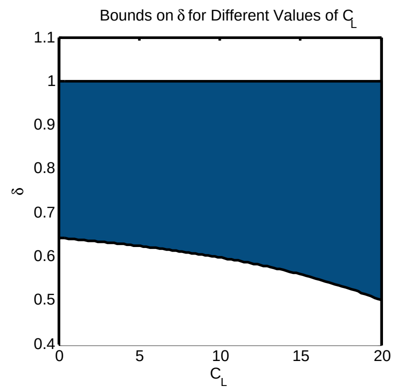

Figure 3.1: CLvsDelta

3.7 Efficiency of the *PPE*

Here we discuss the efficiency of our *PPE* by focusing on the feasibility of production at different fixed cost levels while maintaining efficiency. We proceed with the discussion in a case by case basis.

1. $\sigma^*|_{h^t} = ((E,N),(E,N),\boldsymbol{p}^*,\boldsymbol{y}\in\boldsymbol{Y})$ when $(1-\eta_L)\Pi^{H^-} \leq F \leq (1-\eta_L)\Pi^{L^-}$. Clearly, this can be supported as a Nash equilibrium of the repeated game since we deduced earlier that this strategy profile forms a Nash equilibrium of the one-shot stage game. As long as F is within these bounds firms can continue playing the stage game strategy profile irrespective of the announcements and monitoring mechanism for infinite time periods. It is obvious that this will constitute a *PPE* of the repeated game.

2. If $F > (1-\eta_L)\Pi^{L^-}$ then no one enters the market, i.e., play $(\sigma^*|_{h^t}) = ((N,N),(N,N),\boldsymbol{p}^*,\boldsymbol{y}\in\boldsymbol{Y})$. This strategy is self explanatory since the fixed cost is too high for any type of firm to earn positive expected profit in any situation.

3. If $F < (1-\eta_L)\Pi^{H^-}$, both firms expected profits are larger than the fixed cost. So every firm would like to enter even though production efficiency will be sacrificed. However, as we discovered in our one-shot dynamic game the entry strategy $((E,E),(E,E))$ and the related pricing strategy $p^*((E,E),(E,E))$ cannot be a part of the Nash equilibrium profile. Thus, this poses a problem in order for the strategy profile $\sigma^*|_{h^t} = ((E,E),(E,E),\boldsymbol{p}^*,\boldsymbol{y}\in\boldsymbol{Y})$ to be a *PPE*. The problem is also amplified since a *mimicking* deviation from the collusive agreement could only be partially detected by the monitoring mechanism in this game. So firms of type H will find it

profitable to enter the market *mimicking* the strategy of a type L firm, since there is a chance it will go undetected. However, we show that the penal code designed above prevents such deviation in this repeated game. We use a much simpler variant of the folk theorem proposed by Fudenberg, Levin and Maskin (1994) for this purpose. Note that the simplicity is due to the specific form of public monitoring mechanism we use in this game.

3.8 Conclusion

The proposition, the theorem and the example provided above exhibits that, if firms are sufficiently patient, it is possible for them to collude in a repeated Bertrand duopoly when the cost types are discrete and the firms do not know each other's costs. The monitoring mechanism we have chosen plays an important role in designing a penal code that firms can use as a credible threat in order to enforce the collusive equilibrium. A feature of this mechanism is the asymmetric action of the punishing firm and the deviating firm when the penal code is enforced, which differs from standard APS2 and FLM literature. Secondly, due to the existence of avoidable fixed cost, it becomes possible for firms to collude in our set-up which is a new result compared to the existing literature on repeated as well as one-shot Bertrand games incorporating avoidable fixed costs. Added to this, when costs are unknown and firms depend on a public signal to monitor each other's action, collusion mostly depends on the probability distribution of the types and the probability of detection. With imperfection in detection, if the probability of realizing a high type increases, collusion becomes harder since a firm which realizes low cost would undercut with a higher probability since it will expect the other firm to be high type with increased probability. Moreover, when the difference between the costs increase, the low cost firm will find it easy to initiate punishment and the high cost firm will find it very harsh to do so. Finally, our model preserves efficient production by the duopolists in this collusive equilibrium which could not be achieved without incorporating fixed cost into the model. In our model, only the lowest cost types enter following announcements and produce as long as the game is played. Special mention must be made here about the situation when both firms announce type H and enter the market to collude. Since, there is no firm with a lower cost type in the market, production by the high type indeed preserves efficiency and welfare in the market as compared to staying out.

We note here that, unlike Athey and Bagwell(2001)[3] we do not consider unequal market sharing agreements and quantity restriction by firms in this model to model the penal code for the *on-schedule* deviations in the equilibrium path. We abstract away from such a model due to our innovation of the specific public signaling device which is different from the simple public correlation mechanism by Athey and Bagwell (2001)[3]. However, it will be an interesting avenue to pursue to include simple public correlation mechanism in our model. As another direction, it will also be interesting to look at a situation when the costs are persistent over time. In such situation, with the inclusion of fixed cost, collusion may become harder since firms with low costs may predate than collude when they decide to enter. Persistence of costs will also covert the repeated game into a dynamic game with hidden state variable, a model that was developed by Cole and Kocherlakota (2001)[8]. We can also look at situations where firms could strategically undertake prior investments in order to influence their cost type in future play of the game. This model can be extended to continuum of types and it will be interesting to see if the results hold in a Cournot set-up. Finally, it will also be interesting to explore when firms pay just a one time fixed cost before the repeated play of the Bertrand game with unknown costs.

Our model is novel in the sense that it is a step forward in the direction of endogenous participation of firms and inclusion of avoidable fixed costs in Bertrand games, an area in economics that is not fully developed yet. This model is intended to serve as an anecdote to many other interesting issues arising from such endogeniety in market decision making.

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Chapter 4

Bertrand Competition with One-sided Cost Uncertainty

4.1 Introduction

Due to the existence of the fundamental discontinuity in the profit functions of Bertrand competing firms with homogeneous product with bounded price many existence results are still an active area of research. The classical outcome of competitive profit when firms with symmetric constant marginal cost compete one-shot in prices while having perfect information about each other has been difficult to replicate in different informational and cost setting. For example, when both firm's costs are unknown, Spulber (1995) [5] has shown that, with parameterized asymmetric costs, all but the highest cost firm expect positive profit when costs are drawn from a continuous distribution. Baye and Kovenock (2008) [1] showed that with a fixed cost and constant marginal cost of firms there exists a mixed strategy Nash-equilibria in the full information Bertrand game. They argue that the reason for this existence is that the cost function in this case essentially is concave. Blume(2003)[2], in a seminal paper has indicated that when Bertrand duopolists have different constant marginal costs, have perfect information about each other and the highest marginal cost is below the monopoly price of the lowest cost firm, then in equilibrium the low cost firm will charge a price equal to the higher marginal cost and the higher cost firm would randomize between the two costs. However, observe that this is again a partial mixed strategy equilibrium. In a recent paper, Routledge [4] (2010) showed that in a classical model of Bertrand competition with homogeneous goods and constant marginal costs, only a mixed strategy Nash-equilibrium exists when the marginal costs are unknown and there is symmetric tie-breaking rule.

Unlike these prior studies, we assume that the cost type of one firm is unknown, and that the cost types are asymmetric between firms. The assumptions reflect real life competitions among firms. We characterize the full equilibrium, and show that pure strategy Nash equilibrium can exist in some cases.

This paper is organized as follows. Section 4.2 presents the model set-up. Section 4.3 provides the full characterization of the equilibrium and finally, Section **??** concludes the paper.

4.2 The Model

Two firms, each indexed by $i \in N := \{1, 2\}$, are engaged in a Bertrand price competition with homogeneous goods and equal rationing rule. That is, firms compete by setting their prices simultaneously and independently. Firms that set the lowest price serve all the demand. In case of the tie, firms share the demand equally.

The inverse demand function $D : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following properties. First, there exists a "choke-off price" $p^{\max} \in \mathbb{R}_+$ such that $D'(p) < 0 < D(p)$ for all $p \in (0, p^{\max})$ and that $D(p) = 0$ for all $p \in [p^{\max}, +\infty)$. Second, D is continuous on the entire domain $\mathbb{R}_+$ and twice continuously differentiable on $(0, p^{\max})$. We denote $q^{\max} := D(0)$.

Firm 1's marginal unit cost is publicly known to be c_1. Firm 2, on the other hand, privately observes its own marginal unit cost c_2 which can take two values c_L and c_H with probability η_L and $1 - \eta_L$, respectively. We call firm 2 to be of type L (H) if it faces a marginal cost c_L (c_H) and we assume $0 < c_L < c_H < p^{\max}$. The profit function of a firm is given by $\pi_j(p) := (p - c_j)D(p)$ for each $p \in \mathbb{R}_+$ and for each $j \in \{1, L, H\}$. As is standard, we assume that $(p - c_j)D(p)$ is strictly concave in price p over $[0, p^{\max}]$. We denote by p_j^m, q_j^m, and π_j^m, the monopoly price, quantity, and profit, respectively, for each cost realization: $p_j^m := \operatorname{argmax}_{p \in \mathbb{R}_+} \pi_j(p)$, $q_j^m := D(p_j^m)$, and $\pi_j^m = \pi_j(p_j^m)$ for each $j \in \{1, L, H\}$. Note that p_j^m is well-defined and unique for each $j \in \{1, L, H\}$. Each firm maximizes its expected profit.

