Cohomology of Profinite Groups

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Abstract

The aim of this thesis is to study profinite groups of type FP_n. These are groups G which admit a projective resolution P of $\hat{\mathbb{Z}}$ as a $\hat{\mathbb{Z}}[\![G]\!]$-module such that $P_0, \ldots, P_n$ are finitely generated, so this property can be studied using the tools of profinite group cohomology.

In studying profinite groups it is often useful to consider their cohomology groups with profinite coefficients, but pre-existing theories of profinite cohomology do not allow profinite coefficients in sufficient generality for our purposes. Therefore we develop a new framework in which to study the homology and cohomology of profinite groups, which allows second-countable profinite coefficients for all profinite groups. We prove that many of the results of abstract group cohomology hold here, including Shapiro's Lemma, the Universal Coefficient Theorem and the Lyndon-Hochschild-Serre spectral sequence.

We then use these homology and cohomology theories to study how being of type FP_n controls the structure of a profinite group, and vice versa. We show for all n that the class of groups of type FP_n is closed under extensions, quotients by subgroups of type FP_n, proper amalgamated free products and proper HNN-extensions, and hence that elementary amenable profinite groups of finite rank are of type FP_∞. We construct profinite groups of type FP_n but not FP_{n+1} for all n. Finally, we develop the theory of signed profinite permutation modules, and use these as coefficients for group cohomology to show that torsion-free soluble pro-p groups of type FP_∞ have finite rank.

Contents

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Introduction

0.1 Overview

Profinite groups arise naturally whenever inverse limits of finite groups appear. The original motivation for their study comes from Galois groups: every Galois group can be thought of as a profinite group, and vice versa. These groups are now widely studied, both in their own right, and for their applications to the rest of group theory, where profinite completions characterise the finite quotients of a group.

Cohomology has been an important tool for studying mathematical objects since its development in the first half of the twentieth century. In the case of a group G, one can often recover group properties by considering its cohomology groups $H^n(G, M)$, where M is a G-module: see [4]. The theory of group cohomology has an analogue in the profinite setting. The earliest exposition of the subject is in [27], where Serre studies $H^n(G, M)$ for G a profinite group and M a discrete G-module. More recently, Symonds and Weigel in [31] have considered the case where G is virtually pro-p of type FP_∞ and M is a $\mathbb{Z}_p[\![G]\!]$-module which is either profinite or discrete.

However, the categories of modules in both [27] and [31] do not contain enough information for some purposes. In the former, usefulness is sacrificed for the expediency of having a nice coefficient category in which to work; it cannot give profinite analogues of many results in abstract group cohomology because it does not contain any profinite modules, in particular the group ring $\hat{\mathbb{Z}}[\![G]\!]$ itself. The latter paper is an attempt to fill these deficiencies, in the case of pro-p groups of type FP_∞, by introducing the *Pontryagin category*, tailored to deal with Pontryagin duality groups: an exact category which does contain $\mathbb{Z}_p[\![G]\!]$. It is not abelian and has neither enough projectives nor enough injectives, and therefore once again some basic tools from abstract group cohomology do not apply here.

There are applications for a more complete framework for the study of profinite group cohomology. Here, after developing new, well-behaved categories of topological $\hat{\mathbb{Z}}[\![G]\!]$-modules and using them to develop a theory of profinite group (co)homology which extends the previously existing work, we use this theory to study the type FP_n property for profinite groups.

Recall that an abstract group G is said to be of type FP_n over $\mathbb{Z}$, or just of type FP_n, if $\mathbb{Z}$ has a projective resolution which is finitely generated for the first n steps, considered as an $\mathbb{Z}[G]$-module with trivial G-action. This algebraic condition corresponds roughly to the geometric condition of G having a model for the classifying space which has finite n-skeleton (though there are differences in low dimensions). This condition generalises in a natural way to the profinite case, by replacing $\mathbb{Z}$ and $\mathbb{Z}[G]$ with $\hat{\mathbb{Z}}$ and $\hat{\mathbb{Z}}[\![G]\!]$. It is well-known that the class of groups of type FP_n is closed under extensions, and under quotients by groups of type FP_n [1, Proposition 2.7]. This was also known in the pro-p case, by [31, Proposition 4.2.3]. However, no results of this kind were previously known for profinite groups.

In the abstract case, particularly for certain classes of groups, being of type FP_n places strong constraints on the group structure. In the case $n = 1$, it is well-known that a group is of type FP_1 if and only if it is finitely generated. This is also the case for pro-p groups by [23, Theorem 7.8.1], [31, Proposition 4.2.3]. In the case $n = \infty$, every polycyclic group is of type FP_∞, by [1, Examples 2.6], while in the opposite direction a soluble group of type FP_∞ is constructible by [19, Corollary to Theorem B].

A primary aim of this thesis is to understand the relationship between the type FP_n conditions for profinite groups and the group-theoretic structure, and we obtain results which are often, though not entirely, analogous to those proved for abstract groups.

0.2 Structure of this Thesis

We begin by developing the theoretical background necessary for the applications of our theory to the study of profinite groups, which takes place primarily in the last two chapters.

Chapter 1 takes a category-theoretic approach to defining derived functors, first over quasi-abelian categories, then over functor categories $\mathcal{E}^I$, where $\mathcal{E}$ is abelian and I is any small category. The process of deriving functors over quasi-abelian categories is well understood, but less well known than the abelian case, so we include the relevant background here. The failure of these categories to be abelian means that the derived functors of a functor $F : \mathcal{E} \to \mathcal{F}$ between quasi-abelian categories do not have as $\mathcal{F}$ as their codomain, but rather the heart of t-structure on the category of cochain complexes in $\mathcal{F}$. The case of functor categories presents no new challenges, but we include the work here because we have been unable to find a satisfactory reference which treats the situation in full generality.

In Chapter 2 we develop the basic properties of topological modules over topological rings from scratch, covering such topics as the existence of limits and colimits, free topological modules, tensor products and the compact-open topology on groups of homomorphisms. We then specialise in Section 2.2 to the case of profinite rings Λ, and study modules over them with particular topologies: in this way, the category of profinite Λ-modules is defined as the image of a profinite completion functor from the category of topological Λ-modules which makes the profinite Λ-modules into a reflective subcategory, and similarly the category of Λ-discrete modules is defined as the image of a functor which makes them into a coreflective subcategory of the topological Λ-modules. Through these functors, certain properties proved for categories of topological Λ-modules carry over to the profinite case. As far as we know, this is the first time the theory of profinite and discrete modules for profinite rings has been approached through this use of adjoint functors. In Section 2.3 we extend the classical theory of profinite and discrete Λ-modules to larger categories of topological modules: the ind-profinite and pro-discrete Λ-modules. Many desirable properties carry over to these larger categories, including the former having enough projectives and the latter enough injectives, and the existence of a Pontryagin duality between them.

Chapter 3 combines the work of the previous two chapters to define Ext and Tor functors over profinite rings. Corresponding to the several different categories of topological Λ-modules considered in the previous chapter, there are several different possible definitions for the Ext and Tor functors. We first consider those involving the abelian categories of profinite and discrete Λ-modules – these are well known, and we simply give the necessary definitions – before going on to investigate the properties of Ext and Tor defined over the quasi-abelian categories of ind-profinite and pro-discrete Λ-modules. We show that Pontryagin duality gives a duality between Ext and Tor in this case (Lemma 3.2.2), and that $\mathbb{Q}_p$ is acyclic for Ext in the sense that $\mathrm{Ext}^n_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, -) = \mathrm{Ext}^n_{\hat{\mathbb{Z}}}(-, \mathbb{Q}_p) = 0$ for $n \neq 0$, although $\mathbb{Q}_p$ is not projective or injective (Lemma 3.2.6). We then specialise to consider profinite group homology and cohomology with ind-profinite and pro-discrete coefficients, respectively, and prove analogues of the Universal Coefficient Theorem in Theorem 3.3.4, Shapiro's Lemma in Theorem 3.3.10 and the Lyndon-Hochschild-Serre spectral sequence in Theorem 3.3.12. We finish the chapter by comparing the various definitions of Ext and Tor we have considered, and show that in many cases they are isomorphic.

In the following chapters, we apply this theory to study profinite groups of type FP_n. In Chapter 4 we derive necessary and sufficient homological and cohomological conditions for profinite groups and modules to be of type FP_n over a profinite ring R, analogous to the Bieri-Eckmann criteria for abstract groups: Theorem 4.2.2. This generalises the work of Symonds-Weigel [31, Proposition 4.2.3], giving a necessary and sufficient condition for (virtually) pro-p groups to be of type FP_n. We use these to prove that the class of groups of type FP_n is closed under extensions, quotients by subgroups of type FP_n, proper amalgamated free products and proper HNN-extensions, for each n (Theorem 4.3.8, Proposition 4.3.11, Proposition 4.3.14). We show, as a consequence of this, that by Proposition 4.4.2 elementary amenable profinite groups of finite rank are of type FP_∞ over all profinite rings R. For any class $\mathcal{C}$ of finite groups closed under subgroups, quotients and extensions, we also construct pro-$\mathcal{C}$ groups of type FP_n but not of type FP_{n+1} over $\mathbb{Z}_{\hat{\mathcal{C}}}$ for each n in Proposition 4.4.6.

Finally, in Chapter 5 we construct a large class of profinite groups $\widehat{\mathbf{L'H}_R\mathfrak{F}}$, including all soluble profinite groups and profinite groups of finite cohomological dimension over R. We study the properties of a class of profinite $R[\![G]\!]$-modules called signed permutation modules, and use these to show in Theorem 5.5.2 that, if $G \in \widehat{\mathbf{L'H}_R\mathfrak{F}}$ is of type FP_∞ over R, then there is some n such that $H^n_R(G, R[\![G]\!]) \neq 0$. We can then extend the work of King, who shows in [17, Corollary D] that an abelian-by-(poly-procyclic) pro-p group of type FP_∞ over $\mathbb{Z}_p$ has finite rank, by deducing in Theorem 5.6.9 that torsion-free soluble pro-p groups of type FP_∞ over $\mathbb{Z}_p$ have finite rank, thus answering the torsion-free case of a conjecture of Kropholler, [23, Open Question 6.12.1]. We finish the chapter by adapting a construction of Damian in [10] to exhibit a profinite group of type FP_∞ which is not countably based (and hence not finitely generated), and by showing that a natural analogue of the usual condition measuring when pro-p groups are of type FP_n fails for general profinite groups.

Chapter 1

Derived Functors

1.1 Quasi-Abelian Categories

A *preadditive* category is a category enriched over the category *Ab* of abelian groups; that is, one in which every set of morphisms $\mathrm{mor}(A, B)$ is an abelian group, such that composition of morphisms is distributive over addition. In other words, a morphism $B \to B'$ induces a homomorphism $\mathrm{mor}(A, B) \to \mathrm{mor}(A, B')$ of abelian groups, and similarly a morphism $A' \to A$ induces a homomorphism $\mathrm{mor}(A, B) \to \mathrm{mor}(A', B)$. Over preadditive categories we can define additive functors: functors F enriched over *Ab*, so that $\mathrm{mor}(A, B) \to \mathrm{mor}(F(A), F(B))$ is a group homomorphism.

An *additive* category is a preadditive category that has a zero object (that is, an object that is both terminal and initial in its category) and finite products; it follows that any finite product is the coproduct of the same objects. Over additive categories we can define kernels and cokernels of morphisms; see [32, Appendix A].

Suppose now that $\mathcal{E}$ is an additive category with kernels and cokernels. Given a morphism $f : A \to B$ in $\mathcal{E}$, we write $\mathrm{coim}(f) = \mathrm{coker}(\ker(f))$ for the coimage of f and $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$ for the image of f. Now f induces a unique canonical map $g : \mathrm{coim}(f) \to \mathrm{im}(f)$ such that f factors as

$$A \to \mathrm{coim}(f) \xrightarrow{g} \mathrm{im}(f) \to B,$$

and if g is an isomorphism we say f is *strict*. We say $\mathcal{E}$ is a *quasi-abelian* category if it satisfies the following two conditions:

(QA) in any pull-back square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B, \end{array}$$

if f is strict epic then so is f';

(QA*) in any push-out square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B', \end{array}$$

if f is strict monic then so is f'.

The names of these axioms are taken from [26, Definition 1.1.3]. In a quasi-abelian category, the canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is always a bimorphism.

Finally, an abelian category is a quasi-abelian category in which all maps are strict.

Abelian categories are the usual setting for performing homological algebra, but the weaker axioms of quasi-abelian categories allow most of the same results to be proved in this context. We give a brief sketch of the machinery needed to derive functors in quasi-abelian categories. See [22] and [26] for details.

To establish a notational convention: in a chain complex (A, d) in a quasi-abelian category, unless otherwise stated, d_n will be the map $A_{n+1} \to A_n$. Dually, if (A, d) is a cochain complex, d^n will be the map $A^n \to A^{n+1}$.

Given a quasi-abelian category $\mathcal{E}$, let $\mathcal{K}(\mathcal{E})$ be the category whose objects are cochain complexes in $\mathcal{E}$ and whose morphisms are maps of cochain complexes up to homotopy; this makes $\mathcal{K}(\mathcal{E})$ into a triangulated category. Given a cochain complex A in $\mathcal{E}$, we say A is *strict exact in degree* n if the map $d^{n-1} : A^{n-1} \to A^n$ is strict and $\mathrm{im}(d^{n-1}) = \ker(d^n)$. We say A is *strict exact* if it is strict exact in degree n for all n. Then, writing $N(\mathcal{E})$ for the full subcategory of $\mathcal{K}(\mathcal{E})$ whose objects are strict exact, we get that $N(\mathcal{E})$ is a null system, so we can localise $\mathcal{K}(\mathcal{E})$ at $N(\mathcal{E})$ to get the derived category $\mathcal{D}(\mathcal{E})$. We also define $\mathcal{K}^+(\mathcal{E})$ to be the full subcategory of $\mathcal{K}(\mathcal{E})$ whose objects are bounded below and $\mathcal{K}^-(\mathcal{E})$ to be the full subcategory whose objects are bounded above, and write $\mathcal{D}^+(\mathcal{E})$, $\mathcal{D}^-(\mathcal{E})$ for their localisations, respectively. We say a map of complexes in $\mathcal{K}(\mathcal{E})$ is a *strict quasi-isomorphism* if its cone is in $N(\mathcal{E})$.

Defining derived functors in quasi-abelian categories uses the machinery of *t-structures*: for background on these, see [22, Section 1.3]. Every t-structure on a triangulated category has a *heart*, which is, crucially, an abelian category. There are two natural t-structures on $\mathcal{D}(\mathcal{E})$, the left t-structure and the right t-structure, and correspondingly a left heart $\mathcal{LH}(\mathcal{E})$ and a right heart $\mathcal{RH}(\mathcal{E})$. The t-structures and hearts are dual to each other in the sense that there is a natural isomorphism between $\mathcal{LH}(\mathcal{E})^{op}$ and $\mathcal{RH}(\mathcal{E}^{op})$ induced by a natural isomorphism between $\mathcal{D}(\mathcal{E})^{op}$ and $\mathcal{D}(\mathcal{E}^{op})$ (one can check that $\mathcal{E}^{op}$ is quasi-abelian), so we can restrict investigation to $\mathcal{LH}(\mathcal{E})$ without loss of generality. By general category theory, $\mathcal{LH}(\mathcal{E})$ is an abelian category. Explicitly, $\mathcal{LH}(\mathcal{E})$ is the full subcategory of $\mathcal{D}(\mathcal{E})$ whose objects are strict exact in every degree except 0, and the functor

$$LH^0 : \mathcal{D}(\mathcal{E}) \to \mathcal{LH}(\mathcal{E})$$

is given by

$$0 \to \mathrm{coim}(d^{-1}) \to \ker(d^0) \to 0.$$

Every object of $\mathcal{LH}(\mathcal{E})$ is isomorphic to a complex

$$0 \to E^{-1} \xrightarrow{f} E^0 \to 0$$

of $\mathcal{E}$ with E^0 in degree 0 and f monic. Let $\mathcal{I} : \mathcal{E} \to \mathcal{LH}(\mathcal{E})$ be the functor given by

$$E \mapsto (0 \to E \to 0)$$

with E in degree 0. Let $\mathcal{C} : \mathcal{LH}(\mathcal{E}) \to \mathcal{E}$ be the functor given by

$$(0 \to E^{-1} \xrightarrow{f} E^0 \to 0) \mapsto \operatorname{coker}(f).$$

Proposition 1.1.1. *$\mathcal{I}$ is fully faithful and right adjoint to $\mathcal{C}$. In particular, identifying $\mathcal{E}$ with its image under $\mathcal{I}$, we can think of $\mathcal{E}$ as a reflective subcategory of $\mathcal{LH}(\mathcal{E})$. Moreover, given a sequence*

$$0 \to L \to M \to N \to 0$$

in $\mathcal{E}$, its image under $\mathcal{I}$ is a short exact sequence in $\mathcal{LH}(\mathcal{E})$ if and only if the sequence is short strict exact in $\mathcal{E}$.

The functor $\mathcal{I}$ induces functors $\mathcal{D}(\mathcal{I}) : \mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{LH}(\mathcal{E}))$ and $\mathcal{D}^-(\mathcal{I}) : \mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(\mathcal{LH}(\mathcal{E}))$.

Proposition 1.1.2. *$\mathcal{D}(\mathcal{I})$ and $\mathcal{D}^-(\mathcal{I})$ are equivalences of categories which exchange the left t-structures of $\mathcal{D}(\mathcal{E})$ and $\mathcal{D}^-(\mathcal{E})$ with the standard t-structures of $\mathcal{D}(\mathcal{LH}(\mathcal{E}))$ and $\mathcal{D}^-(\mathcal{LH}(\mathcal{E}))$, respectively.*

Thus there are cohomological functors $LH^n : \mathcal{D}(\mathcal{E}) \to \mathcal{LH}(\mathcal{E})$, so that given any distinguished triangle in $\mathcal{D}(\mathcal{E})$ we get long exact sequences in $\mathcal{LH}(\mathcal{E})$. Given an object $(A, d) \in \mathcal{D}(\mathcal{E})$, $LH^n(A)$ is the complex

$$0 \to \operatorname{coim}(d^{n-1}) \to \ker(d^n) \to 0$$

with $\ker(d^n)$ in degree 0.

Everything for $\mathcal{RH}(\mathcal{E})$ is done dually, so in particular we get:

Lemma 1.1.3. *The functors $RH^n : \mathcal{D}(\mathcal{E}^{op}) \to \mathcal{RH}(\mathcal{E}^{op})$ can equivalently be defined as $(LH^{-n})^{op} : \mathcal{D}(\mathcal{E})^{op} \to \mathcal{RH}(\mathcal{E})^{op}$.*

As for $PD(\Lambda)$, we say an object I of $\mathcal{E}$ is injective if, for any strict monomorphism $E \to E'$ in $\mathcal{E}$, any morphism $E \to I$ extends to a morphism $E' \to I$, and we say $\mathcal{E}$ has enough injectives if for every $E \in \mathcal{E}$ there is a strict monomorphism $E \to I$ for some injective I.

Proposition 1.1.4. *$\mathcal{E}$ has enough injectives if and only if $\mathcal{RH}(\mathcal{E})$ does.*

Suppose that $\mathcal{E}$ has enough injectives. Write $\mathcal{I}$ for the full subcategory of $\mathcal{E}$ whose objects are injective in $\mathcal{E}$.

Proposition 1.1.5. *Localisation at $N^+(\mathcal{I})$ gives an equivalence of categories $\mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{E})$.*

We can now define derived functors in the same way as the abelian case. Suppose we are given an additive functor $F : \mathcal{E} \to \mathcal{E}'$ between quasi-abelian categories. Let $Q : \mathcal{K}^+(\mathcal{E}) \to \mathcal{D}^+(\mathcal{E})$ and $Q' : \mathcal{K}^+(\mathcal{E}') \to \mathcal{D}^+(\mathcal{E}')$ be the canonical functors. Then the *right derived functor* of F is a triangulated functor

$$RF : \mathcal{D}^+(\mathcal{E}) \to \mathcal{D}^+(\mathcal{E}')$$

(that is, a functor compatible with the triangulated structure) together with a natural transformation

$$t : Q' \circ \mathcal{K}^+(F) \to RF \circ Q$$

satisfying the property that, given another triangulated functor

$$G : \mathcal{D}^+(\mathcal{E}) \to \mathcal{D}^+(\mathcal{E}')$$

and a natural transformation

$$g : Q' \circ \mathcal{K}^+(F) \to G \circ Q,$$

there is a unique natural transformation $h : RF \to G$ such that $g = (h \circ Q)t$. Clearly if RF exists it is unique up to natural isomorphism.

Suppose we are given an additive functor $F : \mathcal{E} \to \mathcal{E}'$ between quasi-abelian categories, and suppose $\mathcal{E}$ has enough injectives.

Proposition 1.1.6. *For $E \in \mathcal{K}^+(\mathcal{E})$ there is an $I \in \mathcal{K}^+(\mathcal{E})$ and a strict quasi-isomorphism $E \to I$ such that each I^n is injective and each $E^n \to I^n$ is a strict monomorphism.*

We say such an I is an *injective resolution* of E.

Proposition 1.1.7. *In the situation above, the right derived functor of F exists and $RF(E) = \mathcal{K}^+(F)(I)$ for any injective resolution I of E.*

We write R^nF for the composition $RH^n \circ RF$.

This construction generalises to the case of additive bifunctors $F : \mathcal{E} \times \mathcal{E}' \to \mathcal{E}''$ where $\mathcal{E}$ and $\mathcal{E}'$ have enough injectives: the right derived functor

$$RF : \mathcal{D}^+(\mathcal{E}) \times \mathcal{D}^+(\mathcal{E}') \to \mathcal{D}^+(\mathcal{E}'')$$

exists and is given by $RF(E, E') = s\mathcal{K}^+(F)(I, I')$ where I, I' are injective resolutions of E, E' and $s\mathcal{K}^+(F)(I, I')$ is the total complex of the double complex $\{\mathcal{K}^+(F)(I^p, I'^q)\}_{pq}$ in which the vertical maps with p odd are multiplied by -1.

Projectives are defined dually to injectives, left derived functors are defined dually to right derived ones, and if a quasi-abelian category $\mathcal{E}$ has enough projectives then an additive functor from $\mathcal{E}$ to another quasi-abelian category has a left derived functor which can be calculated by taking projective resolutions. Similarly for bifunctors.

We state here, for future reference, some results on spectral sequences; see [32, Chapter 5] for more details. All of the following results have dual versions obtained by passing to the opposite category, and we will use these dual results interchangeably with the originals. Suppose that $A = A^{pq}$ is a first quadrant double cochain complex in $\mathcal{E}$. By dualising Proposition 1.1.2, we can equivalently think of A as a first quadrant double complex in the abelian category $\mathcal{RH}(\mathcal{E})$. Then we can use the usual spectral sequences for double complexes:

Proposition 1.1.8. *There are two bounded spectral sequences*

$$\left.\begin{array}{rcl} {}^{I}E_2^{pq} & = & RH_h^p RH_v^q(A) \\ {}^{II}E_2^{pq} & = & RH_v^p RH_h^q(A) \end{array}\right. \Rightarrow RH^{p+q}\operatorname{Tot}(A),$$

naturally in A.

Proof. [32, Section 5.6] □

Suppose we are given an additive functor $F : \mathcal{E} \to \mathcal{E}'$ between quasi-abelian categories, and consider the case where $A \in \mathcal{D}^+(\mathcal{E})$. Suppose $\mathcal{E}$ has enough injectives, so that $\mathcal{RH}(\mathcal{E})$ does too. Thinking of A as an object in $\mathcal{D}^+(\mathcal{RH}(\mathcal{E}))$, we can take a Cartan-Eilenberg resolution I of A. Then we can apply Proposition 1.1.8 to the first quadrant double complex $F(I)$ to get the following result.

Proposition 1.1.9. *There are two bounded spectral sequences*

$$\begin{array}{rcl} {}^{I}E_2^{pq} & = & RH^p(R^qF(A)) \\ {}^{II}E_2^{pq} & = & (R^pF)(RH^q(A)) \end{array} \Rightarrow R^{p+q}F(A),$$

naturally in A.

Proof. [32, Section 5.7] □

Suppose now that we are given additive functors $G : \mathcal{E} \to \mathcal{E}'$, $F : \mathcal{E}' \to \mathcal{E}''$ between quasi-abelian categories, where $\mathcal{E}$ and $\mathcal{E}'$ have enough injectives. Suppose G sends injective objects of $\mathcal{E}$ to injective objects of $\mathcal{E}'$.

Theorem 1.1.10 (Grothendieck Spectral Sequence)**.** *For $A \in \mathcal{D}^+(\mathcal{E})$ there is a natural isomorphism $R(FG)(A) \to (RF)(RG)(A)$ and a bounded spectral sequence*

$${}^{I}E_2^{pq} = (R^pF)(R^qG(A)) \Rightarrow R^{p+q}(FG)(A),$$

naturally in A.

Proof. Let I be an injective resolution of A. There is a natural transformation $R(FG) \to (RF)(RG)$ by the universal property of derived functors; it is an isomorphism because, by hypothesis, each $G(I^n)$ is injective and hence

$$(RF)(RG)(A) = F(G(I)) = R(FG)(A).$$

For the spectral sequence, apply Proposition 1.1.9 with $A = G(I)$. We have

$${}^{I}E_2^{pq} = RH^p(R^qF(G(I))) \Rightarrow R^{p+q}F(G(I));$$

by the injectivity of the $G(I^n)$, $R^qF(G(I)) = 0$ for $q > 0$, so the spectral sequence collapses to give

$$R^{p+q}F(G(I)) \cong RH^{p+q}(FG(I)) = R^{p+q}(FG)(A).$$

On the other hand,

$${}^{II}E_2^{pq} = (R^pF)(RH^q(G(I))) = (R^pF)(R^qG(A))$$

and the result follows. □

We consider once more the case of an additive bifunctor

$$F : \mathcal{E} \times \mathcal{E}' \to \mathcal{E}''$$

for $\mathcal{E}, \mathcal{E}'$ and $\mathcal{E}''$ quasi-abelian: this induces a triangulated functor

$$\mathcal{K}^+(F) : \mathcal{K}^+(\mathcal{E}) \times \mathcal{K}^+(\mathcal{E}') \to \mathcal{K}^+(\mathcal{E}''),$$

in the sense that a distinguished triangle in one of the variables, and a fixed object in the other, maps to a distinguished triangle in $\mathcal{K}^+(\mathcal{E}'')$. Hence for a fixed $A \in \mathcal{K}^+(\mathcal{E})$, $\mathcal{K}^+(F)$ restricts to a triangulated functor $\mathcal{K}^+(F)(A, -)$, and if $\mathcal{E}'$ has enough injectives we can derive this to get a triangulated functor

$$R(F(A, -)) : \mathcal{D}^+(\mathcal{E}') \to \mathcal{D}^+(\mathcal{E}'').$$

Now maps $A \to A'$ in $\mathcal{K}^+(\mathcal{E})$ induce natural transformations $R(F(A, -)) \to R(F(A', -))$, so in fact we get a functor which we denote by

$$R_2 F : \mathcal{K}^+(\mathcal{E}) \times \mathcal{D}^+(\mathcal{E}') \to \mathcal{D}^+(\mathcal{E}'').$$

We know R_2F is triangulated in the second variable, and it is triangulated in the first variable too because, given $B \in \mathcal{D}^+(\mathcal{E}')$ with an injective resolution I, $R_2F(-, B) = \mathcal{K}^+(F)(-, I)$ is a triangulated functor $\mathcal{K}^+(\mathcal{E}) \to \mathcal{D}^+(\mathcal{E}'')$. Similarly, we can define a triangulated functor

$$R_1 F : \mathcal{D}^+(\mathcal{E}) \times \mathcal{K}^+(\mathcal{E}') \to \mathcal{D}^+(\mathcal{E}'')$$

by deriving in the first variable, if $\mathcal{E}$ has enough injectives.

Proposition 1.1.11. *(i) Suppose that $\mathcal{E}'$ has enough injectives and $F(-, J) : \mathcal{E} \to \mathcal{E}''$ is strict exact for J injective. Then $R_2F(-, B)$ sends quasi-isomorphisms to isomorphisms; that is, we can think of R_2F as a functor $\mathcal{D}^+(\mathcal{E}) \times \mathcal{D}^+(\mathcal{E}') \to \mathcal{D}^+(\mathcal{E}'')$.*

(ii) Suppose in addition that $\mathcal{E}$ has enough injectives. Then R_2F is naturally isomorphic to RF.

Similarly with the variables switched.

Proof. (i) Take an injective resolution I of B. Then we have $R_2F(-, B) = \mathcal{K}^+(F)(-, I)$. Given a quasi-isomorphism $A \to A'$ in $\mathcal{K}^+(\mathcal{E})$, consider the map of double complexes $\mathcal{K}^+(F)(A, I) \to \mathcal{K}^+(F)(A', I)$ and apply Proposition 1.1.8 to show that this map induces a quasi-isomorphism of the corresponding total complexes.

(ii) This holds by the same argument as (i), taking A' to be an injective resolution of A.

□

Finally, we consider the special case where the quasi-abelian category $\mathcal{E}$ is abelian. In this case, we can recover the standard results of homological algebra: for $\mathcal{E}$ abelian, the functor $\mathcal{I} : \mathcal{E} \to \mathcal{LH}(\mathcal{E})$ is an equivalence of categories.

1.2 Functor Categories

Given a category $\mathcal{E}$ and a small category I, one can construct a category whose objects are the functors $I \to \mathcal{E}$, and whose morphisms are the natural transformations between these functors. Such a category is called a functor category, and written $\mathcal{E}^I$.

We can think of objects in $\mathcal{E}^I$ as diagrams in $\mathcal{E}$, that is, pairs

$$(\{A^i \in \mathrm{ob}(\mathcal{E}) : i \in \mathrm{ob}(I)\}, \{(\alpha^{ij} : A^i \to A^j) \in \mathrm{mor}(\mathcal{E}) : (i \to j) \in \mathrm{mor}(I)\}).$$

When it is clear, we may write $(\{A^i\}, \{\alpha^{ij}\})$ or just $\{A^i\}$ for this. Similarly, a morphism

$$f : (\{A^i\}, \{\alpha^{ij}\}) \to (\{B^i\}, \{\beta^{ij}\})$$

in $\mathcal{E}^I$ consists of a set $\{f^i : i \in I\}$ of morphisms in $\mathcal{E}$ such that for all $i, j \in I$ the square

$$\begin{array}{ccc} A^i & \xrightarrow{f^i} & B^i \\ {\scriptstyle \alpha^{ij}}\downarrow & & \downarrow{\scriptstyle \beta^{ij}} \\ A^j & \xrightarrow{f^j} & B^j \end{array}$$

commutes. When it is clear, we may write $\{f^i : i \in I\}$, or just $\{f^i\}$, for f. We call the A^is the *components* of $\{A^i\}$ and the f^is the *components* of f.

In this section, I will always be a small category.

Suppose now that $F : \mathcal{E} \to \mathcal{F}$ is a functor between categories $\mathcal{E}$ and $\mathcal{F}$. For a morphism

$$f : (\{A^i\}, \{\alpha^{ij}\}) \to (\{B^i\}, \{\beta^{ij}\})$$

in $\mathcal{E}^I$, we have that

$$\{F(f^i)\} : (\{F(A^i)\}, \{F(\alpha^{ij})\}) \to (\{F(B^i)\}, \{F(\beta^{ij})\})$$

is a morphism in $\mathcal{F}^I$ because the square

$$\begin{array}{ccc} A^i & \xrightarrow{f^i} & B^i \\ {\scriptstyle \alpha^{ij}}\downarrow & & \downarrow{\scriptstyle \beta^{ij}} \\ A^j & \xrightarrow{f^j} & B^j \end{array}$$

commutes, so

$$\begin{array}{ccc} F(A^i) & \xrightarrow{F(f^i)} & F(B^i) \\ {\scriptstyle F(\alpha^{ij})}\downarrow & & \downarrow{\scriptstyle F(\beta^{ij})} \\ F(A^j) & \xrightarrow{F(f^j)} & F(B^j) \end{array}$$

does too. It is clear from the definition that composition of morphisms is preserved by this. Thus we get the following results.

Proposition 1.2.1. *Define $F^I : \mathcal{E}^I \to \mathcal{F}^I$ by the maps*

$$(\{A^i\}, \{\alpha^{ij}\}) \mapsto (\{F(A^i)\}, \{F(\alpha^{ij})\})$$

and

$$((\{A^i\}, \{\alpha^{ij}\}) \to (\{B^i\}, \{\beta^{ij}\})) \mapsto$$
$$((\{F(A^i)\}, \{F(\alpha^{ij})\}) \to (\{F(B^i)\}, \{F(\beta^{ij})\})).$$

Then F^I is a functor, which we call the exponent *of F by I.*

Lemma 1.2.2. *Given functors $F, G : \mathcal{E} \to \mathcal{F}$ and a natural transformation $\eta : F \to G$, we get a natural transformation*

$$\eta^I : F^I \to G^I,$$

where, for each $A \in \mathcal{E}^I$,

$$\eta^I_A : F^I(A) \to G^I(A)$$

is the map with ith component

$$\eta_{A^i} : F(A^i) \to G(A^i).$$

Proof. To show that each η^I_A is a morphism in $\mathcal{F}^I$, we need to check that the squares

$$\begin{array}{ccc} F(A^i) & \xrightarrow{\eta_{A^i}} & G(A^i) \\ {\scriptstyle F(\alpha^{ij})}\downarrow & & \downarrow{\scriptstyle G(\alpha^{ij})} \\ F(A^j) & \xrightarrow{\eta_{A^j}} & G(A^j) \end{array}$$

commute, which holds because η is a natural transformation. To show η^I is a natural transformation, it remains to show that, for a morphism $f : A \to B$ in $\mathcal{E}^I$, the squares

$$\begin{array}{ccc} F^I(A) & \xrightarrow{F^I(f)} & F^I(B) \\ {\scriptstyle \eta^I_A}\downarrow & & \downarrow{\scriptstyle \eta^I_B} \\ G^I(A) & \xrightarrow{G^I(f)} & G^I(B) \end{array}$$

commute; it suffices to show that each component commutes, which is just another application of the naturality of η. $\square$

For the rest of Chapter 1, we will assume that our categories $\mathcal{E}$ and $\mathcal{F}$ are abelian. Similar statements hold in the quasi-abelian case, but we will not use this fact.

Given that $\mathcal{E}$ is abelian (resp. additive), it is known that $\mathcal{E}^I$ is abelian (resp. additive) (see [32, Functor Categories 1.6.4]). We want to show that exact sequences in $\mathcal{E}^I$ are just sequences in $\mathcal{E}^I$ which are exact at each component. To show this, we need a preliminary lemma.

Lemma 1.2.3. *Suppose $A = (\{A^i\}, \{\alpha^{ij}\}), B = (\{B^i\}, \{\beta^{ij}\})$, and consider $f : A \to B$ in $\mathcal{E}^I$.*

(i) The kernel $\ker(f)$ *is the object $(\{\ker(f^i)\}, \{\gamma^{ij}\})$ together with the morphism $g : \ker(f) \to A$, where g^i is the canonical map $\ker(f^i) \to A^i$ in $\mathcal{E}$, and γ^{ij} is the (unique) morphism* $\ker(f^i) \to \ker(f^j)$ *given by the universal property of* $\ker(f^j)$ *in the diagram*

$$\begin{array}{ccc} \ker(f^i) & \xrightarrow{f^i} & A^i \\ {\scriptstyle \gamma^{ij}}\downarrow & & \downarrow{\scriptstyle \alpha^{ij}} \\ \ker(f^j) & \xrightarrow{f^j} & A^j. \end{array}$$

(ii) Similarly for coker(f)*.*

Proof. We will prove (i), and leave it to the reader to check that $\ker(f)$ really is an element of $\mathcal{E}^I$, that g really is a morphism, and that (ii) goes through in the same way.

It is clear that

$$fg : \ker(f) \to B$$

is the zero map, since (one may check) the zero element 0^I of $\mathcal{E}^I$ is the element with all its components the zero element 0 of $\mathcal{E}$, with identity morphisms between them. Suppose we have a morphism

$$h : E = (\{E^i\}, \{\varepsilon^{ij}\}) \to A$$

such that $fh = 0$. By definition, to show that $(\{\ker(f^i)\}, \{\gamma^{ij}\})$ is the kernel of f, we need to show that there is a unique

$$k : E \to (\{\ker(f^i)\}, \{\gamma^{ij}\})$$

such that $h = gk$. Now for each $i \in I$, $f^i h^i = 0$ in $\mathcal{E}$, so again by definition of the kernel there is some unique

$$k^i : E^i \to \ker(f^i)$$

such that $h^i = g^i k^i$. To show that h factors through $k = \{k^i\}$, we just need to check that the squares

$$\begin{array}{ccc} E^i & \xrightarrow{k^i} & \ker(f^i) \\ \downarrow{\scriptstyle \varepsilon^{ij}} & & \downarrow{\scriptstyle \gamma^{ij}} \\ E^j & \xrightarrow{k^j} & \ker(f^j) \end{array}$$

commute. Then uniqueness follows from uniqueness of the k^i.

To see this, note that

$$g^j k^j \varepsilon^{ij} = h^j \varepsilon^{ij} = \alpha^{ij} h^i = \alpha^{ij} g^i k^i = g^j \gamma^{ij} k^i,$$

so, as g^j is monic, it follows that $k^j \varepsilon^{ij} = \gamma^{ij} k^i$. □

Similarly, other operations in $\mathcal{E}$ such as taking direct sums and addition of morphisms is calculated componentwise. Details are left to the reader.

Lemma 1.2.4. *Given a sequence* $L \xrightarrow{f} M \xrightarrow{g} N$ *in* $\mathcal{E}^I$ *such that* $gf = 0$,

it is exact at M *iff the canonical map* $\mathrm{im}(f) \to \ker(g)$ *is an isomorphism*
iff the canonical map $\mathrm{im}(f^i) \to \ker(g^i)$ *is an isomorphism for all* i
iff the sequence $L^i \xrightarrow{f^i} M^i \xrightarrow{g^i} N^i$ *is exact at* M^i *for all* i.

Proof. This follows from Lemma 1.2.3. □

Given a triangulated category $\mathcal{T}$, we can give the functor category $\mathcal{T}^I$ the structure of a triangulated category by defining its distinguished triangles to be the triangles which are distinguished on each component and its translation functor to be translation on each component. Moreover, given another triangulated category $\mathcal{T}'$ and a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$, the exponent functor F^I is triangulated because it is triangulated on each component.

Now it is easy to check that the functor $\mathcal{K}(\mathcal{E})^I \to \mathcal{K}(\mathcal{E}^I)$ which sends $A \in \mathcal{K}(\mathcal{E})^I$ to the object $B \in \mathcal{K}(\mathcal{E}^I)$ defined by $(B^n)^i = (A^i)^n$ gives an equivalence of categories. Moreover, this functor is clearly triangulated. Similarly restricting this functor gives a triangulated equivalence $\mathcal{K}^+(\mathcal{E})^I \to \mathcal{K}^+(\mathcal{E}^I)$. Lemma 1.2.4 shows that this functor extends to equivalences $\mathcal{D}(\mathcal{E})^I \to \mathcal{D}(\mathcal{E}^I)$ and $\phi^I_{\mathcal{E}} : \mathcal{D}^+(\mathcal{E})^I \to \mathcal{D}^+(\mathcal{E}^I)$ of triangulated categories. Then suppose $\mathcal{E}$ has enough injectives, and define the triangulated functor $R(F^I)$ by

$$R(F^I)\phi^I_{\mathcal{E}} = \phi^I_{\mathcal{F}}(RF)^I : \mathcal{D}^+(\mathcal{E})^I \to \mathcal{D}^+(\mathcal{F}^I).$$

Suppose now that we are given additive functors $G : \mathcal{E} \to \mathcal{E}'$, $F : \mathcal{E}' \to \mathcal{E}''$ between abelian categories, where $\mathcal{E}$ and $\mathcal{E}'$ have enough injectives. Suppose G sends injective objects of $\mathcal{E}$ to injective objects of $\mathcal{E}'$.

Proposition 1.2.5. *For $A \in \mathcal{D}^+(\mathcal{E}^I)$ there is an isomorphism $R(F^I G^I)(A) \to (R(F^I))(R(G^I))(A)$, natural in A, and a bounded spectral sequence*

$${}^I E_2^{pq} = (R^p(F^I))(R^q(G^I)(A)) \Rightarrow R^{p+q}(F^I G^I)(A),$$

naturally in A.

Proof. By Theorem 1.1.10 and the definitions, it only remains to note that $(RF)^I(RG)^I = (RF \circ RG)^I$, which is clear. □

Finally, we consider the case of an additive bifunctor

$$F : \mathcal{E} \times \mathcal{E}' \to \mathcal{E}''$$

for $\mathcal{E}, \mathcal{E}'$ and $\mathcal{E}''$ abelian. For small categories I and J, we can define an exponent functor

$$F : \mathcal{E}^I \times (\mathcal{E}')^J \to (\mathcal{E}'')^{I\times J}$$

in the obvious way. Suppose $\mathcal{E}'$ has enough injectives, and define the triangulated functor $R_2(F^{I\times J})$ by

$$R_2(F^{I\times J})(\phi^I_{\mathcal{E}} \times \phi^J_{\mathcal{E}'}) = \phi^{I\times J}_{\mathcal{E}''}(R_2F)^{I\times J} : \mathcal{K}^+(\mathcal{E})^I \times \mathcal{D}^+(\mathcal{E}')^J \to \mathcal{D}^+(\mathcal{F}^{I\times J}).$$

If instead $\mathcal{E}$ has enough injectives, we can define $R_1(F^{I\times J})$ similarly.

Proposition 1.2.6. *(i) Suppose that $\mathcal{E}'$ has enough injectives and $F(-, J) : \mathcal{E} \to \mathcal{E}''$ is exact for J injective. Then $R_2(F^{I\times J})(-, B)$ sends quasi-isomorphisms to isomorphisms; that is, we can think of $R_2(F^{I\times J})$ as a functor $\mathcal{D}^+(\mathcal{E}^I) \times \mathcal{D}^+((\mathcal{E}')^J) \to \mathcal{D}^+((\mathcal{E}'')^{I\times J})$.*

(ii) Suppose in addition that $\mathcal{E}$ has enough injectives. Then $R_2(F^{I\times J})$ is naturally isomorphic to $R(F^{I\times J})$.

Similarly with the variables switched.

Proof. By Lemma 1.2.4, quasi-isomorphisms in $\mathcal{K}^+(\mathcal{E}^I)$ are maps which are quasi-isomorphisms in each component (considered as maps in $\mathcal{K}^+(\mathcal{E})^I$). Then the result follows from Proposition 1.1.11. □

Chapter 2

Topological Modules for Profinite Rings

2.1 Topological Groups, Rings and Modules

A group G whose underlying set is endowed with a topology T is said to be a topological group if T makes multiplication $m : G \times G \to G$ and inversion $i : G \to G$ continuous, where $G \times G$ is given the product topology. Some authors require in addition that T be Hausdorff; we do not. A morphism of topological groups is a group homomorphism which is also a continuous map of the underlying topological spaces. For abelian topological groups, we may refer to the multiplication operation as addition.

A ring Λ whose underlying set is endowed with a topology T is said to be a topological ring if T makes Λ into a topological abelian group under addition such that multiplication $\Lambda \times \Lambda \to \Lambda$ is continuous. A morphism of topological rings is a ring homomorphism which is also a continuous map of the underlying spaces.

Now suppose Λ is a topological ring. A topological left Λ-module A is a Λ-module endowed with a topology that makes A a topological abelian group, such that the scalar multiplication map $s : \Lambda \times A \to A$ is continuous, where $\Lambda \times A$ is given the product topology. A morphism of topological modules is a module homomorphism which is also a continuous map of the underlying spaces. When R is a commutative topological ring, a topological R-algebra is a topological ring Λ equipped with a canonical morphism $R \to \Lambda$ in $TRng$ into the centre of Λ.

We fix some notation. *Set*, *Grp*, *Ab*, *Rng* and $Mod(\Lambda)$ will be the categories of sets, groups, abelian groups, rings and left Λ-modules respectively, for $\Lambda \in Rng$. Top, $TGrp$, TAb, $TRng$ and $T(\Lambda)$ will be the categories of topological spaces, topological groups, topological abelian groups, topological rings and topological left Λ-modules for $\Lambda \in TRng$ respectively. We identify the category of topological right Λ-modules with $T(\Lambda^{op})$, and similarly for other categories of right modules. We will write U for each of the forgetful functors from Top, $TGrp$, TAb, $TRng$ and $T(\Lambda)$ for $\Lambda \in TRng$ to *Set*, *Grp*, *Ab*, *Rng* and $Mod(U(\Lambda))$ respectively which forget the topology; we will write V for each of the forgetful functors from $TGrp$, TAb, $TRng$ and $T(\Lambda)$

for $\Lambda \in TRng$ to Top and from Grp, Ab, Rng and $Mod(\Lambda)$ for $\Lambda \in Rng$ to Set which forget the algebraic structure. Then TAb, $TRng$ and $T(\Lambda)$ all inherit from Ab, Rng and $Mod(U(\Lambda))$ the structure of additive categories, and we will see later that they have all kernels and cokernels too, though they are not in general abelian. In particular, suppose $R \in TRng$ is commutative and $\Lambda \in TRng$ is a topological R-algebra, and consider $T(\Lambda)$. We write $\mathrm{Hom}_\Lambda(A, B)$ for $mor(A, B)$, where $A, B \in T(\Lambda)$. Then $\mathrm{Hom}_\Lambda(A, B)$ is naturally a sub-$U(R)$-module of $\mathrm{Hom}_{Mod(U(\Lambda))}(U(A), U(B))$, the morphisms $U(A) \to U(B)$ in $Mod(U(\Lambda))$. This makes $\mathrm{Hom}_\Lambda(-, -)$ into a contra-/covariant additive bifunctor

$$T(\Lambda) \times T(\Lambda) \to Mod(U(R)),$$

or equivalently a co-/covariant additive bifunctor

$$T(\Lambda)^{op} \times T(\Lambda) \to Mod(U(R)).$$

Lemma 2.1.1. *Suppose A is a topological abelian group with the discrete topology, and suppose $\Lambda \in TAb$. Then a Λ-action $f : \Lambda \times A \to A$ makes A a discrete Λ-module if and only if for every $a \in A$ the stabiliser of a in Λ is open.*

Proof. We have for each a that

$$f^{-1}(a) = \{(g, b) \in G \times A : gb = a\} = \bigcup_{(g,b)} (\mathrm{stab}_G(a)g) \times \{b\},$$

which is open if $\mathrm{stab}_G(a)$ is because A is discrete. The opposite implication is clear. □

Lemma 2.1.2. *Suppose C is Top, $TGrp$, TAb, $TRng$ or $T(\Lambda)$ for $\Lambda \in TRng$. Suppose $A \in U(C)$, $B \in C$, and we have a morphism $f : A \to U(B)$ in $U(C)$. Then the collection of open sets*

$$T = \{f^{-1}(O) : O \subseteq_{open} B\}$$

is a topology on A which makes A into an element of C.

Proof. This is easy for $C = Top$.

For $C = TGrp$ or TAb: We write A_T for A endowed with topology T. We need to check that the multiplication map

$$m_A : A_T \times A_T \to A_T$$

and the inversion map

$$i_A : A_T \to A_T$$

are continuous. Given an open set $f^{-1}(O)$,

$$m_A^{-1} f^{-1}(O) = (f \times f)^{-1} m_B^{-1}(O)$$

is open, and

$$i_A^{-1} f^{-1}(O) = f^{-1} i_B^{-1}(O)$$

is open, as required.

For $C = TRng$: We need to check in addition to the $C = TAb$ case that multiplication is continuous. This holds in the same way as the continuity of m in the $C = TGrp$ case.

For $C = T(\Lambda)$ for $\Lambda \in TRng$: We need to check in addition to the $C = TAb$ case that scalar multiplication

$$s_A : \Lambda \times A_T \to A_T$$

is continuous. Given an open set $f^{-1}(O)$,

$$s_A^{-1} f^{-1}(O) = (id_\Lambda \times f)^{-1} s_B^{-1}(O)$$

is open, as required. □

Proposition 2.1.3. *The categories Top, TGrp, TAb, TRng and $T(\Lambda)$ for $\Lambda \in TRng$ have all small*

(i) limits;

(ii) colimits.

Proof. (i) It is easy to check that products in any of these categories are given by endowing the product in *Set*, *Grp*, *Ab*, *Rng* or $Mod(U(\Lambda))$ respectively with the product topology. Given a functor $F : I \to C$, where I is a small category and C is any of *Top*, *TGrp*, *TAb*, *TRng* and $T(\Lambda)$, take the product in C of the objects $F(i)$ such that $i \in I$. Now take the subobject consisting of tuples $(x_i) \in \prod_I F(i)$ such that for every morphism $f : i \to j$ in I $F(f)(x_i) = x_j$, endowed with the subspace topology: one can check that this is the limit of F.

(ii) We will start from the well-known fact that *Set*, *Grp*, *Ab*, *Rng* and $Mod(U(\Lambda))$ have all small colimits. Given a functor $F : I \to C$, where I is a small category and C is any of *Top*, *TGrp*, *TAb*, *TRng* and $T(\Lambda)$, we can think of F as an object of C^I, and apply the exponent functor $U^I : C^I \to U(C)^I$ defined in Section 1.2. We know $U^I(F) \in U(C)^I$ has a colimit.

For $C = Top$: Let S be the set of topologies on $\operatorname{colim}_I U^I(F)$ making all the canonical maps

$$\phi_i : F(i) \to \operatorname{colim}_I U^I(F)$$

continuous for each $i \in I$.

For $C = TGrp$ or TAb: Let S be the set of topologies on $\operatorname{colim}_I U^I(F)$ such that it is a topological group making all the canonical maps

$$\phi_i : F(i) \to \operatorname{colim}_I U^I F(I)$$

continuous for each $i \in I$.

For $C = TRng$: Let S be the set of topologies on $\operatorname{colim}_I U^I(F)$ such that it is a topological ring making all the canonical maps

$$\phi_i : F(i) \to \operatorname{colim}_I U^I F(I)$$

continuous for each $i \in I$.

For $C = T(\Lambda)$: Let S be the set of topologies on $\mathrm{colim}_I\, U^I(F)$ such that it is a topological Λ-module making all the canonical maps

$$\phi_i : F(i) \to \mathrm{colim}_I\, U^I F(I)$$

continuous for each $i \in I$.

In each case we can make S into a poset ordered by the fineness of the topology. Write $(\mathrm{colim}_I\, U^I(F))_{T'}$ for $\mathrm{colim}_I\, U^I(F)$ endowed with a topology $T' \in S$. By Lemma 2.1.2 and the universal property, in each case, if S has a maximal element T finer than all other topologies in S, $(\mathrm{colim}_I\, U^I(F))_T$ is the colimit we are looking for. Now $S \neq \varnothing$ since it contains the indiscrete topology. Define T to be the topology generated by the subbase $\bigcup_S T'$. We claim $T \in S$: then we will be done.

For $C = Top$: We need to check $\phi_i^{-1}(O)$ is open in $F(i)$ for each i and O open in $(\mathrm{colim}_I\, U^I(F))_T$. It is sufficient to check this when O is in the subbase, and so open in $(\mathrm{colim}_I\, U^I(F))_{T'}$ for some T', where this is clear.

For $C = TGrp$ or TAb: We need to check in addition to the $C = Top$ case that $(\mathrm{colim}_I\, U^I(F))_T$ is a topological group, that is, that the multiplication map

$$m : (\mathrm{colim}_I\, U^I(F))_T \times (\mathrm{colim}_I\, U^I(F))_T \to (\mathrm{colim}_I\, U^I(F))_T$$

and the inversion map

$$i : (\mathrm{colim}_I\, U^I(F))_T \to (\mathrm{colim}_I\, U^I(F))_T$$

are continuous. It suffices to check the inverse images of each open set O in the subbase; if O is open in $(\mathrm{colim}_I\, U^I(F))_{T'}$, its inverse image under m is open in

$$(\mathrm{colim}_I\, U^I(F))_{T'} \times (\mathrm{colim}_I\, U^I(F))_{T'},$$

and hence in

$$(\mathrm{colim}_I\, U^I(F))_T \times (\mathrm{colim}_I\, U^I(F))_T;$$

its inverse image under i is open in $(\mathrm{colim}_I\, U^I(F))_{T'}$, and hence open in $(\mathrm{colim}_I\, U^I(F))_T$ too.

For $C = TRng$: We need to check in addition to the $C = TAb$ case that multiplication is continuous. This holds in the same way as the continuity of m in the $C = TGrp$ case.

For $C = T(\Lambda)$: We need to check in addition to the $C = TAb$ case that scalar multiplication

$$s : \Lambda \times (\mathrm{colim}_I\, U^I(F))_T \to (\mathrm{colim}_I\, U^I(F))_T$$

is continuous. It suffices to check the inverse images of each open set O in the subbase; if O is open in $(\mathrm{colim}_I\, U^I(F))_{T'}$, then its inverse image is open in $\Lambda \times (\mathrm{colim}_I\, U^I(F))_{T'}$, so open in $\Lambda \times (\mathrm{colim}_I\, U^I(F))_T$.

□

There are free topological groups and modules.

Proposition 2.1.4. *(i) $V : TGrp \to Top$ has a left adjoint $Top \to TGrp$.*

(ii) *Suppose* $\Lambda \in TRng$. *Then* $V : T(\Lambda) \to Top$ *has a left adjoint* $L : Top \to T(\Lambda)$.

Proof. We give a proof for (ii); the proof of (i) is similar, using the existence of abstract free groups.

It is well-known that for the abstract ring $U(\Lambda)$

$$V : Mod(U(\Lambda)) \to Set$$

has a left adjoint

$$L' : Set \to Mod(U(\Lambda)).$$

Explicitly, given a topological space X, then on the set $U(X)$ we have a free module $U(\Lambda)[U(X)]$. Now let S be the set of topologies on $U(\Lambda)[U(X)]$ such that it is a topological Λ-module making the canonical map

$$i : X \to U(\Lambda)[U(X)]$$

continuous. We can make S into a poset ordered by the fineness of the topology. Write $(U(\Lambda)[U(X)])_{T'}$ for $U(\Lambda)[U(X)]$ endowed with a topology $T' \in S$. By Lemma 2.1.2 and the universal property, if S has a maximal element T finer than all other topologies in S, $(U(\Lambda)[U(X)])_T$ is the Λ-module we are looking for; we will write $\Lambda[X]$ for this module. Now $S \neq \varnothing$ since it contains the indiscrete topology. Define T to be the topology generated by the subbase $\bigcup_S T'$. We claim $T \in S$: then we will be done.

$i : X \to (U(\Lambda)[U(X)])_T$ is continuous: We need to check $i^{-1}(O)$ is open in X for each O open in $(U(\Lambda)[U(X)])_T$. It is sufficient to check this when O is in the subbase, and so open in $(U(\Lambda)[U(X)])_{T'}$ for some T', where this is clear.