The Equilibrium Concept we use in this analysis is the *Pure Strategy Bayesian Nash equilibrium (PSBNE)* of this Bertrand game. A Bayesian Nash Equilibrium in this game is a triple $(p_1^*(c_1) \equiv p_1^*, p_2^*(c_L) \equiv p_L^*, p_2^*(c_H) \equiv p_H^*)$ where $p_1^*(c_1)$ is the price that firm 1 sets, $p_2^*(c_L)$ is the price that a type L firm 2 sets and $p_2^*(c_H)$ is the price that a type H firm sets in equilibrium. We denote by E^* the set of pure strategy *PSBNE*.

4.3 The Full Characterization of *PSBNE*

Lemma 4.3.1. *No firms price below their marginal cost.*

Proof. Pricing below marginal costs will generate a negative profit, if the firm wins the competition. Because the game is static, the firm cannot gain future benefits. Such strategy violates the individual rationality constraint, because the firm can always produce nothing. □

Proposition 4.3.1. *If $c_1 < c_L$, then $E^* = \{(c_L, p_L, p_H) \mid p_L > c_L, p_H > c_L\}$.*

Proof. First, according to Lemma 4.3.1, firm 1 prices at or above c_1. Second, as long as firm 1 sets price above c_L, type L firm 2 optimally responds by cutting price by a small ϵ and captures the whole market.[1] Given any pricing strategy of type L firm 2, firm 1 is willing to undercut type L firm 2 as long as capturing the entire market generates higher expected profit than pricing at p_1^m (assuming $p_1^m \le c_H$) but losing the market when facing

[1] The monopolistic price of firm 1 in this case is lower than that of type L firm 2, which follows from our model assumptions.

type L firm 2. Thus the optimal strategy for firm 1 depends on the monopolistic price p_1^m. We discuss below in detail.

Case 1 ($p_1^m \in [c_1, c_L)$): Observe that if firm 1 sets its price in the interval $[c_1, c_L)$ then firm 1's optimal price would be to set $p_1 = p_1^m$. Then, both type L and type H firm 2 would set their respective prices strictly above p_1. Given the best response of both types of firm 2, firm 1 does not have any profitable deviation, which satisfies the conditions for an equilibrium.

Case 2 ($p_1^m = c_L$): Consider firm 1 setting a price $p_1 = p_1^m = c_L$. In such a case a type L firm 2 would best respond by charging price $p_L \geq c_L$ and and a type H firm 2 would set $p_H > c_L$, both types of firm 2 making a 'zero profit'. If a type L firm 2 chooses a price at c_L, Firm 1 in will make a profit of $\frac{1}{2}(c_L - c_1)D(c_L)$. But firm 1 can profitably deviate to charging a price $c_L - \epsilon$ and making monopoly profit, given the best responses of firm 2 irrespective of its type. Thus (p_1^m, c_L, p_H) cannot be a *PSBNE*. Now consider the possibility when a type L firm 2 charges a price $p_L > c_L$. In such a case, firm 1 will have no profitable deviation since it is making monopoly profit by charging c_L and given that both types of firm 2 are best responding by charging strictly above $p_1^m = c_L$. A type H firm 2 is making a 'zero' profit in this strategy profile. In order to gain any market share a type H firm 2 will have to price below c_L but earn negative profit in such a case. So a type H firm 2 does not have a profitable deviation given the pricing strategy of firm 1. Finally, when a type L firm 2 sets its price $p_L > c_L$, it receives no market share and as such receives an expected profit of 'zero' which is no less than what it receives when it sets a price on or below c_L. Thus the pricing strategy profile $E^* = (p_1^* = p_1^m = c_L, p_L^* > c_L, p_H^* > c_L)$ is an equilibrium strategy profile when ($p_1^m = c_L$).

Case 3 ($p_1^m > c_L$): For this case, we first inspect if there exists an equilibrium pricing strategy profile such that $p_1^* \in (c_L, c_H)$. Given p_1^*, type L firm 2 will optimally undercut firm 1. The expected profit of firm 1, given that type L firm 2 undercuts, is $(1 - \eta_L)(p_1^* - c_1)D(p_1^*)$, which implies $p_1^* = p_1^m$. The question becomes, does firm 1 undercuts type L firm 2? The indifference condition is

$$(1 - \eta_L)(p_1^m - c_1)D(p_m^*) = (p_1^s - c_1)D(p_1^s),$$

where Firm 1 is willing to undercut only to p_1^s.

If $p_1^s < c_L$, then type L firm 2 prices at c_L and firm 1 prices at marginally below c_L. If type L firm 2 prices above c_L, firm 1 optimally responds by pricing below type L firm 2. Iterative argument implies that firm 2 prices at c_L. Given firm 2's strategy, firm 1 prices at marginally below c_L.

If $p_1^s \geq c_L$, then type L firm 2 prices at c_L, but firm 1 prices at $\min\{\max\{c_L +$

$\epsilon, p_1^s\}, c_H - \epsilon\}$.

□

Proposition 4.3.2. *If $c_1 = c_L$, then $E^* = \emptyset$.*

Proof. Notice that if firm 1 sets a price below c_1 in equilibrium then it would obtain a negative profit. A profitable deviation for firm 1 would be to charge a price on or above c_1. In addition, firm 1 cannot also charge a price above c_H. Doing so results in being undercut by firm 2 irrespective of its type. So the only interesting case to look at is the pricing between $[c_L, c_H]$.

Firm 1 obtains positive profit, because it can set a price in (c_L, c_H) to win the market when firm 2 draws a high type. Then, firm 2 with low type must also obtain positive profit. If not, firm 2 with type L can profitably deviate by taking $p_L \in (c_L, p_1]$. This implies that $p_1 = p_L (> c_1)$. It is, however, impossible because each firm can profitably deviate by slightly undercutting. Hence, the game does not have a *PSBNE*.

The result is striking since by pricing larger than c_L, firm 1 can earn positive expected profit. Given firm 1's strategy, firm 2 of type L always wants to marginally undercut firm 1. Given this best response by firm 2 of type L, it is always marginally better for firm 1 to lower its price further to capture the market. This is a profitable deviation for firm 1 since for any given price of firm 2 of type L, the expected profit when not undercutting is $\pi_H(p_1 - c_L)D(p_1)$ where $p_1 = p_L + \epsilon$, which is always smaller than the expected profit if firm 1 undercuts is $(p_1 - c_L)D(p_1)$. The result hinges crucially on the assumption that firm 1 has the same cost as firm 2's lower type, i.e., $c_1 = c_L$ (see the next case). It also depends on the assumption that firm 2 takes firm 1's strategy as given and does not account for the effect of its deviation on firm 1's response.

□

Proposition 4.3.3. *If $c_1 \in (c_L, c_H)$, then*

$$E^* = \begin{cases} \{(p_1^m, p_L^m, p_H) \mid p_H > p_1^m\} & \text{if } p_L^m < p_1^m \leq c_H \\ \emptyset & \text{if } p_L^m < p_1^m \text{ and } c_H < p_1^m \end{cases}.$$

Proof. As is in the previous case, in equilibrium it must be the case that $c_L \leq p_L \leq c_H \leq p_H$. The first and third inequalities are due to participation constraints. The second inequality is due to firm 1 will undercut if both types of firm 2 set price larger than c_H. For firm 1, it must be that $c_1 \leq p_1 \leq c_H$.

We discuss two cases. First, $p_1 \geq p_L^m$. Since the costs of the two firms are different, firm 2 does not necessarily want to set the price right below the price of firm 1. We explain this result below.

In this case, it is optimal for firm 2 to set price $p_L = p_L^m$. The reason is $p_1 \geq p_L^m$, so firm 2 of type L captures the market and earns monopolistic profit. Given this, the question is whether firm 1 has incentive to deviate. Firm 1 prefers to deviate from p_1 if and only if $(1-\eta_L)(p_1 - c_1)D(p_1) \leq (p_L^m - c_1)D(p_L^m)$.The break-even price, if it exists, must be larger than p_L^m since $1 > \pi_H > 0$ and $p_1^m > p_L^m$. Call this price p_1^s. If p_1^s exists, then firm 1 will NOT deviate by undercutting. If p_1^s does not exist, firm 1 will deviate by undercutting and this goes on until price reaches c_1 but given c_1, firm 1 would want to deviate to earn positive profit.

In this case, pure strategy Nash equilibrium exists if and only if there exists a price such that $\pi_H(p_1 c_1)D(p_1) = (p_L^m c_1)D(p_L^m)$, which is equivalent to $\pi_H(p_1^m c_1)D(p_1^m) > (p_L^m c_1)D(p_L^m)$ if $p_1^m < c_H$ and $\pi_H(c_H c_1)D(c_H) > (p_L^m c_1)D(p_L^m)$ if $p_1^m \geq c_H$. Firm 1 charges $p_1 = \min\{p_1^m, c_H\}$. Firm 2 charges its monopolistic price.