$(U(\Lambda)[U(X)])_T \in TAb$: We need to show addition

$$+ : (U(\Lambda)[U(X)])_T \times (U(\Lambda)[U(X)])_T \to (U(\Lambda)[U(X)])_T$$

and inversion

$$i : (U(\Lambda)[U(X)])_T \to (U(\Lambda)[U(X)])_T$$

are continuous. It suffices to check the inverse images of each open set O in the subbase; if O is open in $(U(\Lambda)[U(X)])_{T'}$, its inverse image under $+$ is open in

$$(U(\Lambda)[U(X)])_{T'} \times (U(\Lambda)[U(X)])_{T'},$$

hence in

$$(U(\Lambda)[U(X)])_T \times (U(\Lambda)[U(X)])_T;$$

its inverse image under i is open in $(U(\Lambda)[U(X)])_{T'}$, hence in $(U(\Lambda)[U(X)])_T$.

$(U(\Lambda)[U(X)])_T \in T(\Lambda)$: We need to show scalar multiplication

$$s : \Lambda \times (U(\Lambda)[U(X)])_T \to (U(\Lambda)[U(X)])_T$$

is continuous. It suffices to check the inverse images of each open set O in the subbase; if O is open in $(U(\Lambda)[U(X)])_{T'}$, then its inverse image is open in $\Lambda \times (U(\Lambda)[U(X)])_{T'}$, so open in $\Lambda \times (U(\Lambda)[U(X)])_T$. □

We write $\Lambda[X]$ for $L(X)$.

Similarly, given a commutative ring $R \in TRng$, $G \in TGrp$, we can define the topological group ring $R[G]$ to be the topological R-algebra with the universal property that for any $S \in TRng$ and any continuous group homomorphism $G \to S^\times$, the group of units of S, this extends to a continuous R-algebra homomorphism $R[G] \to S$. As before, this is the abstract group ring $U(R)[U(G)]$ endowed with the strongest topology making $U(R)[U(G)]$ a topological R-algebra such that the canonical group homomorphism $G \to U(R)[U(G)]$ is continuous. Details are left to the reader.

Finally, we have topological tensor products. Suppose R is a commutative topological ring and Λ is a topological R-algebra. Given a right Λ-module A, a left Λ-module B and an R-module M, a bilinear map $b : A \times B \to M$ is a map of sets, such that for all $x, x' \in A, y, y' \in B$ and $\lambda \in \Lambda$:

(i) $b(x + x', y) = b(x, y) + b(x', y)$;

(ii) $b(x, y + y') = b(x, y) + b(x, y')$;

(iii) $b(x\lambda, y) = b(x, \lambda y)$.

Proposition 2.1.5. *Suppose $R \in TRng$ is commutative, $\Lambda \in TRng$ is a topological R-algebra, $A \in T(\Lambda^{op})$, $B \in T(\Lambda)$. Then there is an object $A \otimes_\Lambda B \in T(R)$ and a continuous bilinear map*

$$\otimes_\Lambda : A \times B \to A \otimes_\Lambda B$$

satisfying the following universal property: given a continuous bilinear map $b : A \times B \to M$ for $M \in T(R)$, there is a unique continuous homomorphism $f : A \otimes_\Lambda B \to M$ in $T(R)$ such that $f \otimes_\Lambda = b$.

Proof. Take the abstract tensor product of $U(A)$ and $U(B)$, $U(A) \otimes_{U(\Lambda)} U(B)$. Let S be the set of topologies on $U(A) \otimes_{U(\Lambda)} U(B)$ such that it is a topological R-module making the canonical map

$$\otimes_\Lambda : A \times B \to U(A) \otimes_{U(\Lambda)} U(B)$$

continuous. We can make S into a poset ordered by the fineness of the topology. Write $(U(A) \otimes_{U(\Lambda)} U(B))_{T'}$ for $U(A) \otimes_{U(\Lambda)} U(B)$ endowed with a topology $T' \in S$. By Lemma 2.1.2 and the universal property, if S has a maximal element T finer than all other topologies in S, $(U(A) \otimes_{U(\Lambda)} U(B))_T$ is the R-module we are looking for; we will write $A \otimes_\Lambda B$ for this module. Now $S \neq \varnothing$ since it contains the indiscrete topology. Define T to be the topology generated by the subbase $\bigcup_S T'$. We claim $T \in S$: then we will be done.

$\otimes_\Lambda : A \times B \to (U(A) \otimes_{U(\Lambda)} U(B))_T$ is continuous: We need to check $\otimes_\Lambda^{-1}(O)$ is open in $A \times B$ for each O open in $(U(A) \otimes_{U(\Lambda)} U(B))_T$. It is sufficient to check this when O is in the subbase, and so open in $(U(A) \otimes_{U(\Lambda)} U(B))_{T'}$ for some T', where this is clear.

$(U(A) \otimes_{U(\Lambda)} U(B))_T \in TAb$: We need to show addition

$$+ : (U(A) \otimes_{U(\Lambda)} U(B))_T \times (U(A) \otimes_{U(\Lambda)} U(B))_T \to (U(A) \otimes_{U(\Lambda)} U(B))_T$$

and inversion

$$i : (U(A) \otimes_{U(\Lambda)} U(B))_T \to (U(A) \otimes_{U(\Lambda)} U(B))_T$$

are continuous. It suffices to check the inverse images of each open set O in the subbase; if O is open in $(U(A) \otimes_{U(\Lambda)} U(B))_{T'}$, its inverse image under $+$ is open in

$$(U(A) \otimes_{U(\Lambda)} U(B))_{T'} \times (U(A) \otimes_{U(\Lambda)} U(B))_{T'},$$

hence in

$$(U(A) \otimes_{U(\Lambda)} U(B))_T \times (U(A) \otimes_{U(\Lambda)} U(B))_T;$$

its inverse image under i is open in $(U(A) \otimes_{U(\Lambda)} U(B))_{T'}$, hence in $(U(A) \otimes_{U(\Lambda)} U(B))_T$.

$(U(A) \otimes_{U(\Lambda)} U(B))_T \in T(R)$: We need to show scalar multiplication

$$s : R \times (U(A) \otimes_{U(\Lambda)} U(B))_T \to (U(A) \otimes_{U(\Lambda)} U(B))_T$$

is continuous. It suffices to check the inverse images of each open set O in the subbase; if O is open in $(U(A) \otimes_{U(\Lambda)} U(B))_{T'}$, then its inverse image is open in $R \times (U(A) \otimes_{U(\Lambda)} U(B))_{T'}$, so open in $R \times (U(A) \otimes_{U(\Lambda)} U(B))_T$. □

It is clear from the universal property that $\otimes_\Lambda$ is a co-/covariant bifunctor

$$T(\Lambda^{op}) \times T(\Lambda) \to T(R).$$

2.2 Profinite and Discrete Modules

We now define a completion functor $c : TGrp \to TGrp$. Given $G \in TGrp$, consider the diagram in $TGrp$ whose objects are the continuous finite discrete quotient groups G_i of G, and whose morphisms are the ones making the diagrams

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ G_i & \longrightarrow & G_j. \end{array}$$

commute, where the maps $G \to G_i$ are quotients. Then we define $c(G)$ to be the limit in $TGrp$ of this diagram, and the universal property of limits gives a canonical morphism $G \to c(G)$ in $TGrp$. Given a morphism $f : G \to H$ in $TGrp$, for every continuous finite discrete quotient H_i of H, the composites

$$G \xrightarrow{f} H \to H_i$$

give us a collection of morphisms $f_i : G \to H_i$ making

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ H_i & \longrightarrow & H_j \end{array}$$

commute. Now for each H_i, $G/\ker(f_i)$ is a continuous finite discrete quotient of G, since the continuity of f_i and discreteness of H_i imply that $\ker(f_i)$ is a clopen subgroup of G of finite index. Hence we have a collection of morphisms

$$c(f_i) : c(G) \to G/\ker(f_i) \to H_i$$

making

$$\begin{array}{ccc} c(G) & & \\ \downarrow & \searrow & \\ H_i & \longrightarrow & H_j \end{array}$$

commute, and hence a unique $c(f) : c(G) \to c(H)$, making c a functor. We will refer to this as the profinite completion functor.

At last, we define the category $PGrp$ to be the full subcategory of $TGrp$ whose objects are those isomorphic to $c(G)$ for some G in $TGrp$; we call objects of $PGrp$ profinite groups. Thus in fact c is a functor $TGrp \to PGrp$. Observe that $c \circ c$ is naturally isomorphic to c, and so c, restricted to $PGrp$, is naturally isomorphic to the identity functor on $PGrp$.

We denote by t the inclusion functor $PGrp \to TGrp$.

Proposition 2.2.1. *c is a left adjoint of t.*

Proof. Suppose $G \in TGrp, H \in PGrp$. Then c induces a map of sets

$$mor_{TGrp}(G, t(H)) \to mor_{PGrp}(c(G), H);$$

this map is surjective because, given $f' : c(G) \to H$, we have

$$f = tf'c : G \to c(G) \to H \to t(H)$$

such that $c(f) = f'$. To see it is injective, consider $f : G \to t(H)$ such that $c(f) = 0$. From the construction, this implies that each $G/\ker(f_i) \to H_i$ is 0, hence that each $f_i : G \to H_i$ is too, and hence that f is, in the notation used above. □

This adjunction makes $PGrp$ a reflective subcategory of $TGrp$, and hence we can characterise profinite completion by the following universal property: for $G \in TGrp, H \in PGrp, f : G \to t(H)$, there is a unique $f' : c(G) \to H$ such that $f = tf'c$.

In exactly the same way we define profinite completion functors c on Top, TAb, $TRng$ and $T(\Lambda)$ for $\Lambda \in TRng$. In each case, the profinite completion of an object is the limit of its finite quotient objects. We define Pro, PAb, $PRng$ and $P(\Lambda)$ to be the full subcategories of Top, TAb, $TRng$ and $T(\Lambda)$, respectively, whose objects are isomorphic to the image under c of some object in that category; objects of Pro, PAb, $PRng$ and $P(\Lambda)$ will be called profinite spaces, profinite abelian groups, profinite rings and profinite Λ-modules, respectively. In each case, c is a left adjoint of the inclusion functor t, so that each of these profinite categories is a reflective subcategory of the corresponding topological one.

As before we write U for each of the forgetful functors from Pro, $PGrp$, PAb, $PRng$ and $P(\Lambda)$ for $\Lambda \in TRng$ to Set, Grp, Ab, Rng and $Mod(U(\Lambda))$ respectively which forget the topology, and V for each of the forgetful functors from $PGrp$, PAb, $PRng$ and $P(\Lambda)$ for $\Lambda \in TRng$ to Pro which forget the algebraic structure.

By [23, Theorem 1.1.12, Theorem 2.1.3, Proposition 5.1.2], profinite spaces, groups and rings are exactly the topological spaces, groups and rings respectively which are compact, Hausdorff and totally disconnected. By [23, Lemma

5.1.1], profinite Λ-modules are just the compact, Hausdorff, totally disconnected topological Λ-modules when Λ is profinite. So a closed subgroup of a profinite group, or a quotient of a profinite group by a closed subgroup, or an extension of one profinite group by another, will again be profinite, and similar statements hold for profinite spaces, rings, and Λ-modules for $\Lambda \in PRng$. Therefore, when we talk of profinite groups, subgroups and quotient groups will always be assumed to be profinite, unless stated otherwise, and similarly for spaces, etc.

Note that $PAb, PRng$ and $P(\Lambda)$ for $\Lambda \in PRng$ inherit the structure of additive categories via c; the results on topology show that they also have kernels and cokernels. In fact PAb and $P(\Lambda)$ are abelian categories: this follows from the fact that continuous bijections of compact, Hausdorff spaces are homeomorphisms, and that the compact subspaces of compact, Hausdorff spaces are exactly the closed sets.

Example 2.2.2. Consider the abelian group $\mathbb{Z}$, with the discrete topology. We write $\hat{\mathbb{Z}}$ for the abelian profinite group $c(\mathbb{Z})$. Since the finite discrete quotient groups of $\mathbb{Z}$ as a group are also the finite discrete quotient rings of $\mathbb{Z}$ considered as a ring, $\hat{\mathbb{Z}}$ also has the structure of a profinite ring.

Proposition 2.2.3. *For $\Lambda \in TAb$ and A a profinite Λ-module, there is a canonical $c(\Lambda)$-action on A making it a profinite $c(\Lambda)$-module. In particular, a profinite abelian group is a profinite $\hat{\mathbb{Z}}$-module.*

Proof. A is the limit of its finite discrete quotient Λ-modules A_i. Because A_i is finite discrete and the Λ-action $\Lambda \times A_i \to A_i$ is continuous, it follows from Lemma 2.1.1 that there is some open (two-sided) ideal I of Λ whose action on A_i is multiplication by 0. So the Λ-action factors as $\Lambda/I \times A_i \to A_i$, and hence the canonical map $c(\Lambda) \to \Lambda/I$ makes A_i into a finite discrete $c(\Lambda)$-module. Taking limits over i, A is a profinite $c(\Lambda)$-module.

For the rest of the statement, let $\Lambda = \mathbb{Z}$ with the discrete topology, and observe that topological abelian groups are automatically topological $\mathbb{Z}$-modules. □

Using the actions on finite quotients, it is easy to check that this construction is functorial: given a map $f : A \to B$ of Λ-modules, f is compatible with the $c(\Lambda)$-action defined above, so we get a functor $P(\Lambda) \to P(c(\Lambda))$. Moreover, the restriction functor $P(c(\Lambda)) \to P(\Lambda)$ is inverse to this. So Proposition 2.2.3 gives equivalences $P(\Lambda) \to P(c(\Lambda))$ and $PAb \to P(\hat{\mathbb{Z}})$.

We can now deduce several nice properties of our profinite categories from the corresponding topological ones, using our adjunctions.

Proposition 2.2.4. *The categories Pro, PGrp, PRng and $P(\Lambda)$ for $\Lambda \in PRng$ have all small colimits, and the colimit of a diagram in any of these categories is the profinite completion of the colimit of the same diagram in Top, TGrp, TRng or $T(\Lambda)$, respectively.*

Proof. This follows from Proposition 2.1.3 by [32, Theorem 2.6.10], using the adjoint functors c and t. □

There are free profinite groups and modules, which are the profinite completions of free topological groups and modules.

Proposition 2.2.5. *(i) $V : PGrp \to Pro$ has a left adjoint $Pro \to PGrp$.*

(ii) Suppose $\Lambda \in PRng$. Then $V : P(\Lambda) \to Pro$ has a left adjoint $L : Pro \to P(\Lambda)$.

Proof. We give a proof for (ii); the proof of (i) is similar

Write L_{Top} and V_{Top} for the adjoint pair $Top \to T(\Lambda)$ and $T(\Lambda) \to Top$ defined earlier. Then V is the composite $P(\Lambda) \xrightarrow{t} T(\Lambda) \xrightarrow{V_{Top}} Top$, whose image is in Pro. Suppose $A \in Pro$, $B \in P(\Lambda)$. Then

$$\begin{aligned} \mathrm{mor}_{Pro}(A, V(B)) &= \mathrm{mor}_{Pro}(A, V_{Top}t(B)) \\ &= \mathrm{mor}_{T(\Lambda)}(L_{Top}(A), t(B)) \\ &= \mathrm{mor}_{P(\Lambda)}(cL_{Top}(A), B), \end{aligned}$$

so let $L = cL_{Top}$. □

Writing FX for the free profinite group on a profinite space X, we can now define $G \in PGrp$ to be finitely generated (respectively, d-generated) if there is an epimorphism $FX \to G$ for some finite X (respectively, such that $|X| \leq d$).

We also write $\Lambda[\![X]\!]$ for the free profinite Λ-module on X, and define $A \in \Lambda$ to be finitely generated if there is an epimorphism $\Lambda[\![X]\!] \to A$ for some finite X.

Now (ii) shows that the abelian category $P(\Lambda)$ has enough projectives: images under L are projective by [23, Proposition 2.2.2], which gives the existence of continuous sections of profinite group homomorphisms.

Similarly, given a commutative ring $R \in PRng$, $G \in PGrp$, we can define the profinite group ring $R[\![G]\!]$ to be a profinite R-algebra, i.e. a profinite ring which is also a topological R-algebra, with the universal property that for any profinite R-algebra S and any continuous group homomorphism $G \to S^\times$, the group of units of S, this extends to a continuous R-algebra homomorphism $R[\![G]\!] \to S$. The universal property of profinite completions shows that this is just the profinite completion of the topological group ring $R[G]$. Details are left to the reader.

Finally, we have profinite tensor products.

Proposition 2.2.6. *Suppose $R \in PRng$ is commutative, Λ is a profinite R-algebra, $A \in P(\Lambda^{op})$, $B \in P(\Lambda)$. Then there is an object $A\hat{\otimes}_\Lambda B \in P(R)$ and a continuous bilinear map*

$$\hat{\otimes}_\Lambda : A \times B \to A\hat{\otimes}_\Lambda B$$

satisfying the following universal property: given a continuous bilinear map $b : A \times B \to M$ for $M \in PM(R)$, there is a unique continuous homomorphism $f : A\hat{\otimes}_\Lambda B \to M$ in $P(R)$ such that $f\hat{\otimes}_\Lambda = b$.

Proof. The universal property of profinite completions shows that this is just the profinite completion of the topological tensor product $A \otimes_\Lambda B$. □

It is clear from the universal property that $\hat{\otimes}_\Lambda$ is a co-/covariant bifunctor

$$P(\Lambda^{op}) \times P(\Lambda) \to P(R).$$

We write DAb and $D(\Lambda)$, $\Lambda \in TRng$, for the categories of discrete torsion abelian groups and discrete torsion topological left Λ-modules respectively, whose morphisms are continuous group/Λ-module homomorphisms. In other words, they are the full subcategories of TAb and $T(\Lambda)$ containing the objects whose topology is discrete, whose underlying abelian groups are torsion, and (in the latter case) whose scalar multiplication map $\Lambda \times A \to A$ is continuous. As before we write U for the forgetful functors $DAb \to Ab$ and $D(\Lambda) \to Mod(U(\Lambda))$.

We define a functor $d : TAb \to TAb$ in the following way. Given $A \in TAb$, consider the diagram in TAb whose objects A_i are finite subgroups of A, with the discrete topology, and whose morphisms are inclusions of one of these subgroups into another. Then we define $d(A)$ to be the colimit of this diagram in TAb.

Lemma 2.2.7. *$d(A)$ is in DAb.*

Proof. We can identify the underlying group of $d(A)$ with the torsion subgroup of A, so we just need to check that the colimit topology on $d(A)$ is discrete. Clearly every map $A_i \to d(A)$ is continuous when $d(A)$ is given the discrete topology, and the discrete topology is necessarily the strongest topology for which this is the case, so the result follows. □

Morphisms are given by restriction to the torsion subgroup of A, whose image under any morphism is torsion.

We also have a functor $d : T(\Lambda) \to D(\Lambda)$, defined similarly: given $A \in T(\Lambda)$, consider the diagram whose objects are finite Λ-submodules A_i of A, with the discrete topology, such that the discrete topology makes A_i a topological Λ-module; morphisms are inclusions of one submodule into another. Then $d(A)$ is the colimit of this diagram in $T(\Lambda)$. The same argument as Lemma 2.2.7 shows that $d(A)$ is discrete, except that, by Lemma 2.1.1, the underlying module of $d(A)$ is the submodule of the torsion submodule of A on which the stabilisers of the Λ-action are open.

Proposition 2.2.8. *(i) $d : TAb \to DAb$ is right adjoint to inclusion $DAb \to TAb$.*

(ii) $d : T(\Lambda) \to D(\Lambda)$ is right adjoint to inclusion $D(\Lambda) \to T(\Lambda)$.

Therefore DAb and $D(\Lambda)$ are coreflective subcategories of TAb and $T(\Lambda)$, respectively, and have all small limits.

Proof. (i) Given $A \in DAb$, $B \in TAb$, any homomorphism of groups $f : A \to B$ will be continuous, and its image will be torsion. So f factors through $A \to d(B)$.

(ii) Given $A \in D(\Lambda)$, $B \in T(\Lambda)$, the argument is the same as for (i), after noting that for any $x \in B$ in the image of A, it will be the image of some element whose stabiliser is open, so the stabiliser of x will be open, so $x \in d(B)$.

TAb and $T(\Lambda)$ have all small limits by Proposition 2.1.3. Then our adjunctions give all small limits in DAb and $D(\Lambda)$ by [32, Theorem 2.6.10]. □

Proposition 2.2.9. *(i) $P(\Lambda)$ is closed under taking limits in $T(\Lambda)$. That is, given a diagram in $P(\Lambda)$, its limit as a diagram in $T(\Lambda)$ will be a profinite Λ-module. Hence $P(\Lambda)$ has all small limits.*

(ii) $D(\Lambda)$ is closed under taking colimits in $T(\Lambda)$. That is, given a diagram in $D(\Lambda)$, its colimit as a diagram in $T(\Lambda)$ will be a discrete Λ-module. Hence $D(\Lambda)$ has all small colimits.

Proof. (i) Products of compact, Hausdorff, totally disconnected spaces are compact, Hausdorff and totally disconnected, by [23, Proposition 1.1.3], so products of profinite Λ-modules are profinite. The limit of a diagram in $T(\Lambda)$ can be constructed as a submodule of the product of the objects in the diagram, with the subspace topology of the product topology, and [23, Lemma 1.1.2] shows that this submodule is closed, and hence profinite.

(ii) Suppose we have a diagram $\{A_i\}$ in $D(\Lambda)$. Take the colimit A of this diagram considered as a diagram of abstract modules, and give it the discrete topology. Clearly every map $A_i \to A$ is continuous, so if A is a topological Λ-module, it is easy to show that it satisfies the required universal property to be the colimit of $\{A_i\}$ in $T(\Lambda)$. The colimit of a diagram in $T(\Lambda)$ can be constructed as a quotient module of the direct sum of the objects in the diagram, with the colimit topology; so given $a \in A$, we know a is the image of some $(a_i) \in \bigoplus A_i$, with only finitely many a_i non-zero. By Lemma 2.1.1, the stabiliser of each non-zero a_i is open in Λ, so the finite intersection of these stabilisers is open and stabilises a, so A is a topological Λ-module.

□

The same argument as (i) shows that Pro, $PGrp$ and $PRng$ have all small limits too.

2.2.1 Pontryagin Duality

From now on, unless stated otherwise, R will always be a commutative profinite ring, and Λ will always be a profinite R-algebra.

Recall the additive bifunctor

$$\mathrm{Hom}_\Lambda(-,-) : T(\Lambda)^{op} \times T(\Lambda) \to Mod(U(R)),$$

where $R \in PRng$ is commutative and Λ is a profinite R-algebra. It will often be given the compact-open topology: we define the sets

$$O_{K,U} = \{f \in \mathrm{Hom}_\Lambda(A, B) : f(K) \subseteq U\}$$

to be open, whenever $K \subseteq A$ is compact and $U \subseteq B$ is open. Then the $O_{K,U}$ form a subbase for the topology, which makes $\mathrm{Hom}_\Lambda(A, B)$ into a topological abelian group; when this topology is used, we will write $\mathbf{Hom}_\Lambda(A, B)$. This topology makes the R-module homomorphism

$$\mathbf{Hom}_\Lambda(A, B) \to \mathbf{Hom}_\Lambda(A', B')$$

induced by morphisms $A' \to A, B \to B'$ continuous. So $\mathbf{Hom}_\Lambda$ becomes a bifunctor

$$T(\Lambda)^{op} \times T(\Lambda) \to TAb.$$

Now let $\mathbb{Q}/\mathbb{Z}$ be given the discrete topology, and write * for the contravariant functor $\mathbf{Hom}_{TAb}(-, \mathbb{Q}/\mathbb{Z}) : TAb \to TAb$. Moreover, given a topological Λ-module A, define a right Λ-action on A^* by $(f\lambda)(a) = f(\lambda a)$. If A is finite and discrete, A^* will be too, and in this case the Λ-action defined on A^* will be continuous, by Lemma 2.1.1.

Theorem 2.2.10 (Pontryagin Duality)**.** *The functor* * *sends profinite abelian groups to discrete ones and vice versa. Indeed, if* $A \in PAb$ *is the limit of its finite discrete quotient groups* A_i*, then* A^* *is the colimit of its finite discrete subgroups* A_i^**, and vice versa. Then* $^* \circ ^*$ *is naturally isomorphic to the identity on* PAb *and* DAb*, and hence* * *gives a dual equivalence between* DAb *and* PAb*.*

Proof. [23, Theorem 2.9.6] □

Remark 2.2.11. In fact the Pontryagin duality theorem holds more generally, giving a dual equivalence of the category of locally compact abelian groups with itself. We will not use this.

For $A \in P(\Lambda)$, A is the limit of its finite discrete quotient modules A_i. Then each A_i^* is a finite discrete right Λ-module, with the action defined above, so the colimit of these is a discrete right Λ-module by Proposition 2.2.9. Similarly, for $A \in D(\Lambda^{op})$, its Pontryagin dual is a profinite left Λ-module. So in fact * gives functors $P(\Lambda) \to D(\Lambda^{op})$ and $D(\Lambda^{op}) \to P(\Lambda)$. Also, by the previous theorem, we still have that both compositions $^* \circ ^*$ are naturally isomorphic to the identity. So we get:

Corollary 2.2.12. * *gives a dual equivalence between* $P(\Lambda)$ *and* $D(\Lambda^{op})$*.*

This duality immediately gives information about $D(\Lambda)$ from what we know of $P(\Lambda)$.

Corollary 2.2.13. *(i)* $D(\Lambda)$ *is an abelian category with enough injectives.*

(ii) $D(\hat{\mathbb{Z}})$ *is equivalent to* DAb*.*

More generally, to prove results about $D(\Lambda)$ and $P(\Lambda)$, it will often be enough just to give a proof for one of the categories and use Pontryagin duality to deduce the other.

2.2.2 Inverse and Direct Limits

Suppose $\mathcal{E}$ is a category and I is a small category. We call a functor $F : I \to \mathcal{E}$ an *inverse system* in $\mathcal{E}$ if, for each $i, j \in I$, there is at most one morphism in $\mathrm{mor}(i, j) \cup \mathrm{mor}(j, i)$, and there is some $k \in I$ such that there are morphisms $k \to i$ and $k \to j$ in I. In this case, the limit of F is called an *inverse limit.* We call functors $I \to \mathcal{E}$ direct systems if functors $I^{op} \to \mathcal{E}$ are inverse systems, and call their colimits direct limits. We denote inverse limits variously by $\varprojlim_{\mathcal{E},I}$, $\varprojlim_{\mathcal{E}}$, $\varprojlim_I$ or $\varprojlim$, and similarly for direct limits $\varinjlim$.

We call a poset I' directed if for every $i, j \in I'$ there is some $k \in I'$ such that $k \geq i, j$. There is a one-to-one correspondence between categories I over which limits are inverse limits and directed posets, in the following way: given a category I, take the set $\mathrm{ob}(I)$ of objects of I and define a partial order on it by $i \geq j$ whenever there is a morphism $i \to j$; conversely, given a directed poset, take its underlying set to be the objects of a category I and give I a morphism $i \to j$ whenever $i \geq j$ and no others. So we can equivalently think of inverse limits as limits of diagrams of directed posets. The case of direct limits can be thought of similarly. We will use the two equivalent definitions interchangeably.

It is easy to check that if I and J are directed posets, $I \times J$ is again directed, when we define $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$. This allows us to take an inverse limit over directed posets I and J simultaneously by taking the inverse limit over $I \times J$, and similarly for direct limits.

Observe that for $G \in TGrp$ $c(G)$ is the inverse limit of the continuous finite discrete quotients of G. To see this, consider any two such quotients G_i, G_j. Write H_i, H_j for $\ker(G \to G_i), \ker(G \to G_j)$ respectively. Then H_i, H_j are finite index clopen normal subgroups of G, so $H_i \cap H_j$ is too, and hence $G_k = G/(H_i \cap H_j)$ is a continuous finite discrete quotient of G such that

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ G_k & \longrightarrow & G_i \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ G_k & \longrightarrow & G_j \end{array}$$

commute. Similarly one can check that objects in Pro, $PRng$ and $P(\Lambda)$ are the inverse limits of their continuous finite discrete quotients, and that objects in $D(\Lambda)$ are the direct limits of their finite submodules.