Second, $p_1 < p_L^m$. Then firm 2 of type L will deviate by undercutting firm 1. This goes on until price is at c_1 in which case firm 1 will deviate. Hence there is no pure strategy NE. □

Proposition 4.3.4. *If $c_1 = c_H$, then $E^* = \{(p_1, p_L, p_H) \mid p_1 \geq c_1, p_L < p_1, p_H \geq c_H\}$.*

Proof. Consider $p_1 < c_L$. Then, both types of firm 2 would best respond by $p_L, p_H > p_1$. Now consider $p_1 = c_L$. Then a type L firm 2 would best respond with $p_L \geq c_1$ [?] and a type H firm 2 would best respond with $p_H > c_1$. Now, consider firm 1 charging a price $p_1 \in (c_L, c_H)$. In such a case, a type L firm 2 would best respond by $p_L = p_1 - \epsilon$ and a type H firm 1 would best respond by $p_H > c_H$. Given the best responses in the above cases, firm 1 would earn a negative expected profit [2]. However, firm 1 can profitably deviate in all the aforesaid cases by charging a price $p_1 \geq c_H$ where it can earn a minimum expected profit of 'zero'. So we establish that in equilibrium firm 1 would not set a price $p_1 < c_H$. Now, consider firm 1 setting $p_1 > c_H$. Then firm 2 would best respond with $p_1 - \epsilon$ irrespective of its type leading firm 1 to earn a 'zero' profit [3]. In such a case, firm 1 setting a $p_1 = min\{p_L, p_H\} - \epsilon$ would be a profitable deviation. Using standard Bertrand argument of spiral undercutting of prices, any price strictly above c_H would yield a profitable deviation for firm 1. So the only surviving candidate price is $p_1 = c_H$. Now, consider $p_1 = c_H$. In such a case a type L firm 2 would best respond by $p_L < p_1$ and a type H and a type H firm 2 would best respond with $p_H \geq c_H$. Now consider the pricing strategy profile $\{(p_1, p_L, p_H) \mid p_1 = c_1 = c_H, p_L = c_H - \epsilon, p_H \geq c_H\}$. Observe that, given this pricing profile no firm will have any profitable deviation. So, this is a Nash-equilibrium pricing strategy profile. □

[2] Since firm 1 would make a positive sale when firm 2 is type H, which occurs with positive probability. In the case firm 2 is type L firm 1 would earn a zero profit.

[3] observe that this result holds irrespective of the location of the monopoly price of both the cost types

Proposition 4.3.5. *If* $c_1 > c_H$*, then* $E^* = \{(p_1, c_H, p_H) \mid p_1 > c_H, p_H \geq c_H\}$.

Proof. Using the arguments from the previous section it is immediate that $p_1 \not< c_L$ in equilibrium. Now consider $p_1 = c_L$. In such a case a type L firm 1 will best respond by setting $p_L \geq p_1$ and a type H firm 2 will best respond by $p_H > p_1$. In this case firm 1 would earn a negative profit. But firm 1 can set its price at c_1 and make a 'zero' profit which is a profitable deviation. So in equilibrium it must be that $p_1 \neq c_L$. Now consider $p_1 \in (c_L, c_H)$. In this case firm 2 will best respond with $p_L = p_1 - \epsilon$ and $p_H > p_1$. Arguing similarly as the last section we can conclude that firm 1 will not set a price $p_1 \in (c_L, c_H)$ in equilibrium. Now consider $p_1 = c_H$. In this case $p_L = p_1 - \epsilon$ and $p_H \geq p_1$. Observe that in expectation, firm 1 will also gain positive market share in this situation and earn negative profit. As such, firm 1 can profitably deviate by setting $p_1 = c_1$. Now consider firm 1 setting $p_1 \in (c_H, c_1)$. In this case both types of firm 2 will undercut p_1, forcing firm 1 to earn zero profit. Observe that firm 1 can consider deviating only by setting a larger p_1. But in such a case this price will be undercut and firm 1 will earn 'zero' profit. So there is no strictly profitable deviation for firm 1. So we can conclude that in equilibrium firm 1 will set $p_1 \geq c_1$. Now consider an equilibrium price $p_1^* = c_1$. Given this, both firm 1 and firm 2 would charge a price $p_L = p_H = p_1^* - \epsilon$. In such a case firm 1 would earn a 'zero' profit by not earning any market share. Continuing with that argument, consider the pricing profile $(p_1^* = c_1, p_L^* = p_H^* = min\{c_1, p_L^m\})$. Its clear that there is no profitable deviation for any firm in this since firm 2 will always gain the market and firm 1 will always price weakly above its marginal cost. □

4.4 Conclusion

We have provided a full characterization of the pure strategy equilibrium in a standard Bertrand game with one-sided cost uncertainty. We find that one-sided cost uncertainty and bounded known cost type are sufficient to guarantee the existence of the *PSBNE* in the Bertrand game when costs of one firm is a stochastic draw from some known distribution. Note that when the cost of the certain firm type (firm 1 in our case) matches the lowest cost of the uncertain type (firm 2 in our case) there is no equilibrium. This is a strong feature of our equilibrium which is not present in a standard Cournot game in similar setting [3]. This is largely due to the fact that price as a strategic variable is quite sensitive to the cost of a firm than quantity as such. In a Cournot game the firm with uncertain types takes advantage of the fact that it can change its production according to its cost type while the certain type cannot change the information about its cost and as such production. So while the certain type will be making its decision about the quantity level in expectation, the uncertain type will be able to make a more informed decision than the certain type which basically drives the equilibrium result. Note that, there is no real effect of the order of costs in the Cournot game on its equilibrium characterization.

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Part II

Statistical Fluctuations along the Lennard-Jones Melting Curve

Chapter 5

Statistical Fluctuations along the Lennard-Jones Melting Curve

5.1 INTRODUCTION

Any thermodynamic system over time exhibits a distribution of thermodynamic state variable values which depend on the ensemble adopted. Such fluctuations in first order thermodynamic quantities can be used as a route to calculate second order thermodynamic quantities (*e.g.*, heat capacity and compressibility), and they have been used in molecular simulation studies over many decades to perform this task, [1]. Relatively recently a new use for system property fluctuations has been proposed, and that is to identify those states on the phase diagram that have (to a good approximation) an underlying scale invariance which has been called *isomorphism*, [3, 4, 5, 6, 7, 2] because of similar underlying assembly structures of these thermodynamic state points. Consider a point in the configurational phase space of N molecules which may represent the molecules in a periodic simulation periodic cell, where $\underline{r}_i$ is the coordinate of molecule i, and the configurational phase state point is represented in concise form by, $\underline{r}^N \equiv \underline{r}_1\underline{r}_2 \cdots \underline{r}_N$. If ρ is the number density of molecules, and $\tilde{r}_i \equiv \rho^{1/3} r_i$ is a non-dimensionalised coordinate, two state points on the phase diagram (*e.g.*, defined by density and temperature) are said to be isomorphic if the probability distribution function of these states, $P(\tilde{\underline{r}}^N)$, for all $\tilde{\underline{r}}^N$ in the two thermodynamic state points are the same. By extension an isomorphic line on the phase diagram (typically, defined by the density and temperature points) is one along which all state points have the same $P(\tilde{\underline{r}}^N)$.

Assuming pair-wise additivity of the potential energy surface, the analytic form of the pair potential is, in addition to the density and temperature, the most important factor in controlling the extent of isomorphic behaviour. The Lennard-Jones (LJ) potential is one of the most used classical representations of model condensed phase systems, and is defined by, $\phi_{LJ}(r) = 4\epsilon[(\sigma/r)^{12} - (\sigma/r)^6]$, where ϵ and σ define the characteristic energy of interaction and diameter of the molecule, respectively, and r is the separation between the centres of two of the molecules. The inverse power potential, $\phi_{IP}(r) = 4\epsilon(\sigma/r)^n$, is another relevant potential in the present context, where n is an exponent which governs the steepness of the potential. The inverse power (IP) fluid and solid are examples of perfectly isomorphic condensed phases in which the isomorphic line is defined through the relationship, $\rho^{n/3}/T = const$, where T is the temperature (using the usual molecule-based reduced units). The LJ potential is the sum of two such inverse power terms, and the $n = 12$ IP fluid or solid could be considered to be a possible reference system for the LJ system, and the $n = 6$ IP attractive part of the potential is taken to be a first order perturbation.

The 'melting line' on the phase diagram is where a transition between a fluid and solid (crystalline) state takes place. In fact it is only a line when plotted in the P, T plane, where P is the pressure. On the ρ, T and ρ, P planes, there are coexisting region 'gaps' between distinct fluid and solid single phase zones. Knowledge of the melting line (ML) of a chemical system is important in various chemically relevant fields as the physical state of the molecules can have a strong influence on the physical behaviour (*e.g.*, flow characteristics) of the system. This is important in, for example, geology and high pressure (elastohydrodynamic) lubrication. The melting line is already known to be almost isomorphic, which in part explains the success of various phenomenological 'rules' of melting for many different types of molecule, [8].

The Pearson coefficient, R_p, between the configurational part of the pressure, P_c, and the potential energy, u, has been used as a convenient measure of the extent to which two state points are isomorphic [3, 4, 5, 6, 7, 9]. If the pressure-energy correlation measure, R_p, is equal to unity, the two states would be completely isomorphic; in reality only IP fluids form isomorphic lines, so $0 \leq R_p \leq 1$ for all other model systems having repulsive and attractive components in their interaction potential. The closer R_p is to unity the more 'isomorphic' the two state points can be said to be.

The statistical analysis of data is carried out in a wide range of disciplines, such as Economics, whose experience could be made use of in the branch of statistical physics associated with isomorphism. The purpose of the present study is to determine the Pearson coefficient and related statistical measures of correlation between a variety of thermodynamic state variables (not just the configurational part of the pressure, P_c, and the total potential energy per particle, u) by Molecular Dynamics computer simulation. The statistical theoretical framework employed in Economics is made use of here. This examination elucidates further the nature of near-isomorphic states and by association the factors that influence the melting line. Such conclusions might eventually lead to improved perturbation theories of the liquid state.

5.2 SIMULATION AND DEFINITIONS

The Lennard-Jones pair potential was used to generate the molecular configurations reported here. All quantities presented are given in LJ reduced units ($i.e.$, ϵ for energy, and σ for distance). The potential energy, pair force and other static properties were obtained by truncating the LJ potential interactions at a molecule pair separation of $r = 2.5$ [10]. The usual mean field long range correction formulas, [10] were added to the potential energy and other static properties. The time step was $0.005/\sqrt{T}$, where T is the temperature, and simulations were conducted for up to 10^6 time steps during the post-equilibration stage. The number of particles in the simulation cell, N, was 2048, which is large enough to have minimal finite size effects. Molecular dynamics (MD) simulations were carried out in the constant temperature ensemble using velocity rescaling. State points on the fluid side of the melting line terminating at the triple point at *ca.* T, ρ values of $0.69, 0.85$ [11] were simulated (ρ is the reduced number density or $N\sigma^3/V$, where V is the volume of the cubic simulation cell). The state points simulated were determined via a polynomial fit to numerous sources of molecular simulation fluid-solid coexistence data taken from the literature ($e.g.$, [11, 12]).

Simulations were carried out using different pair potentials to generate the state points, with some being carried out with the LJ potential. The Weeks-Chandler-Andersen (WCA) decomposition of the LJ potential into a steeply repulsive ('r') and a smoothly varying ('background') attractive ('a') part is respectively as follows, [14, 15, 16, 17, 18, 21, 13, 22, 20, 19] $\phi_{WCA,r}(r) = \phi_{LJ}(r) + \epsilon, \quad r \leq r_c$ and $\phi_{WCA,r}(r) = 0 \quad r > r_c$, where $r_c = 2^{1/6}\sigma$ is the position of the minimum of the LJ potential. Also, $\phi_{WCA,a}(r) = -\epsilon, \quad r \leq r_c$ and $\phi_{WCA,a}(r) = \phi_{LJ}(r) \quad r > r_c$, so $\phi_{LJ}(r) = \phi_{WCA,r}(r) + \phi_{WCA,a}(r)$ for all r. Some simulations were carried with $\phi_{WCA,r}(r)$, and others using the inverse power potential, [23] $\phi(r) = 4\epsilon(\sigma/r)^{12}$ to generate the configurations, for the same values of T and ρ. The values of the thermodynamic properties of the 'virtual' LJ and WCA potential systems were also computed even for state distributions generated by the other two force fields.

The virial expression for the pressure, P, was used in the simulations, [1, 10]

$$P = \frac{1}{3V}\Big[\sum_{i=1}^{N}\frac{1}{m}\underline{p}_i\underline{p}_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} r_{ij}f_{ij}\Big] \qquad (5.1)$$

where V is the volume of the system, $\underline{p}_i$ is the translational momentum of molecule, i, of mass m, $\underline{r}_{ij} = \underline{r}_i - \underline{r}_j$, and $\underline{r}_i$ and $\underline{r}_j$ are the coordinates of molecules using the *nearest image*, i and j, respectively. The pair force between the molecules is $f_{ij} = -d\phi_{ij}/dr_{ij}$, applying the nearest image convention between molecules i and j. The kinetic contribution to the total pressure is the first term in Eq. (5.1), which by equipartition can be written for equilibrium systems as $\underline{P}_k = \rho k_B T$, where k_B is Boltzmann's constant and $\rho = N/V$. The second term on the right hand side of Eq. (5.1), is the configurational part of the pressure, denoted by P_c. The potential energy per particle is $u =< \sum_{i<j} \phi(r_{ij}) > /N$, where $< \cdots >$ represents a simulation average (the configurational part of the pressure is similarly averaged). For the LJ potential this can be decomposed into repulsive ('r') and attractive ('a') parts, *i.e.*, , $u_r = 4 < \sum_{i<j} \epsilon(\sigma/r_{ij})^{12} > /N$ and $u_a = -4 < \sum_{i<j} \epsilon(\sigma/r_{ij})^6 > /N$, respectively. The LJ potential can also be decomposed into the two WCA contribution parts, $u_{WCA,r} =< \sum_{i<j} \phi_{WCA,r}(r_{ij}) > /N$ for the WCA repulsive potential component, and $u_{WCA,a} =< \sum_{i<j} \phi_{WCA,a}(r_{ij}) > /N$ for the attractive component. In the literature, the potential term, $\phi_{WCA,r}$ is often just referred to as the 'WCA' potential.

Three temperature and density fluid states along the LJ melting line were considered. The three temperatures were 0.7, 4.0 and 60, and the corresponding densities were 0.847, 1.229 and 2.289, respectively. The theory of statistical fluctuations relating to linear regression and the Pearson coefficient is covered in Sec. III. Application of this theory to the simulation data is undergone in Sec. IV. The correlation between P_c and u is computed, as these two quantities were first used to test for isomorphism in previous molecular simulation studies [6]. Correlations between two decompositions of the total potential energy are also assessed. The results from IP and WCA (repulsive part only) and full LJ dynamics are compared. Section V is mainly concerned with a time-dependent extension of the Pearson coefficient criterion.

5.3 THEORY AND RESULTS

In this section the directional relationships between the several variables are analysed. The analysis of the variance, Pearson's Rank Correlation Coefficient, [24] which is sometimes referred to as *Pearson's Product-Moment Correlation coefficient* or PCC for short, and Ordinary Least Squares (OLS) regression techniques, [25] are used for this purpose. Pairs of variables are treated and the standard $t-$test, [26] is carried out to establish the statistical significance of the derived relationship. The goodness of fit of the correlation between the variables, for example, $u_{WCA,r}$ and $u_{WCA,a}$ is used to verify to what extent the relationship between these variables is linear.
Several basic statistical concepts and the relationship between them are covered first, in order to interpret properly Pearson's coefficient. One of the most commonly used measures of how the points in a data set are distributed is the second central moment around the mean. The 'variance' of a variable, A, or σ_A^2, is the mean squared deviation from its mean for a given sample of data,

$$\sigma_A^2 \equiv Var(A) = \mathbb{E}[(A - \mathbb{E}[A])^2)] = \mathbb{E}[A^2] - (\mathbb{E}[A])^2, \tag{5.2}$$

where $\mathbb{E}$ is the expectation value of A. ($i.e.$, $\mathbb{E}[A] = \sum_{i=1}^{N} A_i/N$ for the $i-th$ value of A in a data set). The variance measures how spread out about the mean the distribution of data points is. A variance of zero means all the values of A in a data set have the same value, and the variance is always ≥ 0, of course. The 'standard deviation', denoted by σ_A, is the square root of the variance, which in standard notation is,

$$\sigma_A = \sqrt{\mathbb{E}[A^2] - (\mathbb{E}[A])^2}, \tag{5.3}$$

which should not be confused here with the particle diameter, σ, in the potential.
A related quantity, the 'covariance' is a measure of the 'strength' of the linear relationship between two variables A and B,

$$Cov(A, B) = \mathbb{E}[(A - \mathbb{E}[A])(B - \mathbb{E}[B])] = \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B]. \tag{5.4}$$

If $Cov(A, B) > 0$, then on average A increases as B increases and *vice versa*. If $Cov(A, B) < 0$, then A tends to decrease as B increases and *vice versa*. These quantities are important when it comes to defining the PCC, a widely used measure of the correlation relationship between the two variables, which is denoted here by, $R_{p,A,B}$. Correlation is a measure of the directional relationship between the paired elements in two data sets, A and B, and,

$$R_{p,A,B} = \frac{Cov(A,B)}{\sigma_A \sigma_B} = \frac{\mathbb{E}[(A-\overline{A})(B-\overline{B})]}{\sqrt{\mathbb{E}(A^2)-[\mathbb{E}(A)]^2}\sqrt{\mathbb{E}(B^2)-[\mathbb{E}(B)]^2}}, \tag{5.5}$$

where $Cov_{A,B}$ is the covariance between data sets A and B, and σ_A is the standard deviation of data set A (and the same notation for B). The average value of A is denoted by $\overline{A}$ and the average value of B is denoted by $\overline{B}$. Note that the PCC is dimensionless while covariance has units obtained by multiplying the units of the two variables. The PCC is a measure of the 'strength' of the relationship between the two variable sets, but does not define any particular functional relationship (or 'causality') between the two variables taken at the same time or in a particular order. By 'causality' we do not necessarily mean that a value in A leads directly to the corresponding value in B or *vice versa*, but that both quantities may be determined by an underlying third parameter of the system (*e.g.*, the partition function of the system in statistical mechanics). This latter point limits our ability to draw a causal relationship between the two variables, and for this reason an additional procedure, known as 'Regression Analysis', (RA) or in the present context of assumed proportionality between two variables, 'Linear Regression' (LR), which involves minimising the sum-of-the-squares of the errors is widely used to draw inferences about any causal relationship between the variables. The RA involves the method of Ordinary Least Squares (OLS). Below the procedure of regression is defined and used to establish a formal link between Pearson's correlation coefficient and the OLS regression coefficient. Regression analysis is the process of constructing a mathematical model or function that can be used to predict or determine the value of one variable from that of another variable, or other variables. The most elementary regression model is called 'simple regression'. In simple regression, the variable to be predicted is called the dependent variable, and is usually designated by Y. The independent variable, or 'explanatory' variable, usually designated by X is also called the 'predictor'. The procedure of simple regression involves fitting a straight line through a set of N_p points in such a way that the sum of the squared residuals of the model is minimised. The equation of this line is,