Now several of our constructions can be defined in terms of inverse and direct limits. For $X = \varprojlim X_i \in Pro$ with the X_i finite, and $\Lambda = \varprojlim \Lambda_j \in PRng$ with the Λ_j finite,

$$\Lambda[\![X]\!] = \varprojlim \Lambda_j[X_i]$$

by [23, Exercise 5.2.3], where the $\Lambda_j[X_i]$ are the topological free modules defined earlier. For $G = \varprojlim G_i \in PGrp$ with the G_i finite, and $R \in PRng$ commutative,

$$R[\![G]\!] = \varprojlim R_j[G_i]$$

by [23, Lemma 5.3.5]. More examples are given in the next section.

We can also now phrase Pontryagin duality in terms of inverse and direct limits.

Lemma 2.2.14. *If $A = \varprojlim A_i \in P(\Lambda)$, then $A^* = \varinjlim A_i^* \in D(\Lambda^{op})$, and if $A = \varinjlim A_i \in D(\Lambda^{op})$, then $A^* = \varprojlim A_i^* \in P(\Lambda)$.*

Proof. This follows immediately from the dual equivalence between $P(\Lambda)$ and $D(\Lambda^{op}$, Corollary 2.2.12. □

The reason for introducing these special kinds of limits and colimits is for exactness. Inverse limits of short exact sequences of profinite groups are exact, by [23, Proposition 2.2.4], and hence inverse limits are exact in $P(\Lambda)$. By Pontryagin duality, direct limits are exact in $D(\Lambda)$.

2.2.3 Hom and $\hat{\otimes}$

Suppose now that $A = \varprojlim A_i \in P(\Lambda^{op})$, $B = \varprojlim B_j \in P(\Lambda)$ and $C = \varinjlim C_k \in D(\Lambda)$ where the A_i and B_j are finite discrete quotient modules of A and B, and the C_k are finite discrete submodules of C.

Lemma 2.2.15. *(i) $A \hat{\otimes}_\Lambda B = \varprojlim A_i \otimes_\Lambda B_j$, where the $A_i \otimes_\Lambda B_j$ are the abstract tensor products of A_i and B_j with the discrete topology.*

(ii) Each $\mathbf{Hom}_\Lambda(B_j, C_k)$ has the discrete topology, and

$$\mathbf{Hom}_\Lambda(B, C) = \varinjlim \mathbf{Hom}_\Lambda(B_j, C_k).$$

Proof. [23, Lemma 5.5.1, Lemma 5.1.4] □

For (ii), each $\mathbf{Hom}_\Lambda(B_j, C_k)$ is in fact a topological R-module by Lemma 2.1.1, because each map $B_j \to C_k$ is fixed by an open subring of R. Indeed, since C_k is in $D(\Lambda)$, the stabiliser of any element of C_k is open in Λ by Lemma 2.1.1; taking the intersection of these stabilisers, some open subring of Λ fixes C_k pointwise. The intersection of this subring with R fixes $\mathbf{Hom}_\Lambda(B_j, C_k)$ pointwise, as required. So in this situation $\mathbf{Hom}_\Lambda(B, C)$ is a discrete R-module, not just a discrete abelian group. In other words, we have:

Corollary 2.2.16. $\mathbf{Hom}_\Lambda$ *gives an additive bifunctor* $P(\Lambda)^{op} \times D(\Lambda) \to D(R)$.

Similarly, if B is finitely generated, each $\mathbf{Hom}_\Lambda(B, C_k)$ is finite and discrete, and the same argument as before shows that in fact $\mathbf{Hom}_\Lambda(B, C_k)$ is a finite discrete R-module. It follows from the universal property of limits that

$$\mathrm{Hom}_\Lambda(A, B) = \varprojlim \mathrm{Hom}_\Lambda(A, B_k).$$

In fact this gives an isomorphism

$$\mathbf{Hom}_\Lambda(A, B) = \varprojlim \mathbf{Hom}_\Lambda(A, B_k)$$

of topological R-modules: this follows from the fact that the kernels of the maps $B \to B_k$ form a fundamental system of neighbourhoods of 0 in B, so these kernels can be used to calculate the compact-open topology on $\mathbf{Hom}_\Lambda(A, B)$.

Write $P(\Lambda)_0$ for the full subcategory of $P(\Lambda)$ consisting of finitely generated modules. $P(\Lambda)_0$ is additive, but not abelian; when we talk about exactness properties involving $P(\Lambda)_0$, we will mean relative to the class of sequences $0 \to A \to B \to C \to 0$ in $P(\Lambda)_0$ which are exact in $P(\Lambda)$.

Lemma 2.2.17. $\mathbf{Hom}_\Lambda$ *gives an additive bifunctor* $P(\Lambda)_0^{op} \times P(\Lambda) \to P(R)$.

Proof. The topology on the topological R-module $\mathbf{Hom}_\Lambda(A, B)$ is profinite by Proposition 2.2.9. □

Suppose now that Θ, Φ, Ψ are profinite R-algebras. Let $P(\Theta - \Phi)$ be the category of profinite $\Theta - \Phi$-bimodules and continuous $\Theta - \Phi$-bimodule homomorphisms, and let $D(\Theta - \Psi)$ be the category of discrete $\Theta - \Psi$-bimodules and continuous $\Theta - \Psi$-homomorphisms. If L is a profinite $\Theta - \Lambda$-bimodule and M is a profinite $\Lambda - \Phi$-bimodule, one can make $L \hat{\otimes}_\Lambda M$ into a profinite $\Theta - \Phi$-bimodule in the same way as in the abstract case. Similarly, if M is a profinite $\Lambda - \Theta$-bimodule and N is a discrete $\Lambda - \Phi$-bimodule, one can make $\mathbf{Hom}_\Lambda(M, N)$ into a discrete $\Theta - \Phi$-bimodule in the same way as in the abstract case. We leave the details to the reader.

Theorem 2.2.18 (Adjunction isomorphism)**.** *Suppose* $L \in P(\Theta - \Lambda), M \in P(\Lambda - \Phi), N \in D(\Theta - \Psi)$*. Then there is an isomorphism*

$$\mathbf{Hom}_\Theta(L \hat{\otimes}_\Lambda M, N) \cong \mathbf{Hom}_\Lambda(M, \mathbf{Hom}_\Theta(L, N))$$

in $PD(\Phi - \Psi)$*, natural in* L, M, N.

Proof. [23, Proposition 5.5.4(c)] □

It follows that $\mathbf{Hom}_\Lambda$ (considered as a co-/covariant bifunctor $P(\Lambda)^{op} \times D(\Lambda) \to D(R)$) is left-exact in both variables, and that $\hat{\otimes}_\Lambda$ is right-exact in both variables, by [32, Theorem 2.6.1]. It is easy to check also that the other Hom functors we have considered are left-exact in each variable.

If $L \in P(\Theta - \Phi)$, Pontryagin duality gives L^* the structure of a discrete $\Phi - \Theta$-bimodule, and similarly with profinite and discrete switched.

Corollary 2.2.19. *There is a natural isomorphism*

$$(L\hat{\otimes}_\Lambda M)^* \cong \mathbf{Hom}_\Lambda(M, L^*)$$

in $D(\Phi - \Theta)$ *for* $L \in P(\Theta - \Lambda), M \in P(\Lambda - \Phi)$.

Proof. Apply the theorem with $\Psi = \hat{\mathbb{Z}}$ and $N = \mathbb{Q}/\mathbb{Z}$. □

Proposition 2.2.20. *Suppose* $P \in P(\Lambda)$ *is projective and* $I \in D(\Lambda)$ *is injective. Then* $\mathbf{Hom}_\Lambda(P, -)$ *is an exact functor on* $D(\Lambda)$, $-\hat{\otimes}_\Lambda P$ *is an exact functor on* $P(\Lambda^{op})$, *and* $\mathbf{Hom}_\Lambda(-, I)$ *and* $I^*\hat{\otimes}-$ *are exact functors on* $P(\Lambda)$.

Proof. [23, Exercise 5.4.7] and its Pontryagin dual. □

2.2.4 Pro-$\mathcal{C}$ Groups

Let $\mathcal{C}$ be a non-empty class of finite discrete groups, closed under isomorphisms. Given a topological group G, we can define a *pro-$\mathcal{C}$ completion* functor $c_\mathcal{C}$ which sends G to the limit of the quotients of G which are in $\mathcal{C}$, and define the category of *pro-$\mathcal{C}$ groups* to be the image $c_\mathcal{C}(TGrp)$ in $TGrp$. Similarly, one can define pro-$\mathcal{C}$ rings and modules to be ones which are the limit of quotients whose underlying abelian group is in $\mathcal{C}$.

Suppose that $\mathcal{C}$ is closed under taking quotients, and satisfies the following property: if G is a finite discrete group with normal subgroups N_1, N_2 such that G/N_1 and G/N_2 are in C, then $G/(N_1 \cap N_2)$ is in $\mathcal{C}$. (In the terminology of [23], this makes $\mathcal{C}$ a *formation*.) Then, by the same proof as for profinite groups, $c_\mathcal{C}$ is left-adjoint to the inclusion functor $c_\mathcal{C}(TGrp) \to TGrp$.

For sufficiently well-behaved classes $\mathcal{C}$, much of the theory of profinite groups can be carried over to the category of pro-$\mathcal{C}$ groups and continuous homomorphisms. We will primarily be interested in the case where $\mathcal{C}$ is the category of finite discrete p-groups, for some prime p; in this case, we will call such groups *pro-p groups*, write $c_\mathcal{C}$ as c_p and call it *pro-p completion*, and so on. We will call rings and Λ-modules whose underlying abelian group is a p-group p-rings and p-Λ-modules.

Example 2.2.21. Consider the abelian group $\mathbb{Z}$, with the discrete topology. We write $\mathbb{Z}_p$ for the abelian pro-p group $c_p(\mathbb{Z})$. Since the finite discrete quotient p-groups of $\mathbb{Z}$ as a group are also the finite discrete quotient p-rings of $\mathbb{Z}$ considered as a ring, $\mathbb{Z}_p$ also has the structure of a pro-p ring.

Proposition 2.2.22. *For* $\Lambda \in TAb$ *and* A *a pro-p* Λ*-module, there is a canonical* $c_p(\Lambda)$*-action on* A *making it a pro-p* $c_p(\Lambda)$*-module. In particular, a pro-p abelian group is a pro-p* $\mathbb{Z}_p$*-module.*

Proof. A is the limit of its finite discrete quotient p-Λ-modules A_i. Because A_i is a finite discrete p-group and the Λ-action $\Lambda \times A_i \to A_i$ is continuous, it follows from Lemma 2.1.1 that there is some open (two-sided) ideal I of Λ with p-power index whose action on A_i is multiplication by 0. So the Λ-action factors as $\Lambda/I \times A_i \to A_i$, where Λ/I is a finite p-ring, and hence the canonical map $c_p(\Lambda) \to \Lambda/I$ makes A_i into a finite discrete p-$c_p(\Lambda)$-module. Taking limits over i, A is a pro-p $c_p(\Lambda)$-module.

For the rest of the statement, let $\Lambda = \mathbb{Z}$ with the discrete topology, and observe that topological abelian groups are automatically topological $\mathbb{Z}$-modules. □

Using the actions on finite quotients, it is easy to check that this construction is functorial: given a map $f : A \to B$ of Λ-modules, f is compatible with the $c_p(\Lambda)$-action defined above. As for Proposition 2.2.3, this functor is inverse to restriction, so we get equivalences between the categories of pro-p Λ-modules and pro-p $c_p(\Lambda)$-modules, and in particular between the categories of pro-p abelian groups and pro-p $\mathbb{Z}_p$-modules.

In many ways, the class of pro-p groups is better behaved than the class of profinite groups. $\mathbb{Z}_p$ is a principal ideal domain, and a local ring, by [23, Proposition 2.7.1], so it is a discrete valuation ring. Explicitly, the valuation of an element $x \in \mathbb{Z}_p$ is the largest k such that $x \in p^k\mathbb{Z}_p$. On the other hand, $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ by [23, Theorem 2.7.2], so $\hat{\mathbb{Z}}$ is not a domain.

We also write $\mathbb{Q}_p$ for the field of fractions of $\mathbb{Z}_p$, with the topology given by defining the sets $p^n\mathbb{Z}_p$, $n \in \mathbb{Z}$, to be a fundamental system of neighbourhoods of 0.

2.3 Ind-Profinite and Pro-Discrete Modules

2.3.1 Ind-Profinite Modules

We say a topological space X is *ind-profinite* if there is an injective sequence of subspaces X_i, $i \in \mathbb{N}$, whose union is X, such that each X_i is profinite and X has the colimit topology with respect to the inclusions $X_i \to X$. That is, $X = \varinjlim_{IPSpace} X_i$. We write $IPSpace$ for the category of ind-profinite spaces and continuous maps.

Proposition 2.3.1. *Given an ind-profinite space X defined as the colimit of an injective sequence $\{X_i\}$ of profinite spaces, any compact subspace K of X is contained in some X_i.*

Proof. [12, Proposition 1.1] proves this under the additional assumption that the X_i are profinite groups, but the proof does not use this. □

This shows that compact subspaces of X are exactly the profinite subspaces, and that, if an ind-profinite space X is defined as the colimit of a sequence $\{X_i\}$, then the X_i are cofinal in the poset of compact subspaces of X. We call such a sequence a *cofinal sequence for* X: any cofinal sequence of profinite subspaces defines X up to homeomorphism.

A topological space X is called *compactly generated* if it satisfies the following condition: a subspace U of X is closed if and only if $U \cap K$ is closed in K for every compact subspace K of X. See [29] for background on such spaces. By the definition of the colimit topology, ind-profinite spaces are compactly generated. Indeed, a subspace U of an ind-profinite space X is closed if and only if $U \cap X_i$

is closed in X_i for all i, if and only if $U \cap K$ is closed in K for every compact subspace K of X by Proposition 2.3.1.

Lemma 2.3.2. *IPSpace has finite products and coproducts.*

Proof. Given $X, Y \in \mathit{IPSpace}$ with cofinal sequences $\{X_i\}, \{Y_i\}$, we can construct $X \sqcup Y$ using the cofinal sequence $\{X_i \sqcup Y_i\}$. Thanks to Proposition 2.3.1, the ind-profinite space $\varinjlim X_i \times Y_i$ is the product of X and Y in *IPSpace*. Indeed, given an ind-profinite space Z with cofinal sequence $\{Z_i\}$ and continuous maps $f : Z \to X$ and $g : Z \to Y$, write f_i, g_i for f, g restricted to Z_i, respectively. We know $f(Z_i), g(Z_i)$ are compact, so they are contained in some X_{j_i}, Y_{k_i} respectively. Let $l_i = \max\{j_i, k_i\}$. Then f_i and g_i factor through $X_{l_i} \times Y_{l_i}$, so $f = \varinjlim f_i$ and $g = \varinjlim g_i$ factor through $\varinjlim X_{l_i} \times X_{l_i} = \varinjlim X_i \times Y_i$. □

Moreover, by the proposition, $\{X_i \times Y_i\}$ is cofinal in the poset of compact subspaces of $X \times Y$ (with the product topology), and hence $\varinjlim X_i \times Y_i$ is the *k-ification* of $X \times Y$, or in other words it is the product of X and Y in the category of compactly generated spaces – see [29] for details. So we will write $X \times_k Y$ for the product in *IPSpace*. It is not clear whether the space $X \times Y$, with the product topology, is homeomorphic to $X \times_k Y$.

We say an abelian group M equipped with an ind-profinite topology is an *ind-profinite* abelian group if it satisfies the following condition: there is an injective sequence of profinite subgroups M_i, $i \in \mathbb{N}$, which is a cofinal sequence for the underlying space of M. It is easy to see that profinite groups and countable discrete torsion groups are ind-profinite. Moreover $\mathbb{Q}_p$ is ind-profinite via the cofinal sequence

$$\mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p \xrightarrow{\cdot p} \cdots . \tag{$*$}$$

Remark 2.3.3. It is not obvious that ind-profinite abelian groups are topological groups. In fact, we see below that they are. But it is much easier to see that they are k-groups in the sense of [20]: the multiplication map

$$M \times_k M = \varinjlim_{IPSpace} M_i \times M_i \to M$$

is continuous by the definition of colimits. The k-group intuition will often be more useful.

In the terminology of [12] the ind-profinite abelian groups are just the abelian *weakly profinite* groups. We recall some of the basic results of [12].

Proposition 2.3.4. *Suppose M is an ind-profinite abelian group with cofinal sequence $\{M_i\}$.*

(i) Any compact subspace of M is contained in some M_i.

(ii) Closed subgroups N of M are ind-profinite, with cofinal sequence $N \cap M_i$.

(iii) Quotients of M by closed subgroups N are ind-profinite, with cofinal sequence $M_i/(N \cap M_i)$.

(iv) Ind-profinite abelian groups are topological groups.

Proof. [12, Proposition 1.1, Proposition 1.2, Proposition 1.5] □

As before, we call a sequence $\{M_i\}$ of profinite subgroups making M into an ind-profinite group a cofinal sequence for M.

Suppose from now on that R is a commutative profinite ring and Λ is a profinite R-algebra.

Remark 2.3.5. We could define ind-profinite rings as colimits of injective sequences (indexed by $\mathbb{N}$) of profinite rings, and much of what follows does hold in some sense for such rings, but not much is lost by the restriction. In particular, it would be nice to use the machinery of ind-profinite rings to study $\mathbb{Q}_p$, but the sequence $(*)$ making $\mathbb{Q}_p$ into an ind-profinite abelian group does not make it into an ind-profinite ring because the maps are not maps of rings.

We say that M is a left Λ-k-module if M is a k-group equipped with a continuous map $\Lambda \times_k M \to M$. A Λ-k-module homomorphism $M \to N$ is a continuous map which is a homomorphism of the underlying abstract Λ-modules. Because Λ is profinite, $\Lambda \times_k M = \Lambda \times M$, so $\Lambda \times M \to M$ is continuous. Hence if M is a topological group (that is, if multiplication $M \times M \to M$ is continuous) then it is a topological Λ-module.

We say that a left Λ-k-module M equipped with an ind-profinite topology is a left *ind-profinite* Λ*-module* if there is an injective sequence of profinite submodules M_i, $i \in \mathbb{N}$, which is a cofinal sequence for the underlying space of M. So countable discrete Λ-modules are ind-profinite, because finitely generated discrete Λ-modules are finite, and so are profinite Λ-modules. In particular Λ, with left-multiplication, is an ind-profinite Λ-module. Note that, since profinite $\hat{\mathbb{Z}}$-modules are the same as profinite abelian groups, ind-profinite $\hat{\mathbb{Z}}$-modules are the same as ind-profinite abelian groups.

Then we immediately get the following.

Corollary 2.3.6. *Suppose M is an ind-profinite Λ-module with cofinal sequence $\{M_i\}$.*

(i) Any compact subspace of M is contained in some M_i.

(ii) Closed submodules N of M are ind-profinite, with cofinal sequence $N \cap M_i$.

(iii) Quotients of M by closed submodules N are ind-profinite, with cofinal sequence $M_i/(N \cap M_i)$.

(iv) Ind-profinite Λ-modules are topological Λ-modules.

As before, we call a sequence $\{M_i\}$ of profinite submodules making M into an ind-profinite Λ-module a cofinal sequence for M.

Lemma 2.3.7. *Ind-profinite Λ-modules have a fundamental system of neighbourhoods of 0 consisting of open submodules. Hence such modules are Hausdorff and totally disconnected.*

Proof. Suppose M has cofinal sequence M_i, and suppose $U \subseteq M$ is open, with $0 \in U$; by definition, $U \cap M_i$ is open in M_i for all i. Profinite modules have a fundamental system of neighbourhoods of 0 consisting of open submodules, by [23, Lemma 5.1.1], so we can pick an open submodule N_0 of M_0 such that $N_0 \subseteq U \cap M_0$. Now we proceed inductively: given an open submodule N_i of M_i such that $N_i \subseteq U \cap M_i$, let f be the quotient map $M \to M/N_i$. Then $f(U)$ is open in M/N_i by [12, Proposition 1.3], so $f(U) \cap M_{i+1}/N_i$ is open in M_{i+1}/N_i.

Pick an open submodule of M_{i+1}/N_i which is contained in $f(U) \cap M_{i+1}/N_i$ and write N_{i+1} for its preimage in M_{i+1}. Finally, let N be the submodule of M with cofinal sequence $\{N_i\}$: N is open and $N \subseteq U$, as required. □

Write $IP(\Lambda)$ for the category whose objects are left ind-profinite Λ-modules, and whose morphisms $M \to N$ are Λ-k-module homomorphisms. We will identify the category of right ind-profinite Λ-modules with $IP(\Lambda^{op})$ in the usual way. Given $M \in IP(\Lambda)$ and a submodule M', write $\overline{M'}$ for the closure of M' in M. Given $M, N \in IP(\Lambda)$, write $\mathrm{Hom}_{IP(\Lambda)}(M, N)$ for the $U(R)$-module of morphisms $M \to N$: this makes $\mathrm{Hom}_{IP(\Lambda)}(-, -)$ into a functor $IP(\Lambda)^{op} \times IP(\Lambda) \to Mod(U(R))$ in the usual way.

Proposition 2.3.8. *$IP(\Lambda)$ is an additive category with kernels and cokernels.*

Proof. The category is clearly pre-additive; the biproduct $M \oplus N$ is the biproduct of the underlying abstract modules, with the topology of $M \times_k N$. The existence of kernels and cokernels follows from Corollary 2.3.6; the cokernel of $f : M \to N$ is $N/\overline{f(M)}$. □

Remark 2.3.9. The category $IP(\Lambda)$ is not abelian in general. Consider the countable direct sum $\oplus_{\aleph_0}\mathbb{Z}/2\mathbb{Z}$, with the discrete topology, and the countable direct product $\prod_{\aleph_0}\mathbb{Z}/2\mathbb{Z}$, with the profinite topology. Both are ind-profinite $\hat{\mathbb{Z}}$-modules. There is a canonical injective map

$$i : \oplus\mathbb{Z}/2\mathbb{Z} \to \prod \mathbb{Z}/2\mathbb{Z},$$

but $i(\oplus\mathbb{Z}/2\mathbb{Z})$ is not closed in $\prod \mathbb{Z}/2\mathbb{Z}$. Moreover, $\oplus\mathbb{Z}/2\mathbb{Z}$ is not homeomorphic to $i(\oplus\mathbb{Z}/2\mathbb{Z})$, with the subspace topology, because $i(\oplus\mathbb{Z}/2\mathbb{Z})$ is not discrete, by the construction of the product topology.

Recall from Chapter 1 that in an additive category with kernels and cokernels we write $\mathrm{coim}(f)$ for $\mathrm{coker}(\ker(f))$, and $\mathrm{im}(f)$ for $\ker(\mathrm{coker}(f))$. That is, $\mathrm{coim}(f) = f(M)$, with the quotient topology coming from M, and $\mathrm{im}(f) = \overline{f(M)}$, with the subspace topology coming from N. In an abelian category, $\mathrm{coim}(f) = \mathrm{im}(f)$, but the preceding remark shows that this fails in $IP(\Lambda)$.

We say a morphism $f : M \to N$ in $IP(\Lambda)$ is *strict* if $\mathrm{coim}(f) = \mathrm{im}(f)$. In particular strict epimorphisms are surjections. Note that if M is profinite all morphisms $f : M \to N$ must be strict, because compact subspaces of Hausdorff spaces are closed, so that $\mathrm{coim}(f) \to \mathrm{im}(f)$ is a continuous bijection of compact Hausdorff spaces and hence a topological isomorphism.

Proposition 2.3.10. *Morphisms $f : M \to N$ in $IP(\Lambda)$ such that $f(M)$ is a closed subset of N have continuous sections $\mathrm{im}(f) \to M$. So f is strict in this case, and in particular continuous bijections are isomorphisms.*

Proof. [12, Proposition 1.6] □

Corollary 2.3.11 (Canonical decomposition of morphisms)**.** *Every morphism $f : M \to N$ in $IP(\Lambda)$ can be uniquely written as the composition of a strict epimorphism, a bimorphism and a strict monomorphism. Moreover the bimorphism is an isomorphism if and only if f is strict.*

Proof. The decomposition is the usual one

$$M \to \operatorname{coim}(f) \xrightarrow{g} \operatorname{im}(f) \to N,$$

for categories with kernels and cokernels. Clearly $\operatorname{coim}(f) = f(M) \to N$ is injective, so g is too, and hence g is monic. Also the set-theoretic image of $M \to \operatorname{im}(f)$ is dense, so the set-theoretic image of g is too, and hence g is epic. Then everything follows from Proposition 2.3.10. □

Because $IP(\Lambda)$ is not abelian, it is not obvious what the right notion of exactness is. We will say that a chain complex

$$\cdots \to L \xrightarrow{f} M \xrightarrow{g} N \to \cdots$$

is *strict exact at* M if $\operatorname{coim}(f) = \ker(g)$. We say a chain complex is strict exact if it is strict exact at each M.

Proposition 2.3.12. *The category $IP(\Lambda)$ has countable colimits.*

Proof. We show first that $IP(\Lambda)$ has countable direct sums. Given a countable collection $\{M_n : n \in \mathbb{N}\}$ of ind-profinite Λ-modules, write $\{M_{n,i} : i \in \mathbb{N}\}$, for each n, for a cofinal sequence for M_n. Now consider the injective sequence $\{N_n\}$ given by $N_n = \prod_{i=1}^{n} M_{i,n+1-i}$: each N_n is a profinite Λ-module, so the sequence defines an ind-profinite Λ-module N. It is easy to check that the underlying abstract module of N is $\bigoplus_n M_n$, that each canonical map $M_n \to N$ is continuous, and that any collection of continuous homomorphisms $M_n \to P$ in $IP(\Lambda)$ induces a continuous $N \to P$.

Now suppose we have a countable diagram $\{M_n\}$ in $IP(\Lambda)$. Write S for the closed submodule of $\bigoplus M_n$ generated (topologically) by the elements with jth component $-x$, kth component $f(x)$ and all other components 0, for all maps $f : M_j \to M_k$ in the diagram and all $x \in M_j$. By standard arguments, $(\bigoplus M_n)/S$, with the quotient topology, is the colimit of the diagram. □

Remark 2.3.13. We get from this construction that, given a countable collection of short strict exact sequences

$$0 \to L_n \to M_n \to N_n \to 0$$

in $IP(\Lambda)$, their direct sum

$$0 \to \bigoplus L_n \to \bigoplus M_n \to \bigoplus N_n \to 0$$

is strict exact by Proposition 2.3.10, because the sequence of underlying modules is exact. So direct sums preserve kernels and cokernels, and in particular direct sums preserve strict maps, because given a countable collection of strict maps $\{f_n\}$ in $IP(\Lambda)$,

$$\begin{aligned}\operatorname{coim}(\bigoplus f_n) &= \operatorname{coker}(\ker(\bigoplus f_n)) = \bigoplus \operatorname{coker}(\ker(f_n)) \\ &= \bigoplus \ker(\operatorname{coker}(f_n)) = \ker(\operatorname{coker}(\bigoplus f_n)) = \operatorname{im}(\bigoplus f_n).\end{aligned}$$

Lemma 2.3.14. *(i) Given $M, N \in IP(\Lambda)$, pick cofinal sequences $\{M_i\}, \{N_j\}$ respectively. Then*

$$\mathrm{Hom}_{IP(\Lambda)}(M, N) = \varprojlim_i \varinjlim_j \mathrm{Hom}_{IP(\Lambda)}(M_i, N_j),$$

in the category of $U(R)$-modules.

(ii) Given $X \in IPSpace$ with a cofinal sequence $\{X_i\}$ and $N \in IP(\Lambda)$ with cofinal sequence $\{N_j\}$, write $C(X, N)$ for the $U(R)$-module of continuous maps $X \to N$. Then $C(X, N) = \varprojlim_i \varinjlim_j C(X_i, N_j)$.