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i, \tag{5.6}$$

where, $\hat{Y}_i$ is the predicted value of Y_i using a finite number of sample sets, $\hat{\beta}_0$ is the y-intercept of the line of best fit, and $\hat{\beta}_1$ is the slope of the line of best fit. The difference between the actual and predicted value of the dependent variable, called the 'residual', is,

$$\hat{U}_i = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i. \tag{5.7}$$

The residuals are the vertical distances between the points of the data set and the fitted line. Intuitively it is readily appreciated that the smaller the residuals the closer the fit line is to the distribution of input pair values. To avoid the problem of positive residuals offsetting negative residuals, the principle of Ordinary Least Squares (OLS) is employed, which involves finding the values of $\hat{\beta}_0$ and $\hat{\beta}_1$ which minimise the sum of the squares of the residuals, S,

$$S = \sum_{i=1}^{N_p} \hat{U}_i^2 = \sum_{i=1}^{N_p} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{N_p} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2. \tag{5.8}$$

By minimising the above equation with respect to $\hat{\beta}_0$ (intercept) and $\hat{\beta}_1$ (slope) expressions for these two quantities are obtained,

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X}, \tag{5.9}$$

where $\hat{\beta}_0$ is the predicted intercept, and the predicted slope, $\hat{\beta}_1$ is,

$$\hat{\beta}_1 = \frac{\mathbb{E}[(X - \overline{X})(Y - \overline{Y})]}{\mathbb{E}(X^2) - [\mathbb{E}(X)]^2} = \frac{Cov(X,Y)}{Var(X)}, \tag{5.10}$$

for N_p data points, and where, $\overline{X}$ is the mean value of the explanatory variable X, and $\overline{Y}$ is the mean value of the dependent variable, Y.
The relationship between the OLS estimator and Pearson's Correlation coefficient is now derived. The formula for the estimator, $\hat{\beta}_1$, is given by,

$$\hat{\beta}_1 = \frac{Cov(A,B)}{\sigma_B^2} = \frac{\mathbb{E}[(A - \overline{A})(B - \overline{B})]}{\mathbb{E}(B^2) - [\mathbb{E}(B)]^2}. \tag{5.11}$$

The relationship between the PCC and the OLS estimator, $\hat{\beta}_1$, is then,

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\mathbb{E}[(A - \overline{A})(B - \overline{B})]}{\mathbb{E}(B^2) - [\mathbb{E}(B)]^2} \\
&= \frac{\mathbb{E}[(A - \overline{A})(B - \overline{B})]}{\sqrt{\mathbb{E}(A^2) - [\mathbb{E}(A)]^2}\sqrt{\mathbb{E}(B^2) - [\mathbb{E}(B)]^2}} \frac{\sqrt{\mathbb{E}(A^2) - [\mathbb{E}(A)]^2}}{\sqrt{\mathbb{E}(B^2) - [\mathbb{E}(B)]^2}} \\
&= R_{p,A,B} \frac{\sigma_A}{\sigma_B},
\end{aligned} \tag{5.12}$$

which reveals that the regresssion coefficient is Pearson's correlation coefficient times the ratio of the standard deviations of the independent variable divided by that of the dependent variable. This signifies that regression analysis provides additional information when compared to the Pearson coefficient, namely, the relative distribution spreads of the two variables. The OLS coefficient, $\hat{\beta}_1$, will be used here to analyze the relationship between the two variables, in addition to the PCC. Equation (5.12) also proves that the sign of Pearson's correlation coefficient and that of the OLS coefficient are the same, as $\sigma_A \geq 0$ and $\sigma_B \geq 0$. To establish how well the predicted line fits the data, the ratio,

$$R^2 = \frac{Var(\hat{Y})}{Var(Y)}, \tag{5.13}$$

is used, where R^2 measures the fractional variation in the dependent variable given by the model. The predicted value of Y is $\hat{Y}$, and the input value of the treatment is Y.
The so-called t-test, [27] is used here to establish whether the OLS estimator is significantly different from zero, ($i.e.$, the slope is statistically significant based on the number and distribution of data points) through the parameter, t,

$$t = \frac{\hat{\beta}_1}{s.e.(\hat{\beta}_1)} \tag{5.14}$$

where $s.e.(\hat{\beta}_1)$ is the standard error of $\hat{\beta}_1$.
The above analysis is now used to establish the extent of correlation between the following pairs of variables, (u, P_c), (u_r, u_a) and $(u_{WCA,r}, u_{WCA,a})$ where the right entry is taken to be the dependent variable and the left entry to be the independent variable. In the latter two sets it is reasonable to take the repulsive energy term to be the independent variable as this is consistent with perturbation theories of liquids where the structure of the liquid is assumed to be dominated by the repulsive part of the potential. For (u, P_c) there appears to be no clear preference for which of the quantities should be taken to be the independent variable as they are formally different system measures, and both include the repulsive and attractive parts of the potential energy (although weighted differently). The adopted choice is therefore arbitrary.

5.4 REGRESSION AND CORRELATION

The linear regression and Pearson coefficient as defined in Fig. (5.5) are first explored for system states generated using the LJ potential for three state points in the low and high temperature limits, and one in the middle.

Figure 1 shows (a) P_c as a function of u, (b) u_a as a function of u_r and and (c) $u_{WCA,a}$ as a function of $u_{WCA,r}$ respectively from left to right. The state point used has a temperature of 60 and a density of 2.289 in LJ reduced units. The degree of correlation is measured by the extent to which the data points fall on or near the regression straight line. Figures 2 and 3 give the corresponding plots for the temperature, density pairs of $(4.00, 1.229)$ and $(0.70, 0.847)$, respectively. These three state points are on the fluid boundary side of the LJ melting line. The dynamics and state points are generated using the LJ potential. Tables 1-9 give a further statistical analysis of these nine data sets, with specific conclusions for each case made in the figure caption. A number of noteworthy points emerge from this analysis. Of the three sets, the (u, P_c) correlations are strongest, and have a positive slope. The extent of linearity in the correlation between this pair has been used to determine the extent to which lines on the density-temperature planes of the phase diagram are isomorphic ($i.e.$, have an underlying structural invariance),[3, 4, 5, 6, 7]. The u_r and u_a are also quite strongly linked but with a negative slope, which indicates that they are *anticorrelated*. Even at constant temperature, one expects the attractive part of the potential to change in the opposite direction to a change in the repulsive part, as it would do exactly in the microcanonical or NVE ensemble. Both of these trends are evident along the whole of the melting line.

The behaviour of the $u_{WCA,a}, u_{WCA,r}$ pair, in contrast, changes qualitatively along the melting line. The slope goes from being negative to positive as the temperature (density) decreases, and is approximately infinite in the region, $T \sim 10$. This change in behaviour has to be associated with the analytic form of the repulsive and attractive parts of the WCA decomposition of the LJ potential, and the dynamic distribution of near neighbour molecules. As the (temperature) density decreases the near neighbour particles move further apart, and the repulsive part of the potential is weaker. This must surely weaken the anticorrelation coupling between $\phi_{WCA,r}$ and $\phi_{WCA,a}$ energy terms. In fact it becomes slightly correlated close to the triple point.

The radial distribution function for the three fluid state points generated using the LJ potential force field are shown in Fig. 4. The lower set of curves uses the pair separation, r in LJ σ on the abscissa. The top set of radial distribution functions expressed in isomorphic distance units, $\tilde{r} = \rho^{1/3} r$, show excellent isomorphic collapse. The peaks of $g(r)$ shift to smaller distances with increasing density. In fact, the first peak of all three are to varying extents within the truncation distance of $\phi_{WCA,r}$ *i.e.*, $r_c = 2^{1/6}$, which is shown as a vertical line on the figure. For $T = 60$ the whole of the first peak is within r_c while only half of it is in this range for $T = 0.7$. The correlation to anticorrelation transition must surely by attributable to these variations, or more many-body consequences of these trends.

The analysis used to generate Figs. 1-3 was made for systems generated by the LJ potential. Two purely repulsive potentials, which are formed from the repulsive region of the LJ potential are now considered as origins of the system dynamics. One form, called here, 'IP12', is the IP potential with $n = 12$, *i.e.*, $\phi(r) = 4\epsilon(\sigma/r)^{12}$, which does not include any of the attractive part of the LJ potential. The other purely repulsive potential is the repulsive part of the LJ or WCA interaction, $\phi_{WCA,r}$, which does include the short range region of the attractive part of the LJ potential, up to r_c. The potential (or derived force) used to generate the system of states is referred to as the 'force field' here. The results of these simulations are summarised in Figs. 5-7, and in Table 10.

Figure 5 compares the same three pairs of computed property as in Figs. 1-3, given along the rows. Each row is a different force field. The bottom row is derived from LJ potential dynamics. The middle row used the (repulsive) WCA force field, and the top row from the IP12 potential. The temperature and density of the state point are 0.70 and 0.847 for each of the nine frames. First, the figure shows that the WCA and IP12 potentials generate very similar pair-property correlation behaviour to the LJ case. The u_r and u_a are strongly anticorrelated more or less equally for the three force fields. The figure also indicates that the $u_{WCA,r}$, $u_{WCA,a}$ pair are relatively weakly correlated, especially for IP12, indicated by the ellipsoidal pattern of symbols on the figure (top right frame). This weak correlation trend is understandable as the WCA decomposition of the LJ potential was originally chosen to partition it into a strongly repulsive part and a slowly varying component which is weakly correlated with the repulsive decomposition part (acting almost as an 'attractive background'), for use in perturbation theories of the liquid state, [13, 28, 29, 30]. In the perturbation theory the attractive part of the WCA potential is treated as a background term and the structure is governed by the repulsive part of the WCA decomposition.

Figure 6 presents the same set of correlations for a state point in which the temperature and density are 4.00 and 1.229, repectively. The u_r, u_a pair are again strongly anticorrelated for all force fields. The $u_{WCA,r}$, $u_{WCA,a}$ pair are even more weakly correlated than in Fig. 5, for all force fields, as evident by the nearly circular pattern of symbols for all frames in the rightmost column. Superficially at least it appears that the two potential terms are statistically independent, which could be made use of in developing perturbation theories of the liquid state, as these two components appear to be statistically independent over a certain temperature (density) range along the melting curve. Another noteworthy feature is that again the distribution of points for each type of correlation is largely independent of the dynamics generating force field, along the melting line at least.

Figure 7 presents the corresponding data for a temperature of 60 and density of 2.289. The three u_r and u_a are anticorrelated to more or less the same extent as found in the previous two figures. The three $u_{WCA,r}$, $u_{WCA,a}$ pair reveal strong *anticorrelation* this time. This change in behavior could be explained because the repulsive part of the potential becomes relatively more important at higher temperatures (density), and larger fluctuations in this part of the potential energy will take place which will induce oppositely signed fluctuations in the attractive part of the potential ($i.e.$, the constant temperature system is not too far from the microcanonical ensemble in this limit). As discussed above, a key issue in this respect is probably where the first peak in the radial distribution function lies in relation to the minimum in the radial distribution function.

Table 10 gives a summary of the simulation average property values for the three state points considered in Figs. 5-7, which are well separated along the melting line. The table shows that as temperature (density) increases the total LJ potential energy, u, shifts in the positive direction, especially for simulations carried out with the IP potential using $n = 12$ (or 'IP12') force field dynamics. The difference in the total energy from LJ and WCA dynamics is insignificant at $T = 60$ and not very great for $T = 0.7$, a result which is consistent with the aim of using the WCA potential in perturbation theory. The average potential energy, u, values from the LJ and WCA dynamics are not too different, and typically within a few percent of each other, while that of the IP12 force field is much more positive, which becomes more accentuated with increasing temperature along the melting curve. The PCC for the three pairs of quantities are shown in the last three columns of the table, which shows that the Pearson coefficient for the P_c and u pair of quantities is very close to unity for all of the state points considered. Its value increases towards unity with increasing temperature. Just why the Pearson Coefficient is so close to unity for this pair of system quantities is not immediately obvious. One might expect there to be a reasonably strong correlation between u and P_c as the latter has a component of u_r in its definition. In fact, for the Lennard-Jones potential, $P_c/\rho T = 4u_r + 2u_a$, [31] Indeed, all static properties of the LJ system can be expressed as a linear combination of the average repulsive and attractive parts of the potential, apart from some known constants or numerical factors. The strong anticorrelation between u_r and u_a may also contribute to the proximity of the PCC to unity, as then the repulsive and attractive terms can be combined into one effective (less repulsive) quantity. The table shows that the quantity, $R_p[u_r.u_a]$, is close to -1 for the three state points. The behaviour in R_p for the pair, $u_{WCA,r}$ with $u_{WCA,a}$, is quite different, as noted above. The absolute value is much less than unity and for all types of force field dynamics; it is sensitive to state point and R_p goes from positive to negative in the temperature interval between 4 and 60. At higher temperature there is evident more anticorrelation between the positive and attractive parts of the potential. A limitation of linear regression and the Pearson coefficient is that it does not give any indication of the time or chronological persistence of the correlation between the two quantities along the data set. In fact, any randomly sorted array of a two column table would give the same scatter plots and PCC values. However consecutive data values in a table can be correlated with each other, which generally decays to a statistically uncorrelated state between two data points far enough along the table. This is useful information which can give further insights into the underlying physics. An extension of the Pearson correlation concept which gives this additional information is proposed and tested in the next section.

5.5 Time correlation Pearson Modification

The degree of correlation between the same or two different quantities at times separated by an interval, t, can be expressed as,

$$R_{p,A,B}(t) = \frac{< \delta A(0)\delta B(t) >}{< \delta A(0)^2 >^{1/2} < \delta B(0)^2 >^{1/2}}. \tag{5.15}$$

where A and B are again the two system quantities of interest. In Eq. (5.15) the quantity, $\delta A(t) = A(t) - \overline{A}$ and $\delta B(t) = B(t) - \overline{B}$. The function in Eq. (5.15) is an extension of the Pearson coefficient definition (which is the $t = 0$ value) to account for temporal correlations between the two quantities. Such a formula is widely used in economics and is known as regression with $n-$lagged explanatory variable [25]. In liquid state physics, if $A = B$ then this is called an 'autocorrelation' function whereas if $A \neq B$ it is referred to as a 'cross-correlation or perhaps 'Pearson' correlation function in the present context. The quantity defined in Eq. (5.15) is closely related to the time-correlation function used to explore the dynamics and calculate transport coefficients of fluids by MD with Green-Kubo formulae, [32] but in that case the denominator is set to $< \delta A(0)\delta B(0) >$ or unity (respectively) instead. The only significant difference is the normalisation factor used in the denominator. An informative step in the present context is to express time in isomorphic units defined by, $\tilde{t} = \rho^{1/3}T^{1/2}t$. Along an isomorphic line time dependent properties scale with time expressed as $\tilde{t}$. Time dependent properties along an isomorph should collapse onto the same curve if the ordinate quantity is suitably normalised (this is referred to as *isochronal* scaling). The time-dependent function, $R_{p,A,B}(\tilde{t})$ from Eq. (5.15) expressed in terms of isomorphic time quantifies the time persistence of any correlation

between A and B over time, t. It gives some information on how long it takes the correlated values between two variables to become statistically uncorrelated to a strong degree.
The above treatment is another statistical analysis tool which quantifies the relationship between variables, where an explanatory variable can influence the dependent variable even with a time lag. These are known as distributed lag models in the time series literature, and which are formulated as follows,

$$Y_t = \beta_0 + \beta_1 X_{t-1} + U_t \tag{5.16}$$

where Y_t is the functional value at time t for the input vector, X_{t-1} and β_1 measures the effect of the explanatory variable one increment of time in the past to the dependent variable, *ceteris paribus*. The residual vector at time t is denoted by U_t. Many lagged variables $t-1, t-2, \cdots$ as far back as necessary may need to be included in the set of explanatory variables to account fully for memory effects. The extent of the time lag can be chosen by using $t-$tests for every subsequent addition of a lagged explanatory variable. The OLS estimation gives the best fit to the data, the statistical significance of which can be established using the $t-$test and other statistical measures to prove the data is stationary, that is when the mean, variance, autocorrelation of the data are constant within the data statistics.
Figure 8 shows $R_{p,A,B}(\tilde{t})$ for the same fluid state points and quantities as given in Table I, where the LJ force field has been used to generate the dynamics. The three Pearson cross-correlation functions shown on the figure are, $< u(0)P_c(\tilde{t}) >$, $< u_r(0)u_a(\tilde{t}) >$ and $< u_{r,WCA}(0)u_{a,WCA}(\tilde{t}) >$. The first two functions decay monotonically to zero from a positive or negative initial value, and to a very good approximation exhibit isochronal collapse along the studied melting line. The corresponding WCA quantity has a quite different time dependence even when cast in isomorphic units, which is consistent with the data in Table I (*i.e.*, the time equal to zero value of this function). The $t = 0$ value goes from being positive to negative with increasing temperature, and at a certain temperature $R_{p,A,B}(0)$ is zero for each type of dynamics. Simulations carried out at that state point could therefore be useful in informing the development of perturbation theory descriptions of the liquid state. There is a long-time tail in these functions, having not achieved zero by 0.5 isomorphic time units. Figures 9 and 10 show the corresponding $R_p(\tilde{t})$ produced by WCA and IP12 forcefield dynamics. The correlation functions decay more rapidly to zero for WCA (by $\tilde{t} = 0.4$) and especially IP12 (by $\tilde{t} = 0.2$). Otherwise the features and trends are qualitatively the same as for LJ dynamics as shown in Fig. 8.