Proof. (i) Since $M = \varinjlim_{IP(\Lambda)} M_i$, we have

$$\mathrm{Hom}_{IP(\Lambda)}(M, N) = \varprojlim \mathrm{Hom}_{IP(\Lambda)}(M_i, N).$$

Since the N_j are cofinal for N, every continuous map $M_i \to N$ factors through some N_j, so

$$\mathrm{Hom}_{IP(\Lambda)}(M_i, N) = \varinjlim \mathrm{Hom}_{IP(\Lambda)}(M_i, N_j).$$

(ii) Similarly.

□

Given $X \in IPSpace$ as before, define a module $\Lambda[\![X]\!] \in IP(\Lambda)$ in the following way: let $\Lambda[\![X_i]\!]$ be the free profinite Λ-module on X_i. The maps $X_i \to X_{i+1}$ induce maps $\Lambda[\![X_i]\!] \to \Lambda[\![X_{i+1}]\!]$ of profinite Λ-modules, and hence we get an ind-profinite Λ-module with cofinal sequence $\{\Lambda[\![X_i]\!]\}$. Write $\Lambda[\![X]\!]$ for this module, which we will call the *free ind-profinite Λ-module on X*.

Proposition 2.3.15. *Suppose $X \in IPSpace$ and $N \in IP(\Lambda)$. Then we have $\mathrm{Hom}_{IP(\Lambda)}(\Lambda[\![X]\!], N) = C(X, N)$, naturally in X and N.*

Proof. Recall, by Proposition 2.2.5, that $\mathrm{Hom}_{IP(\Lambda)}(\Lambda[\![X]\!], N) = C(X, N)$ when X and N are profinite. Then by Lemma 2.3.14,

$$\begin{aligned}\mathrm{Hom}_{IP(\Lambda)}(\Lambda[\![X]\!], N) &= \varprojlim_i \varinjlim_j \mathrm{Hom}_{IP(\Lambda)}(\Lambda[\![X_i]\!], N_j)\\ &= \varprojlim_i \varinjlim_j C(X_i, N_j)\\ &= C(X, N).\end{aligned}$$

The isomorphism is natural because $\mathrm{Hom}_{IP(\Lambda)}(\Lambda[\![-]\!], -)$ and $C(-,-)$ are both bifunctors. □

We call $P \in IP(\Lambda)$ *projective* if

$$0 \to \mathrm{Hom}_{IP(\Lambda)}(P, L) \to \mathrm{Hom}_{IP(\Lambda)}(P, M) \to \mathrm{Hom}_{IP(\Lambda)}(P, N) \to 0$$

is an exact sequence of $U(R)$-modules whenever

$$0 \to L \to M \to N \to 0$$

is strict exact. We say $IP(\Lambda)$ has *enough projectives* if for every $M \in IP(\Lambda)$ there is a projective P and a strict epimorphism $P \to M$.

Corollary 2.3.16. *$IP(\Lambda)$ has enough projectives.*

Proof. By Proposition 2.3.15 and Proposition 2.3.10, $\Lambda[\![X]\!]$ is projective for all $X \in IPSpace$. So given $M \in IP(\Lambda)$, $\Lambda[\![M]\!]$ has the required property: the identity $M \to M$ induces a canonical 'evaluation map' $\varepsilon : \Lambda[\![M]\!] \to M$, which is strict epic because it is a surjection. □

Lemma 2.3.17. *Projective modules in $IP(\Lambda)$ are summands of free ones.*

Proof. Given a projective $P \in IP(\Lambda)$, pick a free module F and a strict epimorphism $f : F \to P$. By definition, the map $\mathrm{Hom}_{IP(\Lambda)}(P, F) \xrightarrow{f^*} \mathrm{Hom}_{IP(\Lambda)}(P, P)$ induced by f is a surjection, so there is some morphism $g : P \to F$ such that $f^*(g) = gf = \mathrm{id}_P$. Then we get that the map $\ker(f) \oplus P \to F$ is a continuous bijection, and hence an isomorphism by Proposition 2.3.10. □

Remarks 2.3.18. (i) We can also define the class of *strictly free modules* to be free ind-profinite modules on an ind-profinite space X which has the form of a disjoint union of profinite spaces X_i. By the universal properties of coproducts and free modules we immediately get $\Lambda[\![X]\!] = \bigoplus \Lambda[\![X_i]\!]$. Moreover, for every ind-profinite space Y there is some X of this form with a surjection $X \to Y$: given a cofinal sequence $\{Y_i\}$ in Y, let $X_i = Y_i$ and $X = \bigsqcup X_i$, and the identity maps $X_i \to Y_i$ induce the required map $X \to Y$. Then the same argument as before shows that projective modules in $IP(\Lambda)$ are summands of strictly free ones.

(ii) Note that a profinite module in $IP(\Lambda)$ is projective in $IP(\Lambda)$ if and only if it is projective in the category of profinite Λ-modules. Indeed, Proposition 2.3.15 shows that free profinite modules are projective in $IP(\Lambda)$, and the rest follows.

2.3.2 Pro-Discrete Modules

Write $PD(\Lambda)$ for the category of left *pro-discrete* Λ-modules: the objects M in this category are countable inverse limits, as topological Λ-modules, of discrete Λ-modules M^i, $i \in \mathbb{N}$; the morphisms are continuous Λ-module homomorphisms. So discrete torsion Λ-modules are pro-discrete, and so are second-countable profinite Λ-modules by [23, Proposition 2.6.1, Lemma 5.1.1], and in particular Λ, with left-multiplication, is a pro-discrete Λ-module if Λ is second-countable. Moreover $\mathbb{Q}_p$ is a pro-discrete $\hat{\mathbb{Z}}$-module via the sequence

$$\cdots \xrightarrow{\cdot p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Q}_p/\mathbb{Z}_p.$$

We will identify the category of right pro-discrete Λ-modules with $PD(\Lambda^{op})$ in the usual way.

Lemma 2.3.19. *Pro-discrete Λ-modules are first-countable.*

Proof. We can construct $M = \varprojlim M^i$ as a closed subspace of $\prod M^i$. Each M^i is first-countable because it is discrete, and first-countability is closed under countable products and subspaces. □

Remarks 2.3.20. (i) This shows that Λ itself can be regarded as a pro-discrete Λ-module if and only if it is first-countable, if and only if it is second-countable by [23, Proposition 2.6.1]. Rings of interest are often second-countable; this class includes, for example, $\mathbb{Z}_p$, $\hat{\mathbb{Z}}$, $\mathbb{Q}_p$, and the completed group ring $R[\![G]\!]$ when R and G are second-countable.

(ii) Since first-countable spaces are always compactly generated by [29, Proposition 1.6], pro-discrete Λ-modules are compactly generated as topological spaces. In fact more is true. Given a pro-discrete Λ-module M which is the inverse limit of a countable sequence $\{M^i\}$ of finite quotients, suppose X is a compact subspace of M and write X^i for the image of X in M^i. By compactness, each X^i is finite. Let N^i be the submodule of M^i generated by X^i: because X^i is finite, Λ is compact and M^i is discrete torsion, N^i is finite. Hence $N = \varprojlim N^i$ is a profinite Λ-submodule of M containing X. So pro-discrete modules M are compactly generated by their profinite submodules N, in the sense that a subspace U of M is closed if and only if $U \cap N$ is closed in N for all N.

Lemma 2.3.21. *Pro-discrete Λ-modules are metrisable and complete.*

Proof. [3, IX, Section 3.1, Proposition 1] and the corollary to [3, II, Section 3.5, Proposition 10]. □

Note that pro-discrete Λ-modules need not be second-countable in general, because for example $PD(\hat{\mathbb{Z}})$ contains uncountable discrete abelian groups. However, we have the following result.

Lemma 2.3.22. *Suppose a Λ-module M has a topology which makes it pro-discrete and ind-profinite (as a Λ-module). Then M is second-countable and locally compact.*

Proof. As an ind-profinite Λ-module, take a cofinal sequence of profinite submodules M_i. For any discrete quotient N of M, the image of each M_i in N is compact and hence finite, and N is the union of these images, so N is countable. Then if M is the inverse limit of a countable sequence of discrete quotients M^j, each M^j is countable and M can be identified with a closed subspace of $\prod M^j$, so M is second-countable because second-countability is closed under countable products and subspaces. By Proposition 2.3.21, M is a Baire space, and hence by the Baire category theorem one of the M_i must be open. The result follows. □

Proposition 2.3.23. *Suppose M is a pro-discrete Λ-module which is the inverse limit of a sequence of discrete quotient modules $\{M^i\}$. Let $U^i = \ker(M \to M^i)$.*

(i) The sequence $\{M^i\}$ is cofinal in the poset of all discrete quotient modules of M.

(ii) Closed submodules N of M are pro-discrete, with cofinal sequence $N/(N \cap U^i)$.

(iii) Quotients of M by closed submodules N are pro-discrete, with cofinal sequence $M/(U^i + N)$.

Proof. (i) The U^i form a basis of open neighbourhoods of 0 in M, by [23, Exercise 1.1.15]. Therefore, for any discrete quotient D of M, the kernel of the quotient map $f : M \to D$ contains some U^i, so f factors through U^i.

(ii) M is complete, and hence N is complete by [3, II, Section 3.4, Proposition 8]. It is easy to check that $\{N \cap U^i\}$ is a fundamental system of neighbourhoods of the identity, so $N = \varprojlim N/(N \cap U^i)$ by [3, III, Section 7.3, Proposition 2]. Also, since M is metrisable, by [3, IX, Section 3.1, Proposition 4] M/N is complete too. After checking that $(U^i + N)/N$ is a fundamental system of neighbourhoods of the identity in M/N, we get $M/N = \varprojlim M/(U^i + N)$ by applying [3, III, Section 7.3, Proposition 2] again.

□

As a result of (i), we call $\{M^i\}$ a cofinal sequence for M.

As in $IP(\Lambda)$, it is clear from Proposition 2.3.23 that $PD(\Lambda)$ is an additive category with kernels and cokernels. Given $M, N \in PD(\Lambda)$, we write $\mathrm{Hom}_{PD(\Lambda)}(M, N)$ for the $U(R)$-module of morphisms $M \to N$: this makes $\mathrm{Hom}_{PD(\Lambda)}(-, -)$ into a functor

$$PD(\Lambda)^{op} \times PD(\Lambda) \to Mod(U(R))$$

in the usual way. Note that the ind-profinite $\hat{\mathbb{Z}}$-modules in Remark 2.3.9 are also pro-discrete $\hat{\mathbb{Z}}$-modules, so the remark also shows that $PD(\Lambda)$ is not abelian in general.

As before, we say a morphism $f : M \to N$ in $PD(\Lambda)$ is *strict* if $\mathrm{coim}(f) = \mathrm{im}(f)$. Since $\mathrm{im}(f)$ is a closed submodule of N, $f(M) = \{f(m) : m \in M\}$ must be closed for strict f. In particular strict epimorphisms are surjections. We say that a chain complex

$$\cdots \to L \xrightarrow{f} M \xrightarrow{g} N \to \cdots$$

is *strict exact at* M if $\mathrm{coim}(f) = \ker(g)$. We say a chain complex is strict exact if it is strict exact at each M.

Remark 2.3.24. In general, it is not clear whether a map $f : M \to N$ in $PD(\Lambda)$ with $f(M)$ closed in N must be strict, as is the case for ind-profinite modules. However, if in addition M (and hence $\mathrm{coim}(f)$) is second-countable, [16, Chapter 6, Problem R] shows that the continuous bijection $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.

As for ind-profinite modules, we can factorise morphisms in a canonical way.

Corollary 2.3.25 (Canonical decomposition of morphisms)**.** *Every morphism $f : M \to N$ in $IP(\Lambda)$ can be uniquely written as the composition of a strict epimorphism, a bimorphism and a strict monomorphism. Moreover the bimorphism is an isomorphism if and only if the morphism is strict.*

Remark 2.3.26. Suppose we have a short strict exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

in $PD(\Lambda)$. Pick a cofinal sequence $\{M^i\}$ for M. Then, as in Proposition 2.3.23(ii),

$$L = \mathrm{coim}(f) = \mathrm{im}(f) = \varprojlim \mathrm{im}(\mathrm{im}(f) \to M^i),$$

and similarly for N, so we can write the sequence as a surjective inverse limit of short (strict) exact sequences of discrete Λ-modules.

Conversely, suppose we have a surjective sequence of short (strict) exact sequences

$$0 \to L^i \to M^i \to N^i \to 0$$

of discrete Λ-modules. Taking limits we get a sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \qquad (*)$$

of pro-discrete Λ-modules. It is easy to check that

$$\mathrm{im}(f) = \ker(g) = L = \mathrm{coim}(f)$$

and

$$\mathrm{coim}(g) = \mathrm{coker}(f) = N = \mathrm{im}(g),$$

so f and g are strict, and hence $(*)$ is a short strict exact sequence.

Lemma 2.3.27. *Given $M, N \in PD(\Lambda)$, pick cofinal sequences $\{M^i\}, \{N^j\}$ respectively. Then*

$$\mathrm{Hom}_{PD(\Lambda)}(M, N) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{PD(\Lambda)}(M^i, N^j),$$

in the category of $U(R)$-modules.

Proof. Since $N = \varprojlim_{PD(\Lambda)} N^j$, we have by definition that

$$\mathrm{Hom}_{PD(\Lambda)}(M, N) = \varprojlim \mathrm{Hom}_{PD(\Lambda)}(M, N^j).$$

Since the M^i are cofinal for M, every continuous map $M \to N^j$ factors through some M^i, so

$$\mathrm{Hom}_{PD(\Lambda)}(M, N^j) = \varinjlim \mathrm{Hom}_{PD(\Lambda)}(M^i, N^j).$$

□

We call $I \in PD(\Lambda)$ *injective* if

$$0 \to \mathrm{Hom}_{PD(\Lambda)}(N, I) \to \mathrm{Hom}_{PD(\Lambda)}(M, I) \to \mathrm{Hom}_{PD(\Lambda)}(L, I) \to 0$$

is an exact sequence of R-modules whenever

$$0 \to L \to M \to N \to 0$$

is strict exact.

Lemma 2.3.28. *Suppose that I is a discrete Λ-module which is injective in the category of discrete Λ-modules. Then I is injective in $PD(\Lambda)$.*

Proof. We know $\mathrm{Hom}_{PD(\Lambda)}(-, I)$ is exact on discrete Λ-modules. Remark 2.3.26 shows that we can write short strict exact sequences of pro-discrete Λ-modules as surjective inverse limits of short exact sequences of discrete modules in $PD(\Lambda)$, and then, by injectivity, applying $\mathrm{Hom}_{PD(\Lambda)}(-, I)$ gives a direct system of short exact sequences of $U(R)$-modules; the exactness of such direct limits is well known. □

In particular we get that $\mathbb{Q}/\mathbb{Z}$, with the discrete topology, is injective in $PD(\hat{\mathbb{Z}})$ – it is injective among discrete $\hat{\mathbb{Z}}$-modules (i.e. torsion abelian groups by Corollary 2.2.13) by Baer's lemma [32, 2.3.1], because it is divisible.

Given $M \in IP(\Lambda)$, with a cofinal sequence $\{M_i\}$, and $N \in PD(\Lambda)$, with a cofinal sequence $\{N^j\}$, consider the category $T(\Lambda)$ of topological Λ-modules and continuous Λ-module homomorphisms. We can consider $IP(\Lambda)$ and $PD(\Lambda)$ as full subcategories of $T(\Lambda)$, and observe that $M = \varinjlim_{T(\Lambda)} M_i$ and $N = \varprojlim_{T(\Lambda)} N^j$. We write $\mathrm{Hom}_\Lambda(M,N)$ for the $U(R)$-module of morphisms $M \to N$ in $T(\Lambda)$. By the universal properties of limits and colimits:

Lemma 2.3.29. *As $U(R)$-modules,* $\mathrm{Hom}_\Lambda(M,N) = \varprojlim_{i,j} \mathrm{Hom}_\Lambda(M_i, N^j)$.

Then we may make $\mathrm{Hom}_\Lambda(M,N)$ into a topological R-module by identifying it with $\varprojlim_{i,j} \mathbf{Hom}_\Lambda(M_i, N_j)$ with the limit topology: this makes $\mathrm{Hom}_\Lambda(M,N)$ into a pro-discrete R-module which we write as $\mathbf{Hom}_\Lambda(M,N)$. Because M_i and N^j are cofinal sequences, this topology coincides with the compact-open topology, so this notation does not cause any ambiguity. The topology thus constructed is well-defined because the M_i are cofinal for M and the N_j cofinal for N. Moreover, given a morphism $M \to M'$ in $IP(\Lambda)$, this construction makes the induced map $\mathbf{Hom}_\Lambda(M',N) \to \mathbf{Hom}_\Lambda(M,N)$ continuous, and similarly in the second variable, so that $\mathbf{Hom}_\Lambda(-,-)$ becomes a functor $IP(\Lambda)^{op} \times PD(\Lambda) \to PD(R)$. Of course the case when M and N are right Λ-modules behaves in the same way; we may express this by treating M, N as left Λ^{op}-modules and writing $\mathrm{Hom}_{T(\Lambda^{op})}(M,N)$ in this case.

More generally, given a chain complex

$$\cdots \xrightarrow{d_1} M_1 \xrightarrow{d_0} M_0 \xrightarrow{d_{-1}} \cdots$$

in $IP(\Lambda)$ and a cochain complex

$$\cdots \xrightarrow{d^{-1}} N^0 \xrightarrow{d^0} N^1 \xrightarrow{d^1} \cdots$$

in $PD(\Lambda)$, both bounded below, consider the double cochain complex with (p,q)th term $\{\mathbf{Hom}_\Lambda(M_p, N^q)\}$, with the obvious horizontal maps, and with the vertical maps defined in the obvious way except that they are multiplied by -1 whenever p is odd: this makes $\mathrm{Tot}(\mathbf{Hom}_\Lambda(M_p, N^q))$ into a cochain complex which we denote by $\mathbf{Hom}_\Lambda(M,N)$. Each term in the total complex is the sum of finitely many pro-discrete R-modules, because M and N are bounded below, so $\mathbf{Hom}_\Lambda(M,N)$ is in $PD(R)$.

Suppose Θ, Φ are profinite R-algebras. Then let $PD(\Theta - \Phi)$ be the category of pro-discrete $\Theta - \Phi$-bimodules and continuous $\Theta - \Phi$-homomorphisms. If M is an ind-profinite $\Lambda - \Theta$-bimodule and N is a pro-discrete $\Lambda - \Phi$-bimodule, one can make $\mathbf{Hom}_\Lambda(M,N)$ into a pro-discrete $\Theta - \Phi$-bimodule in the same way as in the abstract case. We leave the details to the reader.

2.3.3 Pontryagin Duality

Lemma 2.3.30. *Suppose that I is a discrete Λ-module which is injective in $PD(\Lambda)$. Then $\mathbf{Hom}_\Lambda(-,I)$ sends short strict exact sequences of ind-profinite Λ-modules to short strict exact sequences of pro-discrete R-modules.*

Proof. Proposition 2.3.10 shows that we can write short strict exact sequences of ind-profinite Λ-modules as injective direct limits of short exact sequences of profinite modules in $IP(\Lambda)$, and then Proposition 2.2.20 shows that applying $\mathbf{Hom}_\Lambda(-, I)$ gives a surjective inverse system of short exact sequences of discrete R-modules; the inverse limit of these is strict exact by Remark 2.3.26. $\square$

In particular this applies when $I = \mathbb{Q}/\mathbb{Z}$, with the discrete topology, as a $\hat{\mathbb{Z}}$-module.

Consider $\mathbb{Q}/\mathbb{Z}$ with the discrete topology as an ind-profinite abelian group. Given $M \in IP(\Lambda)$, with a cofinal sequence $\{M_i\}$, we can think of M as an ind-profinite abelian group by forgetting the Λ-action; then $\{M_i\}$ becomes a cofinal sequence of profinite abelian groups for M. Now apply $\mathbf{Hom}_{\hat{\mathbb{Z}}}(-, \mathbb{Q}/\mathbb{Z})$ to get a pro-discrete abelian group. We can endow each $\mathbf{Hom}_{\hat{\mathbb{Z}}}(M_i, \mathbb{Q}/\mathbb{Z})$ with the structure of a right Λ-module, such that the Λ-action is continuous, by Corollary 2.2.12. Taking inverse limits, we can therefore make $\mathbf{Hom}_{\hat{\mathbb{Z}}}(M, \mathbb{Q}/\mathbb{Z})$ into a pro-discrete right Λ-module, which we denote by M^*. As before, * gives a contravariant functor $IP(\Lambda) \to PD(\Lambda^{op})$. Lemma 2.3.30 now has the following immediate consequence.

Corollary 2.3.31. *The functor* $^* : IP(\Lambda) \to PD(\Lambda^{op})$ *maps short strict exact sequences to short strict exact sequences.*

Suppose instead that $M \in PD(\Lambda)$, with a cofinal sequence $\{M^i\}$. As before, we can think of M as a pro-discrete abelian group by forgetting the Λ-action, and then $\{M^i\}$ is a cofinal sequence of discrete abelian groups. Recall that, as $U(\hat{\mathbb{Z}})$-modules,

$$\mathrm{Hom}_{PD(\hat{\mathbb{Z}})}(M, \mathbb{Q}/\mathbb{Z}) \cong \varinjlim_i \mathrm{Hom}_{PD(\hat{\mathbb{Z}})}(M^i, \mathbb{Q}/\mathbb{Z}).$$

We can endow each $\mathrm{Hom}_{PD(\hat{\mathbb{Z}})}(M^i, \mathbb{Q}/\mathbb{Z})$ with the structure of a profinite right Λ-module, by Corollary 2.2.12. Taking direct limits, we can therefore make $\mathrm{Hom}_{PD(\hat{\mathbb{Z}})}(M, \mathbb{Q}/\mathbb{Z})$ into an ind-profinite right Λ-module, which we denote by M_*, and in the same way as before $_*$ gives a functor $PD(\Lambda) \to IP(\Lambda^{op})$.

Note that $_*$ also maps short strict exact sequences to short strict exact sequences, by Lemma 2.3.28 and Proposition 2.3.10. Note too that both * and $_*$ send profinite modules to discrete modules and vice versa; on such modules they give the same result as the usual Pontryagin duality functor of Section 2.2.1.

Theorem 2.3.32 (Pontryagin duality)**.** *The composite functors* $IP(\Lambda) \xrightarrow{-^*} PD(\Lambda^{op}) \xrightarrow{-_*} IP(\Lambda)$ *and* $PD(\Lambda) \xrightarrow{-_*} IP(\Lambda^{op}) \xrightarrow{-^*} PD(\Lambda)$ *are naturally isomorphic to the identity, so that* $IP(\Lambda)$ *and* $PD(\Lambda)$ *are dually equivalent.*

Proof. We give a proof for $^* \circ {}_*$; the proof for $_* \circ {}^*$ is similar. Given $M \in IP(\Lambda)$ with a cofinal sequence M_i, by construction $(M^*)_*$ has cofinal sequence $(M_i^*)_*$. By [23, p.165], the functors * and $_*$ give a dual equivalence between the categories of profinite and discrete Λ-modules, so we have natural isomorphisms $M_i \to (M_i^*)_*$ for each i, and the result follows. $\square$

From now on, by abuse of notation, we will follow convention by writing * for both the functors * and $_*$.

Corollary 2.3.33. *Pontryagin duality preserves the canonical decomposition of morphisms. More precisely, given a morphism $f : M \to N$ in $IP(\Lambda)$, $\operatorname{im}(f)^* = \operatorname{coim}(f^*)$ and $\operatorname{im}(f^*) = \operatorname{coim}(f)^*$. In particular, f^* is strict if and only if f is. Similarly for morphisms in $PD(\Lambda)$.*

Proof. This follows from Pontryagin duality and the duality between the definitions of im and coim. For the final observation, note that, by Corollary 2.3.11 and Corollary 2.3.25,

$$\begin{aligned} f^* \text{ is strict} &\Leftrightarrow \operatorname{im}(f^*) = \operatorname{coim}(f^*) \\ &\Leftrightarrow \operatorname{im}(f) = \operatorname{coim}(f) \\ &\Leftrightarrow f \text{ is strict.} \end{aligned}$$

□

Corollary 2.3.34. *(i) $PD(\Lambda)$ has countable limits.*

(ii) Direct products in $PD(\Lambda)$ preserve kernels and cokernels, and hence strict maps.

(iii) $PD(\Lambda)$ has enough injectives*: for every $M \in PD(\Lambda)$ there is an injective I and a strict monomorphism $M \to I$. A discrete Λ-module I is injective in $PD(\Lambda)$ if and only if it is injective in the category of discrete Λ-modules.*

(iv) Every injective in $PD(\Lambda)$ is a summand of a strictly cofree *one, i.e. one whose Pontryagin dual is strictly free.*

(v) Countable products of strict exact sequences in $PD(\Lambda)$ are strict exact.

(vi) Suppose that P is a profinite Λ-module which is projective in $IP(\Lambda)$. Then $\mathbf{Hom}_\Lambda(P, -)$ sends strict exact sequences of pro-discrete Λ-modules to strict exact sequences of pro-discrete R-modules.

Example 2.3.35. It is easy to check that $\hat{\mathbb{Z}}^* = \mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}_p^* = \mathbb{Q}_p/\mathbb{Z}_p$. Then

$$\mathbb{Q}_p^* = (\varinjlim(\mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p \xrightarrow{\cdot p} \cdots))^* = \varprojlim(\cdots \xrightarrow{\cdot p} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Q}_p.$$

The topology defined on $M^* = \mathbf{Hom}_{\hat{\mathbb{Z}}}(M, \mathbb{Q}/\mathbb{Z})$ when M is an ind-profinite Λ-module coincides with the compact-open topology, because the (discrete) topology on each $\mathbf{Hom}_{\hat{\mathbb{Z}}}(M_i, \mathbb{Q}/\mathbb{Z})$ is the compact-open topology and every compact subspace of M is contained in some M_i by Proposition 2.3.1. Similarly, for a pro-discrete Λ-module N, every compact subspace of N is contained in some profinite submodule L by Remark 2.3.20(ii), so the compact-open topology on $\operatorname{Hom}_{PD(\hat{\mathbb{Z}})}(N, \mathbb{Q}/\mathbb{Z})$ coincides with the limit topology defined on $\varprojlim_{T(\Lambda)} \operatorname{Hom}_{PD(\hat{\mathbb{Z}})}(L, \mathbb{Q}/\mathbb{Z})$, where the limit is taken over all profinite submodules L of N and each $\operatorname{Hom}_{PD(\hat{\mathbb{Z}})}(L, \mathbb{Q}/\mathbb{Z})$ is given the (discrete) compact-open topology.

Proposition 2.3.36. *The compact-open topology on $\operatorname{Hom}_{PD(\hat{\mathbb{Z}})}(N, \mathbb{Q}/\mathbb{Z})$ coincides with the topology defined on N^*.*

Proof. By the preceding remarks, $\mathrm{Hom}_{PD(\hat{\mathbb{Z}})}(N, \mathbb{Q}/\mathbb{Z})$ with the compact-open topology is just $\varprojlim_{\text{profinite } L \leq N} L^*$. So the canonical map $N^* \to \varprojlim L^*$ is a continuous bijection; we need to check it is open. By Lemma 2.3.7, it suffices to check this for open submodules K of N^*. Because K is open, N^*/K is discrete, so $(N^*/K)^*$ is a profinite submodule of N. Therefore there is a canonical continuous map $\varprojlim L^* \to (N^*/K)^{**} = N^*/K$, whose kernel is open because N^*/K is discrete. This kernel is K, and the result follows. □

As topological groups, ind-profinite Λ-modules have the structure of uniform spaces (as they are abelian, the left and right uniformities coincide).