5.6 CONCLUSIONS

The statistical fluctuation behaviour of pairs of thermodynamic properties are examined for the Lennard-Jones (LJ) fluid along its melting curve. Plots of the instantaneous values of the two variables ('scatter plots') are used to determine the slope and intercept using standard linear regression analysis. The Pearson coefficient was also calculated, which has been used recently to determine the extent to which a line on the phase diagram is isomorphic (*i.e.*, has an underlying structural invariance) taking the two variables to be the configurational part of the pressure and the potential energy of the system. The statistical analysis has been extended here to include that between the repulsive and attractive parts of the LJ potential, and also that of its Weeks-Chandler-Andersen (WCA) decomposition. At constant temperature, the former are strongly anticorrelated along the melting line whereas the latter go from being weakly correlated near the triple point to being moderately anticorrelated in the high temperature (density) limit.

The present analysis approach gives new insights into the relative roles of the attractive and repulsive parts of the LJ potential in determining its structure and thermodynamic properties, and could perhaps be used to help develop perturbation theories of the liquid state.

The statistical theoretical framework found useful in Economics is exploited here, and an extension of the Pearson coefficient method to determine time dependent correlations is also proposed, and shown to give new insights into the temporal behaviour of system property correlations.

The statistical trends are shown to be relatively insensitive to the potential used to generate the dynamics if it is purely repulsive and constructed from the LJ potential either as the r^{-12} inverse power part or the repulsive part of the WCA reconstruction of the LJ potential.

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5.7 Tables and Figures

P_c	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
β_1	0.9321	0.03682	25.31	0.000	0.8595 - 1.004718
β_0	0.01063	0.01684	0.63	0.529	-0.02257 - 0.04383
$R^2 = 0.7639,\ N_p = 200$					

Table 5.1: Linear single variable OLS regression of the u and P_c data for the state point values of T, ρ equal to $0.7, 0.8468$, respectively. The top left hand entry is the independent variable, which is P_c for this table. N_p is the number of data point pairs. The above regression produces, $\hat{\beta}_0 = 0.01063$ and $\hat{\beta}_1 = 0.9321$. The p value on $\hat{\beta}_0$ is greater than 0.05, hence $\hat{\beta}_0$ is not a significant predictor of the real y-intercept. The p value on $\hat{\beta}_1$ does not exceed 0.05, and therefore P_c is a significant predictor of u, as P_c increases by one unit u increases by 0.93 units. The value of R^2 tells us that variation in P_c explains 76.39% of the variation in u. The $t-$test value is denoted by 't' in the table heading. 'Conf. Interval' is the confidence interval. The statistical analysis carried out for this table and tables II-IX was carried out using the statistical analysis software package, © STATA [33].

u_a	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	-0.9174	0.02348	-39.08	0.000	-0.9636 - 0.8711
$\hat{\beta}_0$	0.9579	0.01334	71.79	0.000	0.9316 - 0.9843

$R^2 = 0.8852,\ N_p = 200$

Table 5.2: Linear single variable OLS regression of the u_r and u_a data for the state point, T, ρ equal to $0.7, 0.8468$. Both $\hat{\beta}_0$ and $\hat{\beta}_1$ are significant predictors of the real intercept and slope respectively. As u_a increases by one unit u_r decreases by 0.92 units and when the value of u_a is equal to zero u_r equals 0.96. The variation in u_a explains 88.52% of the variation in u_r.

$u_{WCA,a}$	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	0.5565	0.05852	9.51	0.000	0.4410 - 0.6719
$\hat{\beta}_0$	0.1188	0.03020	3.93	0.000	0.05927 - 0.17837
		$R^2 = 0.3135,\ N_p = 200$			

Table 5.3: Linear single variable OLS regression of the $u_{WCA,r}$ and $u_{WCA,a}$ data for T, ρ equal to $0.7, 0.8468$. Both $\hat{\beta}_0$ and $\hat{\beta}_1$ are significant predictors of the real intercept and slope respectively. As $u_{WCA,a}$ increases by one unit $u_{WCA,r}$ increases by 0.56 units and when the value of $u_{WCA,a}$ equals zero, $u_{WCA,r}$ is equal to 0.12. The variation in $u_{WCA,a}$ explains 31.35% of the variation in $u_{WCA,r}$.

P_c	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	1.001	0.00483	207.3	0.000	0.9917 - 1.011
$\hat{\beta}_0$	0.0004956	0.00246	0.20	0.840	-0.004354 - 0.005345
$R^2 = 0.9954,\ N_p = 200$					

Table 5.4: Linear single variable OLS regression of the u and P_c data for T, ρ equal to $4.0, 1.229$. The p value on $\hat{\beta}_0$ is greater than 0.05 hence $\hat{\beta}_0$ is not a significant predictor of the real y-intercept. The p value on $\hat{\beta}_1$ does not exeed 0.05, and therefore P_c is a significant predictor of u, as when P_c increases by one unit, u increases by 1.0 units. The value of R^2 informs us that variation in P_c contributes 99.54% of the variation in u.

u_a	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	-0.8965	0.02223	-40.33	0.000	-0.9403 - 0.8526
$\hat{\beta}_0$	0.9448	0.01155	81.77	0.000	0.9220 - 0.9676

$R^2 = 0.8915,\ N_p = 200$

Table 5.5: Linear single variable OLS regression of the u_r and u_a data for T, ρ equal to $4.0, 1.229$. Both $\hat{\beta}_0$ and $\hat{\beta}_1$ are significant predictors of the real intercept and slope respectively. As u_a increases by one unit u_r decreases by 0.90 units and when the value of u_a equals zero u_r equals 0.94. Variation in u_a explains 89.15% variation in u_r.

$u_{WCA,a}$	Coefficient	Std. Error	t	$P > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	0.2885	0.06280	4.59	0.000	0.1647 - 0.4124
$\hat{\beta}_0$	0.3402	0.03407	9.99	0.000	0.2730 - 0.4074
$R^2 = 0.0963,\ N_p = 200$					

Table 5.6: Linear single variable OLS regression of the $u_{WCA,r}$ and $u_{WCA,a}$ data for T, ρ equal to $4.0, 1.229$. Both $\hat{\beta}_0$ and $\hat{\beta}_1$ are significant predictors of the real intercept and slope respectively. As $u_{WCA,a}$ increases by one unit, $u_{WCA,r}$ increases by 0.29 units and when the value of $u_{WCA,a}$ equals zero $u_{WCA,r}$ equals 0.34. Variation in $u_{WCA,a}$ explains 9.63% of the variation in $u_{WCA,r}$.

P_c	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	0.9992	0.0011964	835.19	0.000	0.9968 - 1.002
$\hat{\beta}_0$	-9.72 $\times 10^{-6}$	0.0005778	-0.02	0.987	-0.001149 - 0.001130
$R^2 = 0.9997,\ N_p = 200$					

Table 5.7: Linear single variable OLS regression of the u and P_c data for T, ρ equal to $60.0, 2.289$. The p value on $\hat{\beta}_0$ is greater than 0.05, hence $\hat{\beta}_0$ is not a significant predictor of the real y-intercept. The p value on $\hat{\beta}_1$ does not exeed 0.05, therefore P_c is a significant predictor of u, as P_c increases by one unit u also increases by 1.0 units. The value of R^2 indicates that the variation in P_c explains 99.97% of variation in u.

u_a	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	-0.8013	0.03259	-24.59	0.000	-0.8655 - 0.7370
$\hat{\beta}_0$	0.8856	0.01782	49.70	0.000	0.8504 - 0.9207
$R^2 = 0.7533,\ N_p = 200$					

Table 5.8: Linear single variable OLS regression of the u_r and u_a data for T, ρ equal to $60.0, 2.289$. Both $\hat{\beta}_0$ and $\hat{\beta}_1$ are significant predictors of the real intercept and slope, respectively. As u_a increases by one unit, u_r decreases by 0.80 units. When the value of u_a equals zero u_r equals 0.88. Variation in u_a explains 75.33% variation in u_r.

$u_{WCA,a}$	Coefficient	Std. Error	t	$p > \|t\|$	95% Conf. Interval
$\hat{\beta}_1$	-0.5398	0.06052	-8.92	0.000	-0.6591 - 0.4204
$\hat{\beta}_0$	0.7041	0.02827	24.91	0.000	0.64839 - 0.7599

$R^2 = 0.2866$, $N_p = 200$

Table 5.9: Linear single variable OLS regression of the $u_{WCA,r}$ and $u_{WCA,a}$ data for T, ρ equal to $60.0, 2.289$. Both $\hat{\beta}_0$ and $\hat{\beta}_1$ are significant predictors of the real intercept and slope respectively. As $u_{WCA,a}$ increases by one unit $u_{WCA,r}$, decreases by 0.54 units. When the value of $u_{WCA,a}$ equals zero $u_{WCA,r}$ equals 0.70. Variation in $u_{WCA,a}$ explains 28.66% of the variation in $u_{WCA,r}$.

Dynamics	T	ρ	u_r	u_a	u	$u_{WCA,r}$	$u_{WCA,a}$	$P_c u$	$u_r u_a$	$u_{WCA,r} u_{WCA,a}$
IP12	0.7	0.847	11.67	-14.03	-2.35	4.11	-6.46	0.995	-0.930	0.425
LJ	0.7	0.847	5.80	-11.94	-6.13	0.601	-6.74	0.959	-0.944	0.537
WCA	0.7	0.847	6.10	-12.06	-5.96	0.712	-6.67	0.972	-0.949	0.603
IP12	4	1.230	58.43	-30.48	27.94	37.27	-9.33	0.999	-0.926	0.151
LJ	4	1.230	24.27	-24.62	-0.349	9.12	-9.46	0.998	-0.951	0.335
WCA	4	1.230	24.39	-24.66	-0.265	9.19	-9.46	0.998	-0.952	0.344
IP12	60	2.289	781.7	-108.67	673.01	690.53	-17.53	1.000	-0.920	-0.298
LJ	60	2.289	293.6	-85.39	208.17	225.43	-17.25	1.000	-0.951	-0.571
WCA	60	2.289	293.4	-85.38	208.02	225.27	-17.25	1.000	-0.952	-0.580

Table 5.10: Thermodynamic averages and the Pearson coefficient, R_p, for three fluid state points the configurations of which are generated according to three force fields specified in the first column. Note that $u = u_r + u_a = u_{WCA,r} + u_{WCA,a}$. The acronym, ‘IP12’ indicates dynamics generated using the IP potential with $n = 12$. The acronym ‘WCA’ indicates that the MD dynamics were produced using the repulsive part of the LJ potential, $i.e.$, $\phi_{WCA,r}$. The R_p values for three quantity correlations are given in the last three columns.

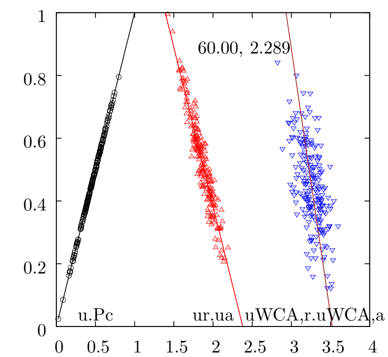

Figure 1

1

Figure 5.1: The correlation from left to right (a) P_c as a function of u (b) u_a as a function of u_r and (c) $u_{WCA,a}$ as a function of $u_{WCA,r}$ where the quantities plotted are the differences from their means. The state point is $T = 60.00$ and $\rho = 2.289$, using the LJ potential to generate the dynamics, which is high up on the melting curve. The data in each frame is normalised to fall within $0 - 1$ for the abscissa and ordinate quantities. The solid lines are least square fits to the data.

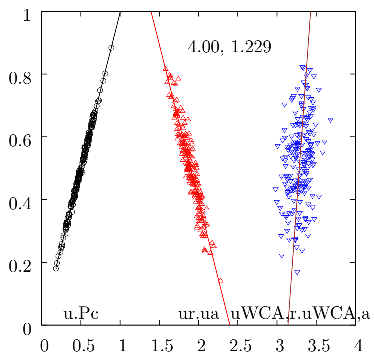

Figure 2

1

Figure 5.2: As for Fig. 1 except that the state point is $T = 4.00$ and $\rho = 1.229$ is used.

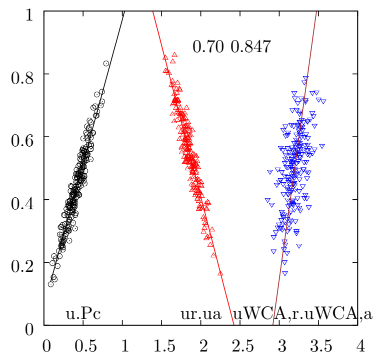

Figure 3

1

Figure 5.3: As for Fig. 1 except that the state point is $T = 0.700$ and $\rho = 0.847$ is used.

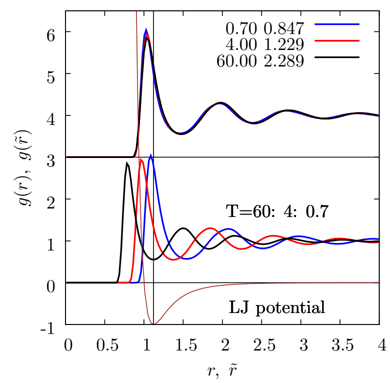

Figure 4

1

Figure 5.4: The radial distribution function, $g(r)$ expressed in LJ distance units, and in isomorphic distance units, $\tilde{r} = \rho^{1/3}r$ for the three state points along the fluid side of the coexistence curve. The upper set curves, shifted upwards by 3 is $g(\tilde{r})$ and the lower set are $g(r)$. The LJ potential is also shown. The vertical line corresponding to the position of the minimum in $\phi_{LJ}(r)$, which is equal to $2^{1/6}$, is shown.

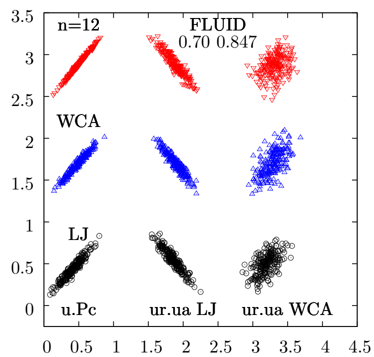

Figure 5

1

Figure 5.5: The correlation between (from left to right columns, respectively), (a) LJ u and P_c, (b) LJ u_r and u_a and (c) $u_{WCA,r}$ and $u_{WCA,a}$ where the quantities are the differences from their means. The rows indicate from bottom to top, (a) LJ, (b) WCA and (c) IP12 force field dynamics. The near-triple point state point of $T = 0.70$ and $\rho = 0.847$ is considered. The data in each frame is normalised to fall within $0 - 1$ for the abscissa and ordinate quantities.

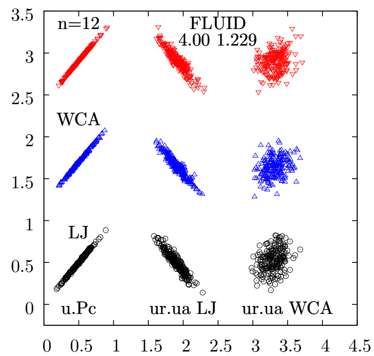

Figure 6

1

Figure 5.6: As for Fig. 5, except the state point $T = 4.00$ and $\rho = 1.229$ is considered.

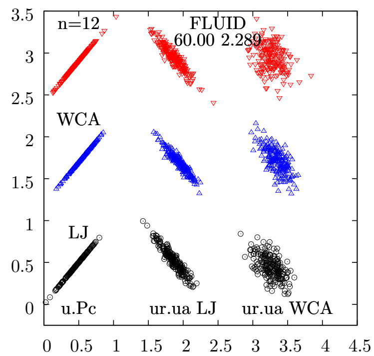

Figure 7

1

Figure 5.7: As for Fig. 5, except the state point $T = 60.0$ and $\rho = 2.289$ is considered.

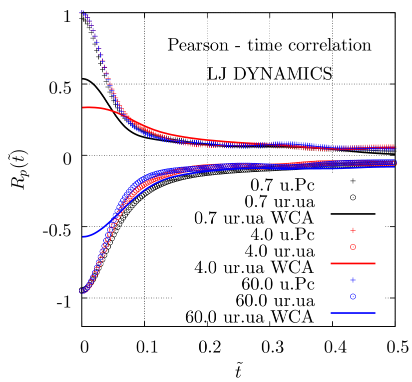

Figure 8

1

Figure 5.8: The function, $R_p(\tilde{t})$ defined in Eq. (5.15) is plotted for three cross-correlations, $< u(0)P_c(\tilde{t}) >$ and $< u_r(0)u_a(\tilde{t}) >$ using the LJ potential terms, and $< u_{r,WCA}(0)u_{a,WCA}(\tilde{t}) >$. The LJ potential was used in each case to calculate the forces used in the equations of motion. For each quantity the difference from the mean is used. Data for the fluid phase ML state points, $[60.00, 2.289]$, $[4.00, 1.229]$ and $[0.70, 0.847]$, are considered, and which are in the same order from bottom to top on the figure in the WCA case.

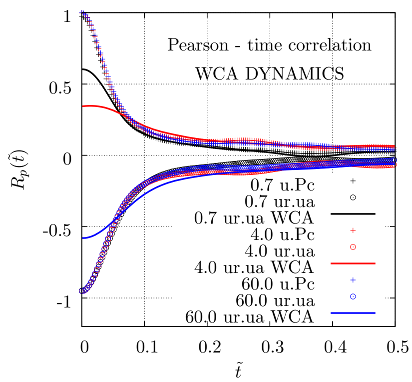

Figure 9

1

Figure 5.9: As for Fig. 8 except that WCA potential was used in each case to generate the dynamics.

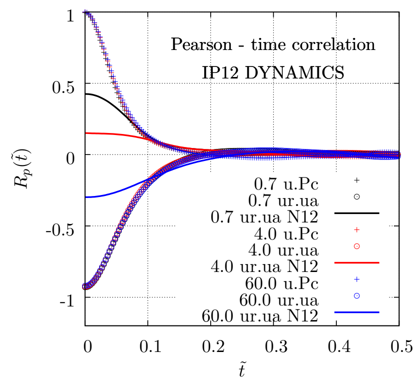

Figure 10

1

Figure 5.10: As for Fig. 8 except that IP potential with $n = 12$ ('IP12') was used for the dynamics.