Corollary 2.3.37. *The uniformity on ind-profinite* Λ*-modules is complete, Hausdorff and totally disconnected.*

Proof. By Lemma 2.3.7 we just need to show the uniformity is complete. Proposition 2.3.36 shows that ind-profinite Λ-modules are the inverse limit of their discrete quotients, and hence that the uniformity on such modules is complete, by the corollary to [3, II, Section 3.5, Proposition 10]. □

Moreover, given ind-profinite Λ-modules M, N, the product $M \times_k N$ is the inverse limit of discrete modules $M' \times_k N'$, where M' and N' are discrete quotients of M and N respectively. But $M' \times_k N' = M' \times N'$, because both are discrete, so

$$M \times_k N = \varprojlim M' \times N' = M \times N,$$

the product in the category of topological modules.

Proposition 2.3.38. *Suppose that* $P \in IP(\Lambda)$ *is projective. Then* $\mathbf{Hom}_\Lambda(P, -)$ *sends strict exact sequences in* $PD(\Lambda)$ *to strict exact sequences in* $PD(R)$.

Proof. For P profinite this is Corollary 2.3.34(vi). For P strictly free, $P = \bigoplus P_i$, we get $\mathbf{Hom}_\Lambda(P, -) = \prod \mathbf{Hom}_\Lambda(P_i, -)$, which sends strict exact sequences to strict exact sequences because $\prod$ and $\mathbf{Hom}_\Lambda(P_i, -)$ do. Now the result follows from Remark 2.3.18. □

Lemma 2.3.39. $\mathbf{Hom}_\Lambda(M, N) = \mathbf{Hom}_{\Lambda^{op}}(N^*, M^*)$ *for all* $M \in IP(\Lambda), N \in PD(\Lambda)$*, naturally in both variables.*

Proof. The functor $-^*$ induces maps of abstract groups

$$\begin{aligned}\mathrm{Hom}_\Lambda(M, N) &\xrightarrow{f_1} \mathrm{Hom}_\Lambda(N^*, M^*)\\ &\xrightarrow{f_2} \mathrm{Hom}_\Lambda(N^{**}, M^{**})\\ &\xrightarrow{f_3} \mathrm{Hom}_\Lambda(N^{***}, M^{***})\end{aligned}$$

such that the compositions $f_2 f_1$ and $f_3 f_2$ are isomorphisms, so f_2 is an isomorphism. In particular, this holds when M is profinite and N is discrete, in which case the topology on $\mathbf{Hom}_\Lambda(M, N)$ is discrete; so, taking cofinal sequences $\{M_i\}$ for M and $\{N^j\}$ for N, we get $\mathbf{Hom}_\Lambda(M_i, N^j) = \mathbf{Hom}_{\Lambda^{op}}(N^{j*}, M_i^*)$ as topological modules for each i, j, and the topologies on $\mathbf{Hom}_\Lambda(M, N)$ and $\mathbf{Hom}_{\Lambda^{op}}(N^*, M^*)$ are given by the inverse limits of these. Naturality is clear. □

Corollary 2.3.40. *Suppose that* $I \in PD(\Lambda)$ *is injective. Then* $\mathbf{Hom}_\Lambda(-, I)$ *sends strict exact sequences in* $IP(\Lambda)$ *to strict exact sequences in* $PD(R)$.

Proposition 2.3.41 (Baer's Lemma)**.** *Suppose $I \in PD(\Lambda)$ is discrete. Then I is injective in $PD(\Lambda)$ if and only if, for every closed left ideal J of Λ, every map $J \to I$ extends to a map $\Lambda \to I$.*

Proof. Think of Λ and J as objects of $PD(\Lambda)$. The condition is clearly necessary. To see it is sufficient, suppose we are given a strict monomorphism $f : M \to N$ in $PD(\Lambda)$ and a map $g : M \to I$. Because I is discrete, $\ker(g)$ is open in M. Because f is strict, we can therefore pick an open submodule U of N such that $\ker(g) = M \cap U$. So the problem reduces to the discrete case: it is enough to show that $M/\ker(g) \to I$ extends to a map $N/U \to I$. In this case, the proof for abstract modules, [32, Baer's Criterion 2.3.1], goes through unchanged. □

Therefore a discrete $\hat{\mathbb{Z}}$-module which is injective in $PD(\hat{\mathbb{Z}})$ is divisible. On the other hand, the discrete $\hat{\mathbb{Z}}$-modules are just the torsion abelian groups with the discrete topology. So, by the version of Baer's Lemma for abstract modules ([32, Baer's Criterion 2.3.1]), divisible discrete $\hat{\mathbb{Z}}$-modules are injective in the category of discrete $\hat{\mathbb{Z}}$-modules, and hence injective in $PD(\hat{\mathbb{Z}})$ too by Corollary 2.3.34(iii). So duality gives:

Corollary 2.3.42. *(i) A discrete $\hat{\mathbb{Z}}$-module is injective in $PD(\hat{\mathbb{Z}})$ if and only if it is divisible.*

(ii) A profinite $\hat{\mathbb{Z}}$-module is projective in $IP(\hat{\mathbb{Z}})$ if and only if it is torsion-free.

Proof. Being divisible and being torsion-free are Pontryagin dual by [23, Theorem 2.9.12]. □

Remark 2.3.43. On the other hand, $\mathbb{Q}_p$ is not injective in $PD(\hat{\mathbb{Z}})$ (and hence not projective in $IP(\hat{\mathbb{Z}})$ either), despite being divisible (respectively, torsion-free). Indeed, consider the monomorphism

$$f : \mathbb{Q}_p \to \prod_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p, x \mapsto (x, x/p, x/p^2, \ldots),$$

which is strict because its dual

$$f^* : \bigoplus_{\mathbb{N}} \mathbb{Z}_p \to \mathbb{Q}_p, (x_0, x_1, \ldots) \mapsto \sum_n x_n/p^n$$

is surjective and hence strict by Proposition 2.3.10. Suppose $\mathbb{Q}_p$ is injective, so that f splits; the map g splitting it must send the torsion elements of $\prod_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p$ to 0 because $\mathbb{Q}_p$ is torsion-free. But the torsion elements contain $\bigoplus_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p$, so they are dense in $\prod_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p$ and hence $g = 0$, giving a contradiction.

Finally, we note that in fact $IP(\Lambda)$ and $PD(\Lambda)$ are *quasi-abelian* categories. Both categories satisfy axiom (QA) because forgetting the topology preserves pullbacks in both, and $Mod(U(\Lambda))$ satisfies (QA). Then both categories satisfy axiom (QA*) by duality, and we have:

Proposition 2.3.44. *$IP(\Lambda)$ and $PD(\Lambda)$ are quasi-abelian categories.*

Moreover, a morphism f in a quasi-abelian category is called *strict* if the canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism, which agrees with our use of the term in $IP(\Lambda)$ and $PD(\Lambda)$ by Corollary 2.3.11 and Corollary 2.3.25.

2.3.4 Tensor Products

As in the profinite case, we can define tensor products of ind-profinite modules. Suppose $L \in IP(\Lambda^{op}), M \in IP(\Lambda), N \in IP(R)$. Then $T \in IP(R)$, together with a continuous bilinear map $\theta : L \times_k M \to T$, is the tensor product of L and M if, for every $N \in IP(R)$ and every continuous bilinear map $b : L \times_k M \to N$, there is a unique morphism $f : T \to N$ in $IP(R)$ such that $b = f\theta$.

If such a T exists, it is clearly unique up to isomorphism, and then we write $L\hat{\otimes}_\Lambda M$ for the tensor product. To show the existence of $L\hat{\otimes}_\Lambda M$, we construct it directly: b defines a morphism $b' : F(L \times_k M) \to N$ in $IP(R)$, where $F(L \times_k M)$ is the free ind-profinite R-module on $L \times_k M$. From the bilinearity of b, we get that the R-submodule K of $F(L \times_k M)$ generated by the elements

$$(l_1 + l_2, m) - (l_1, m) - (l_2, m), (l, m_1 + m_2) - (l, m_1) - (l, m_2), (l\lambda, m) - (l, \lambda m)$$

for all $l, l_1, l_2 \in L, m, m_1, m_2 \in M, \lambda \in \Lambda$ is mapped to 0 by b'. From the continuity of b' we get that its closure $\bar{K}$ is mapped to 0 too. Thus b' induces a morphism $b'' : F(L \times_k M)/\bar{K} \to N$. Then it is not hard to check that $F(L \times_k M)/\bar{K}$, together with b'', satisfies the universal property of the tensor product.

Proposition 2.3.45. *(i)* $-\hat{\otimes}_\Lambda-$ *is an additive bifunctor* $IP(\Lambda^{op}) \times IP(\Lambda) \to IP(R)$.

(ii) There is an isomorphism $\Lambda\hat{\otimes}_\Lambda M = M$ *for all* $M \in IP(\Lambda)$*, natural in* M*, and similarly* $L\hat{\otimes}_\Lambda \Lambda = L$ *naturally.*

(iii) $L\hat{\otimes}_\Lambda M = M\hat{\otimes}_{\Lambda^{op}} L$*, naturally in* L *and* M*.*

(iv) Given L *in* $IP(\Lambda^{op})$ *and* M *in* $IP(\Lambda)$*, with cofinal sequences* $\{L_i\}$ *and* $\{M_j\}$*, there is an isomorphism*

$$L\hat{\otimes}_\Lambda M \cong \varinjlim_{IP(R)} (L_i\hat{\otimes}_\Lambda M_j).$$

Proof. (i) and (ii) follow from the universal property.

(iii) Writing $*$ for the Λ^{op}-actions, a bilinear map $b_\Lambda : L \times M \to N$ (satisfying $b_\Lambda(l\lambda, m) = b_\Lambda(l, \lambda m)$) is the same thing as a bilinear map $b_{\Lambda^{op}} : M \times L \to N$ (satisfying $b_{\Lambda^{op}}(m, \lambda * l) = b_{\Lambda^{op}}(m * \lambda, l)$).

(iv) We have $L \times_k M = \varinjlim L_i \times M_j$ by Lemma 2.3.2. By the universal property of the tensor product, the bilinear map

$$\varinjlim L_i \times M_j \to L \times_k M \to L\hat{\otimes}_\Lambda M$$

factors through $f : \varinjlim L_i\hat{\otimes}_\Lambda M_j \to L\hat{\otimes}_\Lambda M$, and similarly the bilinear map

$$L \times_k M \to \varinjlim L_i \times M_j \to \varinjlim L_i\hat{\otimes}_\Lambda M_j$$

factors through $g : L\hat{\otimes}_\Lambda M \to \varinjlim L_i\hat{\otimes}_\Lambda M_j$. By uniqueness, the compositions fg and gf are both identity maps, so the two sides are isomorphic.

□

More generally, given chain complexes

$$\cdots \xrightarrow{d_1} L_1 \xrightarrow{d_0} L_0 \xrightarrow{d_{-1}} \cdots$$

in $IP(\Lambda^{op})$ and

$$\cdots \xrightarrow{d'_1} M_1 \xrightarrow{d'_0} M_0 \xrightarrow{d'_{-1}} \cdots$$

in $IP(\Lambda)$, both bounded below, define the double chain complex $\{L_p\hat{\otimes}_\Lambda M_q\}$ with the obvious vertical maps, and with the horizontal maps defined in the obvious way except that they are multiplied by -1 whenever q is odd: this makes $\mathrm{Tot}(L_p\hat{\otimes}_\Lambda M_q)$ into a chain complex which we denote by $L\hat{\otimes}_\Lambda M$. Each term in the total complex is the sum of finitely many ind-profinite R-modules, because M and N are bounded below, so $L\hat{\otimes}_\Lambda M$ is a complex in $IP(R)$.

Suppose from now on that Θ, Φ, Ψ are profinite R-algebras. Then let $IP(\Theta-\Phi)$ be the category of ind-profinite $\Theta-\Phi$-bimodules and $\Theta-\Phi$-k-bimodule homomorphisms. We leave the details to the reader, after noting that an ind-profinite R-module N, with a left Θ-action and a right Φ-action which are continuous on profinite submodules, is an ind-profinite $\Theta-\Phi$-bimodule since we can replace a cofinal sequence $\{N_i\}$ of profinite R-modules with a cofinal sequence $\{\Theta\cdot N_i\cdot\Phi\}$ of profinite $\Theta-\Phi$-bimodules. If L is an ind-profinite $\Theta-\Lambda$-bimodule and M is an ind-profinite $\Lambda-\Phi$-bimodule, one can make $L\hat{\otimes}_\Lambda M$ into an ind-profinite $\Theta-\Phi$-bimodule in the same way as in the abstract case.

Theorem 2.3.46 (Adjunction isomorphism)**.** *Suppose* $L \in IP(\Theta-\Lambda), M \in IP(\Lambda-\Phi), N \in PD(\Theta-\Psi)$*. Then there is an isomorphism*

$$\mathbf{Hom}_\Theta(L\hat{\otimes}_\Lambda M, N) \cong \mathbf{Hom}_\Lambda(M, \mathbf{Hom}_\Theta(L, N))$$

in $PD(\Phi-\Psi)$*, natural in* L, M, N*.*

Proof. Given cofinal sequences $\{L_i\}, \{M_j\}, \{N^k\}$ in L, M, N respectively, we have natural isomorphisms

$$\mathbf{Hom}_\Theta(L_i\hat{\otimes}_\Lambda M_j, N_k) \cong \mathbf{Hom}_\Lambda(M_j, \mathbf{Hom}_\Theta(L_i, N_k))$$

of discrete $\Phi-\Psi$-bimodules for each i, j, k by Theorem 2.2.18. Then by Lemma 2.3.29 we have

$$\begin{aligned}\mathbf{Hom}_\Theta(L\hat{\otimes}_\Lambda M, N) &\cong \varprojlim_{PD(\Phi-\Psi)} \mathbf{Hom}_\Theta(L_i\hat{\otimes}_\Lambda M_j, N_k)\\ &\cong \varprojlim_{PD(\Phi-\Psi)} \mathbf{Hom}_\Lambda(M_j, \mathbf{Hom}_\Theta(L_i, N_k))\\ &\cong \mathbf{Hom}_\Lambda(M, \mathbf{Hom}_\Theta(L, N)).\end{aligned}$$

□

It follows that $\mathbf{Hom}_\Lambda$ (considered as a co-/covariant bifunctor $IP(\Lambda)^{op} \times PD(\Lambda) \to PD(R)$) commutes with limits in both variables, and that $\hat{\otimes}_\Lambda$ commutes with colimits in both variables, by [32, Theorem 2.6.10].

If $L \in IP(\Theta-\Phi)$, Pontryagin duality gives L^* the structure of a pro-discrete $\Phi-\Theta$-bimodule, and similarly with ind-profinite and pro-discrete switched.

Corollary 2.3.47. *There is a natural isomorphism*

$$(L\hat{\otimes}_\Lambda M)^* \cong \mathbf{Hom}_\Lambda(M, L^*)$$

in $PD(\Phi - \Theta)$ *for* $L \in IP(\Theta - \Lambda), M \in IP(\Lambda - \Phi)$.

Proof. Apply the theorem with $\Psi = \hat{\mathbb{Z}}$ and $N = \mathbb{Q}/\mathbb{Z}$. □

Properties proved about $\mathbf{Hom}_\Lambda$ in the past two sections carry over immediately to properties of $\hat{\otimes}_\Lambda$, using this natural isomorphism. Details are left to the reader.

Given a chain complex M in $IP(\Lambda)$ and a cochain complex N in $PD(\Lambda)$, both bounded below, if we apply * to the double complex with (p, q)th term $\mathbf{Hom}_\Lambda(M_p, N^q)$, we get a double complex with (q, p)th term $N^{q*}\hat{\otimes}_\Lambda M_p$ – note that the indices are switched. This changes the sign convention used in forming $\mathbf{Hom}_\Lambda(M, N)$ into the one used in forming $N^*\hat{\otimes}_\Lambda M$, so we have $\mathbf{Hom}_\Lambda(M, N)^* = N^*\hat{\otimes}_\Lambda M$ (because * commutes with finite direct sums).

Chapter 3

Derived Functors for Profinite Rings

3.1 $P(\Lambda)$, $D(\Lambda)$ and $Mod(U(\Lambda))$

We now use the framework of Section 1.1 to define derived functors in our categories of interest. It will be necessary to distinguish notationally between the different possible categories we want to derive in, so we write

$\mathbf{Hom}_\Lambda^{(P,D)}$ for $\mathbf{Hom}_\Lambda : P(\Lambda)^{op} \times D(\Lambda) \to D(R)$,
$\mathbf{Hom}_\Lambda^{(P_0,P)}$ for$\mathbf{Hom}_\Lambda : P(\Lambda)_0^{op} \times P(\Lambda) \to P(R)$,
$\mathrm{Hom}_\Lambda^{(P,P)}$ for $\mathrm{Hom}_\Lambda : P(\Lambda)^{op} \times P(\Lambda) \to Mod(U(R))$, and
$\mathbf{Hom}_\Lambda^{(IP,PD)}$ for $\mathbf{Hom}_\Lambda : IP(\Lambda)^{op} \times PD(\Lambda) \to PD(R)$.

When there is no danger of ambiguity, the subscripts may be omitted. We may also write $\hat{\otimes}_\Lambda^P$ and $\hat{\otimes}_\Lambda^{IP}$ for the profinite and ind-profinite tensor products, respectively.

In this section we will consider the functors involving abelian categories; this material is well-known, and we simply define these functors, giving references that provide more detail. We will consider the functors involving quasi-abelian categories in more depth in the next section.

Recall that $P(\Lambda)$ has enough projectives and $D(\Lambda)$ has enough injectives. The classical approach to profinite cohomology is to define the functors

$$\mathrm{Ext}_\Lambda^{(P,D),n} : P(\Lambda)^{op} \times D(\Lambda) \to D(R)$$

by $\mathrm{Ext}_\Lambda^{(P,D),n}(M,N) = H^n(\mathbf{Hom}_\Lambda^{(P,D)}(P,N))$, where P is a projective resolution of M in $P(\Lambda)$ and cohomology is taken in $D(R)$. By Proposition 1.1.11 and Proposition 2.2.20, $\mathrm{Ext}_\Lambda^{(P,D),n}$ can equivalently be defined by an injective resolution of N, and there are long exact sequences in both variables because H^n is a cohomological functor. Similarly, we can define the functors

$$\mathrm{Tor}_{P,n}^\Lambda : P(\Lambda^{op}) \times P(\Lambda) \to P(R)$$

by $\mathrm{Tor}_{P,n}^\Lambda(M,N) = H_n(P \hat{\otimes}_\Lambda^P N)$, where P is a projective resolution of M in $P(\Lambda)$ and homology is taken in $P(R)$. By the same propositions as before,

$\mathrm{Tor}^{\Lambda}_{P,n}$ can equivalently be defined by a projective resolution of N, and there are long exact sequences in both variables. See [23] or [33] for more details.

We say A is *of type* FP_n *over* Λ, $n \leq \infty$, for $A \in P(\Lambda)$, if it has a projective resolution which is finitely generated for the first n steps, and write $P(\Lambda)_n$ for the full subcategory of $P(\Lambda)$ whose objects are of type FP_n. We also write $P(\Lambda)_\infty = \bigcap_n P(\Lambda)_n$ and say that $A \in P(\Lambda)_\infty$ is of type FP_∞. This is equivalent to having a projective resolution with P_n finitely generated for all n, by [1, Proposition 1.5]. When the choice of Λ is clear, we will just say M is of type FP_n. Then we can define the functors

$$\mathrm{Ext}^{(P_\infty,P),n}_{\Lambda} : P(\Lambda)^{op}_\infty \times P(\Lambda) \to P(R)$$

by $\mathrm{Ext}^{(P_\infty,P),n}_{\Lambda}(M,N) = H^n(\mathbf{Hom}^{(P_0,P)}_{\Lambda}(P,N))$, where P is a projective resolution of M in $P(\Lambda)$ with each P_n finitely generated and cohomology is taken in $P(R)$. [31] gives more detail on this. By [31, Theorem 2.2.4, Remark 3.7.3], the $\mathrm{Ext}^{(P_\infty,P)}_{\Lambda}$ functors have long exact sequences in both variables, although $P(\Lambda)$ does not have enough injectives.

We also define the functors

$$\mathrm{Ext}^{(P,P),n}_{\Lambda} : P(\Lambda)^{op} \times P(\Lambda) \to Mod(U(R))$$

by $\mathrm{Ext}^{(P,P),n}_{\Lambda}(M,N) = H^n(\mathrm{Hom}^{(P,P)}_{\Lambda}(P,N))$, where P is a projective resolution of M in $P(\Lambda)$ and cohomology is taken in $Mod(U(R))$. Note that, by Proposition 1.1.11, the $\mathrm{Ext}^{(P,P)}_{\Lambda}$ functors have long exact sequences in both variables because H^n is a cohomological functor, although $P(\Lambda)$ does not have enough injectives.

Since $\mathbf{Hom}^{(P,D)}_{\Lambda}$, $\mathbf{Hom}^{(P_0,P)}_{\Lambda}$ and $\mathrm{Hom}^{(P,P)}_{\Lambda}$ are all left exact functors to abelian categories, we get that $\mathrm{Ext}^0_{\Lambda}(-,-) = \mathrm{Hom}_{\Lambda}(-,-)$ in each case.

Finally, we can define group (co)homology. For a profinite group G, let

$$\begin{aligned} H^{(P,D),n}_R(G,M) &= \mathrm{Ext}^{(P,D),n}_{R[\![G]\!]}(R,M), \ M \in D(R[\![G]\!]),\\ H^R_{P,n}(G,M) &= \mathrm{Tor}^{R[\![G]\!]}_{P,n}(M,R), \quad M \in P(R[\![G]\!]), \text{ and}\\ H^{(P,P),n}_R(G,M) &= \mathrm{Ext}^{(P,P),n}_{R[\![G]\!]}(R,M), \ M \in P(R[\![G]\!]). \end{aligned}$$

See [23, Chapter 6] for more detail on the first two of these.

If R is of type FP_∞ as a profinite $R[\![G]\!]$-module with trivial G-action, we say G is of type FP_∞. In this case, we also let

$$H^{(P_\infty,P),n}_R(G,M) = \mathrm{Ext}^{(P_\infty,P),n}_{R[\![G]\!]}(R,M),$$

for $M \in P(R[\![G]\!])$; these functors are studied in [31].

3.2 $IP(\Lambda)$ and $PD(\Lambda)$

The dual equivalence between $IP(\Lambda)$ and $PD(\Lambda)$ extends to dual equivalences between $\mathcal{D}^-(IP(\Lambda))$ and $\mathcal{D}^+(PD(\Lambda))$ given by applying the functor * to cochain complexes in these categories, by defining $(A^*)^n = (A^{-n})^*$ for a cochain complex A in $PD(\Lambda)$, and similarly for the maps. We will also identify $\mathcal{D}^-(IP(\Lambda))$ with the category of chain complexes A (localised over the strict quasi-isomorphisms) which are 0 in negative degrees by setting $A_n = A^{-n}$. The Pontryagin duality extends to one between $\mathcal{LH}(IP(\Lambda))$ and $\mathcal{RH}(PD(\Lambda))$. Moreover, writing RH^n and LH^n for the nth cohomological functors $\mathcal{D}(PD(R)) \to \mathcal{RH}(PD(R))$ and $\mathcal{D}(IP(R^{op})) \to \mathcal{LH}(IP(R^{op}))$, respectively, the following is just a restatement of Lemma 1.1.3.

Lemma 3.2.1. *$LH^{-n} \circ {}^* = {}^* \circ RH^n$.*

Let

$$R\mathbf{Hom}_\Lambda^{(IP,PD)}(-,-) : \mathcal{D}^-(IP(\Lambda))^{op} \times \mathcal{D}^+(PD(\Lambda)) \to \mathcal{D}^+(PD(R))$$

be the total right derived functor of

$$\mathbf{Hom}_\Lambda^{(IP,PD)}(-,-) : IP(\Lambda)^{op} \times PD(\Lambda) \to PD(R).$$

By Proposition 1.1.7, this exists because $IP(\Lambda)$ has enough projectives and $PD(\Lambda)$ has enough injectives, and $R\mathbf{Hom}_\Lambda^{(IP,PD)}(M,N) = \mathbf{Hom}_\Lambda^{(IP,PD)}(P,I)$, where P is a projective resolution of M and I is an injective resolution of N. Dually, let

$$-\hat{\otimes}_\Lambda^{IP,L}- : \mathcal{D}^-(IP(\Lambda^{op})) \times \mathcal{D}^-(IP(\Lambda)) \to \mathcal{D}^-(IP(R))$$

be the total left derived functor of

$$-\hat{\otimes}_\Lambda^{IP}- : IP(\Lambda^{op}) \times IP(\Lambda) \to IP(R).$$

Then by Proposition 1.1.7 again $M\hat{\otimes}_\Lambda^{IP,L}N = P\hat{\otimes}_\Lambda^{IP}Q$ where P, Q are projective resolutions of M, N respectively.

We also define

$$\mathrm{Ext}_\Lambda^{(IP,PD),n} = RH^n R\mathbf{Hom}_\Lambda^{(IP,PD)} : \mathcal{D}^-(IP(\Lambda))^{op} \times \mathcal{D}^+(PD(\Lambda)) \to \mathcal{RH}(PD(R))$$

and

$$\mathrm{Tor}_{IP,n}^\Lambda = LH^{-n}(-\hat{\otimes}_\Lambda^{IP,L}-) : \mathcal{D}^-(IP(\Lambda^{op})) \times \mathcal{D}^-(IP(\Lambda)) \to \mathcal{LH}(IP(R)).$$

Because LH^n and RH^n are cohomological functors, we get the usual long exact sequences in $\mathcal{LH}(IP(R))$ and $\mathcal{RH}(PD(R))$ coming from strict short exact sequences (in the appropriate category) in either variable, natural in both variables – since these give distinguished triangles in the corresponding derived category.

For the rest of the chapter, for clarity, we will omit the IP and PD subscripts and superscripts on Hom, $\hat{\otimes}$, Ext and Tor functors.

Lemma 3.2.2. *(i) The functors $R\mathbf{Hom}_\Lambda(-,-)$ and $-\hat{\otimes}_\Lambda^L-$ are Pontryagin dual in the sense that, given $M \in \mathcal{D}^-(IP(\Lambda))$ and $N \in \mathcal{D}^+(PD(\Lambda))$, $R\mathbf{Hom}_\Lambda(M,N)^* = N^*\hat{\otimes}_\Lambda^L M$, naturally in M, N.*

(ii) $\mathrm{Ext}_\Lambda^n(M,N)^* = \mathrm{Tor}_n^\Lambda(N^*, M)$.

Proof. (i) Take a projective resolution P of M and an injective resolution I of N, so that by duality I^* is a projective resolution of N^*. Then

$$R\mathbf{Hom}_\Lambda(M,N)^* = \mathbf{Hom}_\Lambda(P,I)^* = I^*\hat{\otimes}_\Lambda P = N^*\hat{\otimes}_\Lambda^L M,$$

naturally by the universal property of derived functors.

(ii)

$$\begin{aligned}
\mathrm{Ext}^n_\Lambda(M,N)^* &= (RH^n R\mathbf{Hom}_\Lambda(M,N))^* \\
&= LH^{-n}(\mathbf{Hom}_\Lambda(M,N)^*) && \text{by Lemma 3.2.1} \\
&= LH^{-n}(N^* \hat{\otimes}^L_\Lambda M) && \text{by (i)} \\
&= \mathrm{Tor}^\Lambda_n(N^*, M).
\end{aligned}$$

□

Proposition 3.2.3. *Suppose* $M \in \mathcal{D}^-(IP(\Lambda))$*,* $N \in \mathcal{D}^+(PD(\Lambda))$*.*

(i) $R\mathbf{Hom}_\Lambda(M,N) = R\mathbf{Hom}_{\Lambda^{op}}(M^*, N^*)$*;*

(ii) $\mathrm{Ext}^n_\Lambda(M,N) = \mathrm{Ext}^n_{\Lambda^{op}}(N^*, M^*)$*;*

(iii) $N^* \hat{\otimes}^L_\Lambda M = M \hat{\otimes}^L_{\Lambda^{op}} N^*$*;*

(iv) $\mathrm{Tor}^\Lambda_n(N^*, M) = \mathrm{Tor}^{\Lambda^{op}}_n(M, N^*)$*;*

naturally in M, N*.*

Proof. (iii) and (iv) follow from (i) and (ii) by Pontryagin duality. To see (i), take a projective resolution P of M and an injective resolution I of N. Then

$$R\mathbf{Hom}_\Lambda(M,N) = \mathbf{Hom}_\Lambda(P,I) = \mathbf{Hom}_{\Lambda^{op}}(I^*, P^*) = R\boldsymbol{Hom}_{\Lambda^{op}}(M^*, N^*),$$

by Lemma 2.3.39. Then (ii) follows by applying LH^{-n}. □

Proposition 3.2.4. $R\mathbf{Hom}_\Lambda$, Ext, $\hat{\otimes}^L_\Lambda$ *and* Tor *can be calculated using a resolution of either variable. That is, given* $M \in \mathcal{D}^-(IP(\Lambda))$ *with a projective resolution* P *and* $N \in \mathcal{D}^+(PD(\Lambda))$ *with an injective resolution* I*,*

$$\begin{aligned}
&R\mathbf{Hom}_\Lambda(M,N) = \mathbf{Hom}_\Lambda(P,N) = \mathbf{Hom}_\Lambda(M,I), \\
&\mathrm{Ext}^n_\Lambda(M,N) = RH^n(\mathbf{Hom}_\Lambda(P,N)) = H^n(\mathbf{Hom}_\Lambda(M,I)), \\
&N^* \hat{\otimes}^L_\Lambda M = N^* \hat{\otimes}_\Lambda P = I^* \hat{\otimes}_\Lambda M \;\; \textit{and} \\
&\mathrm{Tor}^\Lambda_n(N^*, M) = LH^{-n}(N^* \hat{\otimes}_\Lambda P) = LH^{-n}(I^* \hat{\otimes}_\Lambda M).
\end{aligned}$$

Proof. By Proposition 1.1.11, $R\mathbf{Hom}_\Lambda(M,N) = \mathbf{Hom}_\Lambda(M,I)$; everything else follows by some combination of Proposition 3.2.3, taking cohomology and applying Pontryagin duality. □

Example 3.2.5. $\mathbb{Z}_p$ is projective in $IP(\hat{\mathbb{Z}})$ by Corollary 2.3.42. Now consider the sequence

$$0 \to \bigoplus_{\mathbb{N}} \mathbb{Z}_p \xrightarrow{f} \bigoplus_{\mathbb{N}} \mathbb{Z}_p \xrightarrow{g} \mathbb{Q}_p \to 0,$$

where f is given by

$$(x_0, x_1, x_2, \dots) \mapsto (x_0, x_1 - p \cdot x_0, x_2 - p \cdot x_1, \dots)$$

and g is given by $(x_0, x_1, x_2, \dots) \mapsto x_0 + x_1/p + x_2/p^2 + \cdots$. This sequence is exact on the underlying modules, so by Proposition 2.3.10 it is strict exact, and hence it is a projective resolution of $\mathbb{Q}_p$. By applying Pontryagin duality, we also get an injective resolution

$$0 \to \mathbb{Q}_p \to \prod_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p \to \prod_{\mathbb{N}} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

Recall that, by Remark 2.3.43, $\mathbb{Q}_p$ is not projective or injective.

Lemma 3.2.6. *For all $n > 0$ and all $M \in IP(\hat{\mathbb{Z}})$,*

(i) $\mathrm{Ext}^n_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, M^*) = 0$*;*

(ii) $\mathrm{Ext}^n_{\hat{\mathbb{Z}}}(M, \mathbb{Q}_p) = 0$*;*

(iii) $\mathrm{Tor}^{\hat{\mathbb{Z}}}_n(\mathbb{Q}_p, M) = 0$*;*

(iv) $\mathrm{Tor}^{\hat{\mathbb{Z}}}_n(M, \mathbb{Q}_p) = 0$*.*

Proof. By Lemma 3.2.2 and Proposition 3.2.3, it is enough to prove (iii). Since $\mathbb{Q}_p$ has a projective resolution of length 1, the statement is clear for $n > 1$. Now $\mathrm{Tor}^{\hat{\mathbb{Z}}}_1(\mathbb{Q}_p, M) = \ker(f \hat{\otimes}_{\hat{\mathbb{Z}}} M)$, in the notation of the example. Writing M_p for $\mathbb{Z}_p \hat{\otimes}_{\hat{\mathbb{Z}}} M$, $f \hat{\otimes}_{\hat{\mathbb{Z}}} M$ is given by

$$\bigoplus_{\mathbb{N}} M_p \to \bigoplus_{\mathbb{N}} M_p, (x_0, x_1, x_2, \ldots) \mapsto (x_0, x_1 - p \cdot x_0, x_2 - p \cdot x_1, \ldots),$$

because $\hat{\otimes}_{\hat{\mathbb{Z}}}$ commutes with direct sums. But this map is clearly injective, as required. ☐

Remark 3.2.7. By Lemma 3.2.6, $\mathrm{Ext}^0_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, -)$ is an exact functor $\mathcal{RH}(PD(\hat{\mathbb{Z}})) \to \mathcal{RH}(PD(\hat{\mathbb{Z}}))$. In particular, writing $\mathcal{I}$ for the inclusion functor $PD(\hat{\mathbb{Z}}) \to \mathcal{RH}(PD(\hat{\mathbb{Z}}))$, the composite $\mathrm{Ext}^0_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, -) \circ \mathcal{I}$ sends short strict exact sequences in $PD(\hat{\mathbb{Z}})$ to short exact sequences in $\mathcal{RH}(PD(\hat{\mathbb{Z}}))$ by Proposition 1.1.1. On the other hand, by Proposition 1.1.1 again, the composite $\mathcal{I} \circ \mathbf{Hom}_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, -)$ does not send short strict exact sequences in $PD(\hat{\mathbb{Z}})$ to short exact sequences in $\mathcal{RH}(PD(\hat{\mathbb{Z}}))$. Therefore, by [26, Proposition 1.3.10], and in the terminology of [26], $\mathbf{Hom}_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, -)$ is not *RR left exact*: there is some short strict exact sequence

$$0 \to L \to M \to N \to 0$$

in $PD(\hat{\mathbb{Z}})$ such that the induced map $\mathbf{Hom}_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, M) \to \mathbf{Hom}_{\hat{\mathbb{Z}}}(\mathbb{Q}_p, N)$ is not strict. By duality, a similar result holds for tensor products with $\mathbb{Q}_p$.

3.3 (Co)homology for Profinite Groups

We now study the homology of profinite groups with ind-profinite coefficients, and cohomology with pro-discrete coefficients.

We define the category of ind-profinite right G-modules, $IP(G^{op})$, to have as its objects ind-profinite abelian groups M with a continuous map $M \times_k G \to M$, and as its morphisms continuous group homomorphisms which are compatible with the G-action. We define the category of pro-discrete G-modules, $PD(G)$, to have as its objects prodiscrete $\hat{\mathbb{Z}}$-modules M with a continuous map $G \times M \to M$, and as its morphisms continuous group homomorphisms which are compatible with the G-action.

Proposition 3.3.1. *(i) $IP(G^{op})$ and $IP(\hat{\mathbb{Z}}[\![G]\!]^{op})$ are equivalent.*

(ii) An ind-profinite right $R[\![G]\!]$-module is the same as an ind-profinite R-module M with a continuous map $M \times_k G \to M$ such that $(mr)g = (mg)r$ for all $g \in G, r \in R, m \in M$.

(iii) $PD(G)$ and $PD(\hat{\mathbb{Z}}[\![G]\!])$ are equivalent.

(iv) A pro-discrete $R[\![G]\!]$-module is the same as a pro-discrete R-module M with a continuous map $G \times M \to M$ such that $g(rm) = r(gm)$ for all $g \in G, r \in R, m \in M$.

Proof. (i) Given $M \in IP(G^{op})$, take a cofinal sequence $\{M_i\}$ for M as an ind-profinite abelian group. Replacing each M_i with $M_i' = \overline{\langle M_i \cdot G \rangle}$ if necessary, we have a cofinal sequence for M consisting of profinite right G-modules. By [23, Proposition 5.3.6(c)], each M_i' canonically has the structure of a profinite right $\hat{\mathbb{Z}}[\![G]\!]$-module, and with this structure the cofinal sequence $\{M_i'\}$ makes M into an object in $IP(\hat{\mathbb{Z}}[\![G]\!]^{op})$. This gives a functor $IP(G^{op}) \to IP(\hat{\mathbb{Z}}[\![G]\!]^{op})$. Similarly, we get a functor $IP(\hat{\mathbb{Z}}[\![G]\!]^{op}) \to IP(G^{op})$ by taking cofinal sequences and forgetting the $\hat{\mathbb{Z}}$-structure on the profinite elements in the sequence. These functors are clearly inverse to each other.

(ii) Similarly.

(iii) The same proof as (i), after replacing [23, Proposition 5.3.6(c)] with [23, Proposition 5.3.6(e)].

(iv) Similarly.

□

By (ii) of Proposition 3.3.1, given $M \in IP(R)$, we can think of M as an object in $IP(R[\![G]\!]^{op})$ with trivial G-action. This gives a functor, the *trivial module functor*, $IP(R) \to IP(R[\![G]\!]^{op})$, which clearly preserves strict exact sequences.

Given $M \in IP(R[\![G]\!]^{op})$, define the *coinvariant module* M_G by

$$M/\overline{\langle m \cdot g - m, \text{ for all } g \in G, m \in M \rangle}.$$

This makes M_G into an object in $IP(R)$. In the same way as for abstract modules, M_G is the maximal quotient module of M with trivial G-action, and so $-_G$ becomes a functor $IP(R[\![G]\!]^{op}) \to IP(R)$ which is left adjoint to the trivial module functor. $-_G$ is defined similarly for left ind-profinite $R[\![G]\!]$-modules.

By (iv) of Proposition 3.3.1, given $M \in PD(R)$, we can think of M as an object in $PD(R[\![G]\!])$ with trivial G-action. This gives a functor which we also call the trivial module functor, $PD(R) \to PD(R[\![G]\!])$, which clearly preserves strict exact sequences.

Given $M \in PD(R[\![G]\!])$, define the *invariant submodule* M^G by

$$\{m \in M : g \cdot m = m, \text{ for all } g \in G, m \in M\}.$$

It is a closed submodule of M, because

$$M^G = \bigcap_{g \in G} \ker(M \to M, m \mapsto g \cdot m - m).$$

Therefore we can think of M^G as an object in $PD(R)$. In the same way as for abstract modules, M^G is the maximal submodule of M with trivial G-action, and so $-^G$ becomes a functor $PD(R[\![G]\!]) \to PD(R)$ which is right adjoint to the trivial module functor. $-^G$ is defined similarly for right pro-discrete $R[\![G]\!]$-modules.

Lemma 3.3.2. *(i) For $M \in IP(R[\![G]\!]^{op})$, $M_G = M \hat{\otimes}_{R[\![G]\!]} R$.*

(ii) For $M \in PD(R[\![G]\!])$, $M^G = \mathbf{Hom}_{R[\![G]\!]}(R, M)$.

Proof. (i) Let $\{M_i\}$ be a cofinal sequence for M. By [23, Lemma 6.3.3], $(M_i)_G = M_i \hat{\otimes}_{R[\![G]\!]} R$, naturally in M_i. As a left adjoint, $-_G$ commutes with direct limits, so

$$M_G = \varinjlim (M_i)_G = \varinjlim (M_i \hat{\otimes}_{R[\![G]\!]} R) = M \hat{\otimes}_{R[\![G]\!]} R$$

by Proposition 2.3.45.

(ii) Similarly, by [23, Lemma 6.2.1], because $-^G$ and $\mathbf{Hom}_{R[\![G]\!]}(R, -)$ commute with inverse limits.

□

Corollary 3.3.3. *Given $M \in IP(R[\![G]\!]^{op})$, $(M_G)^* = (M^*)^G$.*

Proof. Lemma 3.3.2 and Corollary 2.3.47. □

We now define the nth homology functor of G over R, with ind-profinite coefficients, by

$$H^R_{IP,n}(G, -) = \mathrm{Tor}^{R[\![G]\!]}_n(-, R) : \mathcal{D}^-(IP(R[\![G]\!]^{op})) \to \mathcal{LH}(IP(R))$$

and the nth cohomology functor of G over R, with pro-discrete coefficients, by

$$H^{(IP,PD),n}_R(G, -) = \mathrm{Ext}^n_{R[\![G]\!]}(R, -) : \mathcal{D}^+(PD(R[\![G]\!])) \to \mathcal{RH}(PD(R)).$$

By Lemma 3.2.2 we have $H^R_n(G, M)^* = H^n_R(G, M^*)$ for $M \in \mathcal{D}^-(IP(R[\![G]\!]^{op}))$, naturally in M. When it is clear, we may omit the IP and PD subscripts and superscripts of H^R_n and H^n_R.

Of course, one can calculate all these groups using the projective resolution of R arising from the usual bar resolution, [23, Section 6.2], and this shows that the homology and cohomology groups are the same if we forget the R-module structure and think of M as an object of $\mathcal{D}^-(IP(\hat{\mathbb{Z}}[\![G]\!]^{op}))$; that is, the underlying abelian k-group of $H^R_n(G, M)$ and the underlying topological abelian group of $H^n_R(G, M^*)$ are $H^{\hat{\mathbb{Z}}}_n(G, M)$ and $H^n_{\hat{\mathbb{Z}}}(G, M^*)$, respectively. We therefore write

$$H_n(G, M) = H^{\hat{\mathbb{Z}}}_n(G, M) \text{ and}$$
$$H^n(G, M^*) = H^n_{\hat{\mathbb{Z}}}(G, M^*).$$

Theorem 3.3.4 (Universal Coefficient Theorem)**.** *Suppose $M \in PD(\hat{\mathbb{Z}}[\![G]\!])$ has trivial G-action. Then there are non-canonically split short strict exact sequences*

$$0 \to \mathrm{Ext}^1_{\hat{\mathbb{Z}}}(H_{n-1}(G, \hat{\mathbb{Z}}), M) \to H^n(G, M) \to \mathrm{Ext}^0_{\hat{\mathbb{Z}}}(H_n(G, \hat{\mathbb{Z}}), M) \to 0,$$

$$0 \to \mathrm{Tor}^{\hat{\mathbb{Z}}}_0(M^*, H_n(G, \hat{\mathbb{Z}})) \to H_n(G, M^*) \to \mathrm{Tor}^{\hat{\mathbb{Z}}}_1(M^*, H_{n-1}(G, \hat{\mathbb{Z}})) \to 0.$$

Proof. We prove the first sequence; the second follows by Pontryagin duality. Take a projective resolution P of $\hat{\mathbb{Z}}$ in $IP(\hat{\mathbb{Z}}[\![G]\!])$ with each P_n profinite, so that $H^n(G,M) = RH^n(\mathbf{Hom}_{\hat{\mathbb{Z}}[\![G]\!]}(P,M))$. Because M has trivial G-action, $M = \mathbf{Hom}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}},M)$, where we think of $\hat{\mathbb{Z}}$ as an ind-profinite $\hat{\mathbb{Z}} - \hat{\mathbb{Z}}[\![G]\!]$-bimodule with trivial G-action. So

$$\begin{aligned}\mathbf{Hom}_{\hat{\mathbb{Z}}[\![G]\!]}(P,M) &= \mathbf{Hom}_{\hat{\mathbb{Z}}[\![G]\!]}(P,\mathbf{Hom}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}},M)) \\ &= \mathbf{Hom}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}\hat{\otimes}_{\hat{\mathbb{Z}}[\![G]\!]}P,M) \\ &= \mathbf{Hom}_{\hat{\mathbb{Z}}}(P_G,M).\end{aligned}$$

Note that P_G is a complex of profinite modules, so all the maps involved are automatically strict. Since $-_G$ is left adjoint to an exact functor (the trivial module functor), we get in the same way as for abelian categories that $-_G$ preserves projectives, so each $(P_n)_G$ is projective in $IP(\hat{\mathbb{Z}})$ and hence torsion free by Corollary 2.3.42. Now the profinite subgroups of each $(P_n)_G$ consisting of cycles and boundaries are torsion-free and hence projective in $IP(\hat{\mathbb{Z}})$ by Corollary 2.3.42, so P_G splits. Then the result follows by the same proof as in the abstract case, [32, Section 3.6]. □

Corollary 3.3.5. *For all n,*

(i) $H_n(G,\mathbb{Z}_p) = \mathrm{Tor}_0^{\hat{\mathbb{Z}}}(\mathbb{Z}_p,H_n(G,\hat{\mathbb{Z}})) = \mathbb{Z}_p\hat{\otimes}_{\hat{\mathbb{Z}}}H_n(G,\hat{\mathbb{Z}})$.

(ii) $H_n(G,\mathbb{Q}_p) = \mathrm{Tor}_0^{\hat{\mathbb{Z}}}(\mathbb{Q}_p,H_n(G,\hat{\mathbb{Z}}))$.

(iii) $H^n(G,\mathbb{Q}_p) = \mathrm{Ext}^0_{\hat{\mathbb{Z}}}(H_n(G,\hat{\mathbb{Z}}),\mathbb{Q}_p)$.

Proof. (i) holds because $\mathbb{Z}_p$ is projective; (ii) and (iii) follow from Lemma 3.2.6. □

Suppose now that H is a (profinite) subgroup of G. We can think of $R[\![G]\!]$ as an ind-profinite $R[\![H]\!] - R[\![G]\!]$-bimodule: the left H-action is given by left multiplication by H on G, and the right G-action is given by right multiplication by G on G. We will denote this bimodule by $R[\![{}^{H\searrow}G^{\swarrow G}]\!]$.

If $M \in IP(R[\![G]\!])$, we can restrict the G-action to an H-action. Moreover, maps of G-modules which are compatible with the G-action are compatible with the H-action. So restriction gives a functor

$$\mathrm{Res}^G_H : IP(R[\![G]\!]) \to IP(R[\![H]\!]).$$

Res^G_H can equivalently be defined by the functor $R[\![{}^{H\searrow}G^{\swarrow G}]\!]\hat{\otimes}_{R[\![G]\!]}-$. Similarly, we can define a restriction functor

$$\mathrm{Res}^G_H : PD(R[\![G]\!]^{op}) \to PD(R[\![H]\!]^{op})$$

by $\mathbf{Hom}_{R[\![G]\!]}(R[\![{}^{H\searrow}G^{\swarrow G}]\!],-)$.

On the other hand, given $M \in IP(R[\![H]\!]^{op})$, $M\hat{\otimes}_{R[\![H]\!]}R[\![{}^{H\searrow}G^{\swarrow G}]\!]$ becomes an object in $IP(R[\![G]\!]^{op})$. In this way, $-\hat{\otimes}_{R[\![H]\!]}R[\![{}^{H\searrow}G^{\swarrow G}]\!]$ becomes a functor, induction,

$$\mathrm{Ind}^G_H : IP(R[\![H]\!]^{op}) \to IP(R[\![G]\!]^{op}).$$

Also, $\mathbf{Hom}_{R[\![H]\!]}(R[\![^{H\searrow}G^{\swarrow G}]\!], -)$ becomes a functor, coinduction, which we denote by

$$\mathrm{Coind}_H^G : PD(R[\![H]\!]) \to PD(R[\![G]\!]).$$

Since $R[\![^{H\searrow}G^{\swarrow G}]\!]$ is projective in $IP(R[\![H]\!])$ and $IP(R[\![G]\!])^{op}$, $\mathrm{Res}_H^G, \mathrm{Ind}_H^G$ and Coind_H^G all preserve strict exact sequences. Moreover, Res_H^G and Ind_H^G commute with colimits of ind-profinite modules because tensor products do, and Res_H^G and Coind_H^G commute with limits of pro-discrete modules because Hom does in the second variable.

We can similarly define restriction on right ind-profinite or left pro-discrete $R[\![G]\!]$-modules, induction on left ind-profinite $R[\![G]\!]$-modules and coinduction on right pro-discrete $R[\![G]\!]$-modules, using $R[\![^{G\searrow}G^{\swarrow H}]\!]$. Details are left to the reader.

Suppose an abelian group M has a left H-action together with a topology that makes it into both an ind-profinite H-module and a pro-discrete H-module. For example, this is the case if M is second-countable profinite or countable discrete. Then both Ind_H^G and Coind_H^G are defined. When H is open in G, we get $\mathrm{Ind}_H^G - = \mathrm{Coind}_H^G -$ in the same way as the abstract case, [32, Lemma 6.3.4].

Lemma 3.3.6. *For $M \in IP(R[\![H]\!]^{op})$, $(\mathrm{Ind}_H^G M)^* = \mathrm{Coind}_H^G(M^*)$. For $N \in IP(R[\![G]\!]^{op})$, $(\mathrm{Res}_H^G N)^* = \mathrm{Res}_H^G(N^*)$.*

Proof.

$$\begin{aligned}(\mathrm{Ind}_H^G M)^* &= (M \hat{\otimes}_{R[\![H]\!]} R[\![^{H\searrow}G^{\swarrow G}]\!])^* \\ &= \mathbf{Hom}_{R[\![H]\!]}(R[\![^{H\searrow}G^{\swarrow G}]\!], M^*) = \mathrm{Coind}_H^G(M^*).\end{aligned}$$

$$\begin{aligned}(\mathrm{Res}_H^G N)^* &= (N \hat{\otimes}_{R[\![G]\!]} R[\![^{G\searrow}G^{\swarrow H}]\!])^* \\ &= \mathbf{Hom}_{R[\![H]\!]}(R[\![^{G\searrow}G^{\swarrow H}]\!], N^*) = \mathrm{Res}_H^G(N^*).\end{aligned}$$

□

Lemma 3.3.7. *(i) Ind_H^G is left adjoint to Res_H^G. That is, for $M \in IP(R[\![H]\!])$, $N \in IP(R[\![G]\!])$,*

$$\mathrm{Hom}_{IP(R[\![G]\!])}(\mathrm{Ind}_H^G M, N) = \mathrm{Hom}_{IP(R[\![H]\!])}(M, \mathrm{Res}_H^G N),$$

naturally in M and N.

(ii) Coind_H^G is right adjoint to Res_H^G. That is, for $M \in PD(R[\![G]\!])$, $N \in PD(R[\![H]\!])$,

$$\mathrm{Hom}_{PD(R[\![G]\!])}(M, \mathrm{Coind}_H^G N) = \mathrm{Hom}_{PD(R[\![H]\!])}(\mathrm{Res}_H^G M, N),$$

naturally in M and N.

Proof. (i) and (ii) are equivalent by Pontryagin duality and Lemma 3.3.6. We show (i). Pick cofinal sequences $\{M_i\}, \{N_j\}$ for M, N. Then

$$\begin{aligned}
\mathrm{Hom}_{IP(R[\![G]\!])}(\mathrm{Ind}_H^G M, N) &= \mathrm{Hom}_{IP(R[\![G]\!])}(\varinjlim(\mathrm{Ind}_H^G M_i), \varinjlim N_j) \\
&= \varprojlim_i \varinjlim_j \mathrm{Hom}_{IP(R[\![G]\!])}(\mathrm{Ind}_H^G M_i, N_j) \\
&= \varprojlim_i \varinjlim_j \mathrm{Hom}_{IP(R[\![H]\!])}(M_i, \mathrm{Res}_H^G N_j) \\
&= \mathrm{Hom}_{IP(R[\![H]\!])}(\varinjlim M_i, \varinjlim \mathrm{Res}_H^G N_j) \\
&= \mathrm{Hom}_{IP(R[\![H]\!])}(M, \mathrm{Res}_H^G N)
\end{aligned}$$

by Lemma 2.3.14 and the Pontryagin dual of [23, Lemma 6.10.2], and all the isomorphisms in this sequence are natural. □

Corollary 3.3.8. Ind_H^G *preserves projectives. Dually,* Coind_H^G *preserves injectives.*

Proof. The adjunction of Lemma 3.3.7 shows that, for $P \in IP(R[\![H]\!])$ projective,

$$\mathrm{Hom}_{IP(R[\![G]\!])}(\mathrm{Ind}_H^G P, -) = \mathrm{Hom}_{IP(R[\![H]\!])}(P, \mathrm{Res}_H^G -)$$

sends strict epimorphisms to surjections, as required.

The second statement follows from the first by applying the result for Ind_H^G to $IP(R[\![H]\!]^{op})$, and then using Pontryagin duality. □

Lemma 3.3.9. *For $M \in IP(R[\![H]\!]^{op})$, $N \in IP(R[\![G]\!])$, $\mathrm{Ind}_H^G M \hat{\otimes}_{R[\![G]\!]} N = M \hat{\otimes}_{R[\![H]\!]} \mathrm{Res}_H^G N$ and $\mathbf{Hom}_{R[\![G]\!]}(N, \mathrm{Coind}_H^G(M^*)) = \mathbf{Hom}_{R[\![H]\!]}(\mathrm{Res}_H^G N, M^*)$, naturally in M, N.*

Proof.

$$\mathrm{Ind}_H^G M \hat{\otimes}_{R[\![G]\!]} N = M \hat{\otimes}_{R[\![H]\!]} R[\![{}^{H\searrow}G^{\swarrow G}]\!] \hat{\otimes}_{R[\![G]\!]} N = M \hat{\otimes}_{R[\![H]\!]} \mathrm{Res}_H^G N.$$

The second equation follows by applying Pontryagin duality and Lemma 3.3.6. □

Theorem 3.3.10 (Shapiro's Lemma)**.** *For $M \in IP(R[\![H]\!]^{op})$, $N \in IP(R[\![G]\!])$, we have for all n:*

(i) $\mathrm{Tor}_n^{R[\![G]\!]}(\mathrm{Ind}_H^G M, N) = \mathrm{Tor}_n^{R[\![H]\!]}(M, \mathrm{Res}_H^G N)$;

(ii) $\mathrm{Tor}_n^{R[\![G]\!]^{op}}(N, \mathrm{Ind}_H^G M) = \mathrm{Tor}_n^{R[\![H]\!]^{op}}(\mathrm{Res}_H^G N, M)$;

(iii) $\mathrm{Ext}^n_{R[\![G]\!]^{op}}(\mathrm{Ind}_H^G M, N^*) = \mathrm{Ext}^n_{R[\![H]\!]^{op}}(M, \mathrm{Res}_H^G N^*)$;

(iv) $\mathrm{Ext}^n_{R[\![G]\!]}(N, \mathrm{Coind}_H^G M^*) = \mathrm{Ext}^n_{R[\![H]\!]}(\mathrm{Res}_H^G N, M^*)$;

naturally in M, N.

Proof. We show (i); (ii)-(iv) follow by Lemma 3.2.2 and Proposition 3.2.3. Take a projective resolution P of M. By Corollary 3.3.8, $\operatorname{Ind}_H^G P$ is a projective resolution of $\operatorname{Ind}_H^G M$. Then

$$\begin{aligned}\operatorname{Tor}_n^{R[\![G]\!]}(\operatorname{Ind}_H^G M, N) &= LH^{-n}(\operatorname{Ind}_H^G P \hat{\otimes}_{R[\![G]\!]} N) \\ &= LH^{-n}(P \hat{\otimes}_{R[\![H]\!]} \operatorname{Res}_H^G N) \text{ by Lemma 3.3.9} \\ &= \operatorname{Tor}_n^{R[\![H]\!]}(M, \operatorname{Res}_H^G N),\end{aligned}$$

and all these isomorphisms are natural. □

Corollary 3.3.11. *For $M \in IP(R[\![H]\!]^{op})$,*

$$H_n^R(G, \operatorname{Ind}_H^G M) = H_n^R(H, M) \text{ and}$$
$$H_R^n(G, \operatorname{Coind}_H^G M^*) = H_R^n(H, M^*)$$

for all n, naturally in M.

Proof. Apply Shapiro's Lemma with $N = R$ with trivial G-action – the restriction of this action to H is also trivial. □

If K is a profinite normal subgroup of G, then for $M \in IP(R[\![G]\!]^{op})$, M_K becomes an ind-profinite right $R[\![G/K]\!]$-module, and for $M \in PD(R[\![G]\!])$, M^K becomes a pro-discrete $R[\![G/K]\!]$-module, as in the abstract case. So we may think of $-_K$ as a functor $IP(R[\![G]\!]^{op}) \to IP(R[\![G/K]\!]^{op})$ and write $H_n^R(K, -)$ for its derived functors $\mathcal{D}^-(IP(R[\![G]\!]^{op})) \to \mathcal{D}^-(IP(R[\![G/K]\!]^{op}))$, and similarly for $-^K$.

Theorem 3.3.12 (Lyndon-Hochschild-Serre Spectral Sequence)**.** *Suppose K is a profinite normal subgroup of G. Then there are bounded spectral sequences*

$$E_{rs}^2 = H_r^R(G/K, H_s^R(K, M)) \Rightarrow H_{r+s}^R(G, M)$$

for all $M \in \mathcal{D}^-(IP(R[\![G]\!]^{op}))$ and

$$E_2^{rs} = H_R^r(G/K, H_R^s(K, M)) \Rightarrow H_R^{r+s}(G, M)$$

for all $M \in \mathcal{D}^+(PD(R[\![G]\!]))$, both naturally in M.

Proof. We prove the first statement; then Pontryagin duality gives the second by Lemma 3.2.2. By the universal properties of $-_K$, $-_{G/K}$ and $-_G$, it is easy to see that $(-_K)_{G/K} = -_G$. Moreover, as for abstract modules, $-_K$ is left adjoint to the forgetful functor $IP(R[\![G/K]\!]^{op}) \to IP(R[\![G]\!]^{op})$, which sends strict exact sequences to strict exact sequences, and hence $-_K$ preserves projectives. So the result is just an application of the Grothendieck Spectral Sequence, Theorem 1.1.10. □

3.4 Comparison of the Cohomology Theories

We now compare our various definitions of Ext functors. The case of the Tor functors can be treated similarly.

Recall from Section 1.1 that the inclusion

$$\mathcal{I}^{op} : PD(\Lambda) \to \mathcal{RH}(PD(\Lambda))$$

has a right adjoint $\mathcal{C}^{op}$. We can give an explicit description of these functors by duality: for $M \in PD(\Lambda)$, $\mathcal{I}^{op}(M) = (0 \to M \to 0)$. Each object in $\mathcal{RH}(PD(\Lambda))$ is isomorphic to a complex

$$M' = (0 \to M^0 \xrightarrow{f} M^1 \to 0)$$

in $PD(\Lambda)$, where M^0 is in degree 0 and f is epic, and $\mathcal{C}^{op}(M') = \ker(f)$. Also the functors

$$RH^n : \mathcal{D}(PD(\Lambda)) \to \mathcal{RH}(PD(\Lambda))$$

are given by

$$RH^n(\cdots \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} \cdots) = (0 \to \operatorname{coker}(d^{n-1}) \to \operatorname{im}(d^n) \to 0),$$

with $\operatorname{coker}(d^{n-1})$ in degree 0.

For $M \in P(\Lambda), N \in D(\Lambda)$, we have $\mathbf{Hom}_\Lambda^{(P,D)}(M,N) = \mathbf{Hom}_\Lambda^{(IP,PD)}(M,N)$. Let P be a projective resolution of M in $P(\Lambda)$ and I an injective resolution of N in $D(\Lambda)$: recall that projectives in $P(\Lambda)$ are projective in $IP(\Lambda)$ and injectives in $D(\Lambda)$ are injective in $PD(\Lambda)$ by Lemma 2.3.28.

Proposition 3.4.1. *Suppose $M \in P(\Lambda), N \in D(\Lambda)$. Then*

$$\mathcal{I}^{op} \operatorname{Ext}_\Lambda^{(P,D),n}(M,N) = \operatorname{Ext}_\Lambda^{(IP,PD),n}(M,N)$$

in $\mathcal{RH}(PD(R))$, naturally in M and N.

Proof. We have

$$\operatorname{Ext}_\Lambda^{(IP,PD),n}(M,N) = RH^n(\mathbf{Hom}_\Lambda^{(IP,PD)}(P,N)).$$

Because each P_n is profinite, $\mathbf{Hom}_\Lambda^{(IP,PD)}(P,N) = \mathbf{Hom}_\Lambda^{(P,D)}(P,N)$ is a cochain complex of discrete R-modules; write d^n for the map

$$\mathbf{Hom}_\Lambda^{(IP,PD)}(P_n,N) \to \mathbf{Hom}_\Lambda^{(IP,PD)}(P_{n+1},N).$$

In the abelian category $D(R)$, applying the Snake Lemma to the diagram

$$\begin{array}{ccccccc} & & \operatorname{im}(d^{n-1}) & \longrightarrow & \mathbf{Hom}_\Lambda^{(P,D)}(P_n,N) & \longrightarrow & \operatorname{coker}(d^{n-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker(d^n) & \longrightarrow & \mathbf{Hom}_\Lambda^{(P,D)}(P_n,N) & \longrightarrow & \operatorname{coim}(d^n) & & \end{array}$$

shows that

$$\begin{aligned} H^n(\mathbf{Hom}_\Lambda^{(P,D)}(P,N)) &= \operatorname{coker}(\operatorname{im}(d^{n-1}) \to \ker(d^n)) \\ &= \ker(\operatorname{coker}(d^{n-1}) \to \operatorname{coim}(d^n)). \end{aligned}$$

Next, using once again that $D(R)$ is abelian, we have

$$\begin{aligned} RH^n(\mathbf{Hom}_\Lambda^{(IP,PD)}(P,N)) &= (0 \to \operatorname{coker}(d^{n-1}) \to \operatorname{im}(d^n) \to 0) \\ &= (0 \to \operatorname{coker}(d^{n-1}) \to \operatorname{coim}(d^n) \to 0), \end{aligned}$$

so it is enough to show that the map of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\operatorname{coker}(d^{n-1}) \to \operatorname{coim}(d^n)) & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{coker}(d^{n-1}) & \longrightarrow & \operatorname{coim}(d^n) & \longrightarrow & 0 \end{array}$$

is a strict quasi-isomorphism, or equivalently that its cone

$$0 \to \ker(\operatorname{coker}(d^{n-1}) \to \operatorname{coim}(d^n)) \to \operatorname{coker}(d^{n-1}) \to \operatorname{coim}(d^n) \to 0$$

is strict exact, which is clear. □

If on the other hand we are given $M, N \in P(\Lambda)$ with M of type FP_∞, assume that N is second-countable, so that $\operatorname{Ext}_\Lambda^{(IP,PD),n}(M,N)$ is defined. Because $P(R)$ is an abelian category, the same proof as Proposition 3.4.1 shows:

Proposition 3.4.2. *Suppose $M \in P(\Lambda_\infty), N \in P(\Lambda)$. Then*

$$\mathcal{I}^{op} \operatorname{Ext}_\Lambda^{(P_\infty,P),n}(M,N) = \operatorname{Ext}_\Lambda^{(IP,PD),n}(M,N)$$

in $\mathcal{RH}(PD(R))$, naturally in M and N.

In general, for $M, N \in P(\Lambda)$, with N second-countable and P a projective resolution of M, the cochain complex $\mathbf{Hom}_\Lambda^{(IP,PD)}(P,N)$ need not be strict. In the notation of Proposition 3.4.1,

$$\operatorname{Ext}_\Lambda^{(IP,PD),n}(M,N) = (0 \to \operatorname{coker}(d^{n-1}) \to \operatorname{coim}(d^n) \to 0),$$

but if d^{n-1} is not strict then the underlying module $U(\operatorname{coker}(d^{n-1}))$ is not equal to the cokernel of the underlying map of abstract modules $\operatorname{coker}(U(d^{n-1}))$. Indeed, the former is

$$U(\mathbf{Hom}_\Lambda^{(IP,PD)}(P_n,N)/\overline{d^{n-1}(\mathbf{Hom}_\Lambda^{(IP,PD)}(P_{n-1},N))})$$

and the latter is

$$U(\mathbf{Hom}_\Lambda^{(IP,PD)}(P_n,N))/U(d^{n-1}(\mathbf{Hom}_\Lambda^{(IP,PD)}(P_{n-1},N))).$$

Now

$$U \circ \mathcal{C}^{op} \circ \operatorname{Ext}_\Lambda^{(IP,PD),n}(M,N) = U(\ker(\operatorname{coker}(d^{n-1}) \to \operatorname{coim}(d^n))),$$

and by the Snake Lemma argument of Proposition 3.4.1

$$\operatorname{Ext}_\Lambda^{(P,P),n}(M,N) = \ker(\operatorname{coker}(U(d^{n-1})) \to \operatorname{coim}(U(d^n))),$$

so in general we cannot expect to recover the $\operatorname{Ext}_\Lambda^{(P,P),n}(-,-)$ functors from the $\operatorname{Ext}_\Lambda^{(IP,PD),n}(-,-)$ functors by applying the adjoint functor $\mathcal{C}^{op}$ and forgetting the topology.

However, we can clearly obtain all these functors from the total derived functor $R\mathbf{Hom}_\Lambda^{(IP,PD)}(-,-)$, that is, from the cochain complex $\mathbf{Hom}_\Lambda(P,N)$ in this case.

Chapter 4

Bieri-Eckmann Criteria for Profinite Groups

4.1 Abstract and Profinite Modules

In this chapter we will get information about profinite modules and groups by comparing the effects of profinite and abstract Hom and tensor product functors on them. Since all the topological modules in this chapter will be profinite, nothing is lost if we write Hom_Λ for $\mathrm{Hom}_\Lambda^{(P,P)}$, and similarly for Ext, etc., and we will do this where the meaning is clear.

Write $\otimes_{U(\Lambda)}$ for the standard tensor product of abstract modules

$$Mod(U(\Lambda^{op})) \times Mod(U(\Lambda)) \to Mod(U(R)).$$

The definition of completed tensor products says that there is a unique canonical continuous bilinear map

$$B \times A \to B \hat{\otimes}_\Lambda A,$$

and continuous bilinear maps are clearly bilinear in the abstract sense, so by the universal property of abstract tensor products

$$U(B \times A) \to U(B \hat{\otimes}_\Lambda A)$$

factors canonically (and uniquely) as

$$U(B) \times U(A) \to U(B) \otimes_{U(\Lambda)} U(A) \to U(B \hat{\otimes}_\Lambda A).$$

The map

$$U(B) \otimes_{U(\Lambda)} U(A) \to U(B \hat{\otimes}_\Lambda A)$$

induces a transformation of functors

$$U(-) \otimes_{U(\Lambda)} \to U(- \hat{\otimes}_\Lambda -)$$

which is natural in both variables, by the universal property of $\otimes_{U(\Lambda)}$.

Lemma 4.1.1. *Suppose* $A \in P(\Lambda), B \in P(\Lambda^{op})$.

(i) If A is finitely generated and projective, the canonical map

$$U(B) \otimes_{U(\Lambda)} U(A) \to U(B \hat{\otimes}_\Lambda A)$$

is an isomorphism.

(ii) If A is finitely generated,

$$U(B) \otimes_{U(\Lambda)} U(A) \to U(B \hat{\otimes}_\Lambda A)$$

is an epimorphism.

(iii) If A is finitely presented,

$$U(B) \otimes_{U(\Lambda)} U(A) \to U(B \hat{\otimes}_\Lambda A)$$

is an isomorphism.

Similar results hold for B.

Proof. (i) First suppose A is finitely generated and free. Then the result follows from [23, Proposition 5.5.3 (b),(c)] that $B\hat{\otimes}_\Lambda \Lambda \cong B$ and $B\hat{\otimes}_\Lambda -$ is additive, so that $U(B\hat{\otimes}_\Lambda \Lambda^n)$ and $U(B) \otimes_{U(\Lambda)} U(\Lambda^n)$ are both isomorphic to $U(B^n)$. Since projectives are summands of frees, the result follows for A finitely generated and projective as well.

(ii) Consider the short exact sequence $0 \to K \to F \to A \to 0$ with F free and finitely generated. We get a commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & U(B) \otimes_{U(\Lambda)} U(F) & \longrightarrow & U(B) \otimes_{U(\Lambda)} U(A) & \longrightarrow & 0 \\
& \downarrow \cong & & \downarrow & & \\
\cdots \longrightarrow & U(B\hat{\otimes}_\Lambda F) & \longrightarrow & U(B\hat{\otimes}_\Lambda A) & \longrightarrow & 0,
\end{array}$$

and the result follows by the Five Lemma.

(iii) Consider the short exact sequence $0 \to K \to F \to A \to 0$ with F free and finitely generated, and K finitely generated. We get a commutative diagram

$$\begin{array}{ccccccc}
\cdots \to & U(B) \otimes_{U(\Lambda)} U(K) & \to & U(B) \otimes_{U(\Lambda)} U(F) & \to & U(B) \otimes_{U(\Lambda)} U(A) & \to 0 \\
& \downarrow & & \downarrow \cong & & \downarrow & \\
\cdots \longrightarrow & U(B\hat{\otimes}_\Lambda K) & \longrightarrow & U(B\hat{\otimes}_\Lambda F) & \longrightarrow & U(B\hat{\otimes}_\Lambda A) & \longrightarrow 0,
\end{array}$$

and the result follows by the Five Lemma.

□

We will write the standard Hom-functor of abstract modules as

$$\hom_{U(\Lambda)}(-,-) : Mod(U(\Lambda))^{op} \times Mod(U(\Lambda)) \to Mod(U(R)).$$

There is a canonical natural transformation

$$\mathrm{Hom}_\Lambda^{(P,P)}(-,-) \to \hom_{U(\Lambda)}(U(-), U(-))$$

with each
$$\mathrm{Hom}_{\Lambda}^{(P,P)}(A,B) \to \mathrm{hom}_{U(\Lambda)}(U(A),U(B))$$
given by the inclusion of the group of continuous homomorphisms into the group of abstract homomorphisms.

Lemma 4.1.2. *Suppose $A, B \in P(\Lambda)$. If A is finitely generated, the canonical map*
$$\mathrm{Hom}_{\Lambda}^{(P,P)}(A,B) \to \mathrm{hom}_{U(\Lambda)}(U(A),U(B))$$
is an isomorphism.

Proof. See [33, Lemma 7.2.2]. □

4.2 Direct Systems of Profinite Modules

Recall from Section 2.2.2 that we can think of direct systems in a category $\mathcal{E}$ as functors $I \to \mathcal{E}$ for certain small categories I. In the terminology of Section 1.2, given a category $\mathcal{E}$ and a small category I, we can define the functor category $\mathcal{E}^I$ as the category of functors $I \to \mathcal{E}$ and natural transformations between them. We can also define, for a functor $F : \mathcal{E} \to \mathcal{F}$, the exponent functor $F^I : \mathcal{E}^I \to \mathcal{F}^I$: given $f \in \mathcal{E}^I$, set $F^I(f)(i) = F(f(i))$, and similarly for morphisms. In this chapter we will be interested in the functor category $P(\Lambda)^I$ when I is a directed poset. It is easy to check that, given directed posets I and J, the poset $I \times J$ defined by $(i,j) \geq (i',j') \Leftrightarrow i \geq i'$ and $j \geq j'$ is directed too.

In particular we will need the functor
$$U^I : P(\Lambda)^I \to Mod(U(\Lambda))^I$$
which forgets the topology on each module in a directed system in $P(\Lambda)^I$. Now U is exact, so U^I is also exact, by Lemma 1.2.4. We will also need the direct limit functor $\varinjlim$ which sends a directed system of (abstract) $U(\Lambda)$-modules to their colimit in the category of $U(\Lambda)$-modules. It is well-known that, for a directed poset I, $\varinjlim$ is an exact additive functor $Mod(U(\Lambda))^I \to Mod(U(\Lambda))$. So we can compose these two exact functors; it follows that their composition
$$\varinjlim U^I : P(\Lambda)^I \to Mod(U(\Lambda)),$$
which forgets the topology on a direct system of modules and then takes its direct limit, is exact.

By Proposition 1.2.6(ii), we have a long exact sequence in each variable of the exponent functors
$$\mathrm{Tor}_*^{\Lambda, I\times J} : P(\Lambda^{op})^I \times P(\Lambda)^J \to P(R)^{I\times J},$$
for any posets I and J. By Proposition 1.2.6(i), we have a long exact sequence in each variable of
$$\mathrm{Ext}_{\Lambda}^{*,I\times J} : P(\Lambda)^I \times P(\Lambda)^J \to Mod(U(R))^{I\times J}.$$
When J consists of a single element we may write $\mathrm{Tor}_*^{\Lambda,I}$ and $\mathrm{Ext}_{\Lambda}^{*,I}$; similarly in the other variable.

We can now start proving some results. For our main result of the section, we need this preliminary lemma, whose proof is an easy adaptation of [28, Lemma 2].

Lemma 4.2.1. *For every profinite module $B \in P(\Lambda^{op})$, there is a direct system $\{B^i\}$ of finitely presented modules in $P(\Lambda^{op})$ with a collection of continuous compatible maps $B^i \to B$ such that the induced map*

$$\varinjlim U(B^i) \to U(B)$$

is an isomorphism.

Proof. Let F be the free profinite right Λ-module with basis B. By the universal property of free modules, the identity map $B \to B$ extends to a canonical continuous homomorphism of profinite modules $F \to B$. Consider the set of all pairs (F_S, V) where S is a finite subset of B, F_S is the free profinite submodule of F generated by S and V is a finitely generated profinite submodule of F such that the composite

$$V \hookrightarrow F_S \to B$$

is the zero map. Define a partial order on this set by

$$(F_S, V) \leq (F_T, W) \Leftrightarrow S \subseteq T \text{ and } V \subseteq W.$$

This is clearly directed, so we get a direct system of finitely presented profinite modules F_S/V with the canonical continuous module homomorphisms between them, and canonical compatible continuous module homomorphisms $F_S/V \to B$. Forgetting the topology by applying U, we get a direct system of abstract modules with a compatible collection of module homomorphisms

$$f_{S,V} : U(F_S/V) \to U(B),$$

and hence a module homomorphism

$$f : \varinjlim U(F_S/V) \to U(B).$$

We claim f is an isomorphism. Given $b \in B$, b is in the image of

$$f_{\{b\},0} : U(F_{\{b\}}) \to U(B),$$

and hence it is in the image of f. So f is surjective. Given x in the kernel of f, take a representative x' of x in one of the $U(F_S/V)$, so $f_{S,V}(x') = 0$, and a representative x'' of x' in $U(F_S)$. Now suppose V is generated by $x_1, \ldots, x_n$. Let V' be the profinite submodule of F_S generated by $x_1, \ldots, x_n, x''$, so that V' is finitely generated. Note that the composite

$$V' \hookrightarrow F_S \to B$$

is the zero map, and that $(F_S, V) \leq (F_S, V')$. Finally, note that the image of x' in F_S/V' is 0, and hence the image of x' in $\varinjlim U(F_S/V)$ is 0, so $x = 0$. So f is injective. □

Note that this result is weaker than saying that B can be written as a direct limit of finitely presented profinite modules; indeed, taking the direct limit in $T(\Lambda^{op})$ of the system of profinite modules described above will not in general give a profinite module. This will be a recurring theme throughout the chapter: that we are required to consider certain direct systems of profinite modules whose direct limits as topological modules have underlying abstract modules that are isomorphic, but may not have the same topology. It is in this way that the following theorem is the profinite analogue of [1, Theorem 1.1.3].

Theorem 4.2.2. *Suppose $A \in P(\Lambda)$. The following are equivalent:*

(i) $A \in P(\Lambda)_n$.

(ii) If I is a directed poset, and $B, C \in P(\Lambda^{op})^I$, with a morphism $f : B \to C$ such that

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism, then the induced maps

$$\begin{aligned}\varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(f) : &\varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(B, A) \\ &\to \varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(C, A)\end{aligned}$$

are isomorphisms for $m < n$ and an epimorphism for $m = n$.

(iii) For all products $\prod \Lambda$ of copies of Λ, (ii) holds when C has as each of its components $\prod \Lambda$, with identity maps between them, for some B with each component finitely presented.

(iv) If I is a directed poset, and $B, C \in P(\Lambda)^I$, with a morphism $f : B \to C$ such that

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism, then the induced maps

$$\begin{aligned}\varinjlim \operatorname{Ext}_\Lambda^{m,I}(f) : &\varinjlim \operatorname{Ext}_\Lambda^{m,I}(A, B) \\ &\to \varinjlim \operatorname{Ext}_\Lambda^{m,I}(A, C)\end{aligned}$$

are isomorphisms for $m < n$ and a monomorphism for $m = n$.

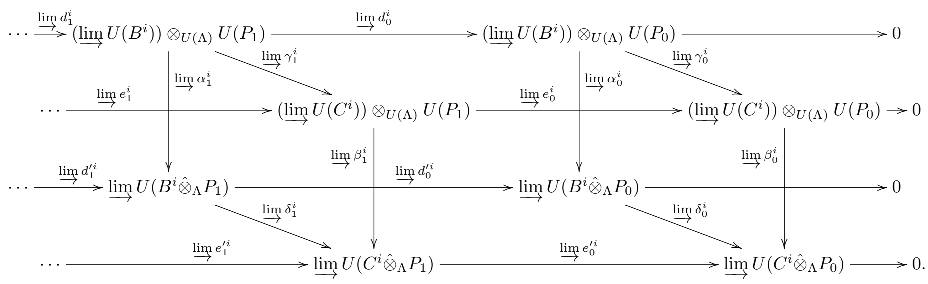
(v) (iv) holds when C has 0 as each of its components, for some B with each component finitely presented.

Proof. (i) $\Rightarrow$ (ii): Take a projective resolution P_* of A with $P_0, \dots P_n$ finitely generated. Then for each $i \in I$ we get a diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{d_1^i} & U(B^i) \otimes_{U(\Lambda)} U(P_1) & \xrightarrow{d_0^i} & U(B^i) \otimes_{U(\Lambda)} U(P_0) & \longrightarrow & 0 \\
\cdots & \xrightarrow{e_1^i} & U(C^i) \otimes_{U(\Lambda)} U(P_1) & \xrightarrow{e_0^i} & U(C^i) \otimes_{U(\Lambda)} U(P_0) & \longrightarrow & 0 \\
\cdots & \xrightarrow{d_1'^i} & U(B^i \hat{\otimes}_\Lambda P_1) & \xrightarrow{d_0'^i} & U(B^i \hat{\otimes}_\Lambda P_0) & \longrightarrow & 0 \\
\cdots & \xrightarrow{e_1'^i} & U(C^i \hat{\otimes}_\Lambda P_1) & \xrightarrow{e_0'^i} & U(C^i \hat{\otimes}_\Lambda P_0) & \longrightarrow & 0
\end{array}$$

with vertical maps $\gamma_k^i : U(B^i) \otimes_{U(\Lambda)} U(P_k) \to U(C^i) \otimes_{U(\Lambda)} U(P_k)$, $\alpha_k^i : U(B^i) \otimes_{U(\Lambda)} U(P_k) \to U(B^i \hat{\otimes}_\Lambda P_k)$, $\beta_k^i : U(C^i) \otimes_{U(\Lambda)} U(P_k) \to U(C^i \hat{\otimes}_\Lambda P_k)$, $\delta_k^i : U(B^i \hat{\otimes}_\Lambda P_k) \to U(C^i \hat{\otimes}_\Lambda P_k)$ ($k = 0, 1$),

where all the squares commute. By Lemma 4.1.1, $\alpha_0^i, \dots, \alpha_n^i, \beta_0^i, \dots, \beta_n^i$ are isomorphisms. Now apply $\varinjlim$. Since $\otimes_{U(\Lambda)}$ commutes with direct limits, we have a commutative diagram



Then as before we have that $\varinjlim \alpha_0^i, \dots, \varinjlim \alpha_n^i, \varinjlim \beta_0^i, \dots, \varinjlim \beta_n^i$ are isomorphisms. By hypothesis $\varinjlim U(B^i) = \varinjlim U(C^i)$, so that $\varinjlim \gamma_0^i, \dots, \varinjlim \gamma_n^i$ are isomorphisms. Hence $\varinjlim \delta_0^i, \dots, \varinjlim \delta_n^i$ are, and the result follows after taking homology.

(ii) $\Rightarrow$ (iii) trivial.

(iii) $\Rightarrow$ (i): Induction on n. First suppose $n = 0$: we want to show $A \in P(\Lambda)_0$. Consider the case where each C^i is a direct product of copies of Λ indexed by X, $\prod_X \Lambda$, for some set X such that there is an injection $\iota : A \to X$. Note that we could just use the set A itself here, but in Lemma 4.2.3 below we will make use of the fact that we only need (iii) to hold for some X with an injection $\iota : A \to X$ to deduce (i), rather than all X, as claimed in the statement of the theorem. Now $B \in P(\Lambda^{op})_1^I$, so by (i) $\Rightarrow$ (ii) of Lemma 4.1.1,

$$\begin{aligned}\varinjlim U^I(B \hat{\otimes}_\Lambda^I A) &= \varinjlim (U^I(B) \otimes_{U(\Lambda)}^I U(A)) \\ &= \varinjlim (U^I(B)) \otimes_{U(\Lambda)} U(A) \\ &= U(\prod_X \Lambda) \otimes_{U(\Lambda)} U(A),\end{aligned}$$

where $\hat{\otimes}_\Lambda^I$ and $\otimes_{U(\Lambda)}^I$ are the exponent functors of $\hat{\otimes}_\Lambda$ and $\otimes_{U(\Lambda)}$, respectively. By hypothesis,

$$\begin{aligned}\varinjlim U^I(f \hat{\otimes}_\Lambda^I -) : \varinjlim U^I(B \hat{\otimes}_\Lambda^I A) &= U(\prod_X \Lambda) \otimes_{U(\Lambda)} U(A) \\ \to \varinjlim U^I(C \hat{\otimes}_\Lambda^I A) &= U(\prod_X \Lambda \hat{\otimes}_\Lambda A) = U(\prod_X A)\end{aligned}$$

is an epimorphism, so there is a

$$c \in U(\prod_X \Lambda) \otimes_{U(\Lambda)} U(A)$$

such that

$$\varinjlim U^I(f \hat{\otimes}_\Lambda^I -)(c)$$

is the 'diagonal' element of $U(\prod_X A)$ whose $\iota(a)$th component is a, for each $a \in A$. Now c has the form

$$\sum_{k=1}^m (\prod_{x \in X} \lambda_k^x) \otimes a_k$$

for some $\lambda_k^x \in \Lambda$ and $a_k \in A$, so

$$\varinjlim U^I(f \hat{\otimes}_\Lambda^I -)(c)$$

has $\iota(a)$th component

$$\sum_{k=1}^{m} \lambda_k^{\iota(a)} a_k = a.$$

So $a_1, \ldots, a_m$ generate A.

For $n > 0$, suppose (iii) $\Rightarrow$ (i) holds for $n-1$. We get A finitely generated as before, and an exact sequence

$$0 \to K \to F \to A \to 0,$$

with F free and finitely generated. Then, using our long exact sequence in the second variable, we get the diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \varinjlim U^I \operatorname{Tor}_n^{\Lambda,I}(B,F) & \longrightarrow & \varinjlim U^I \operatorname{Tor}_n^{\Lambda,I}(B,A) & \longrightarrow & \varinjlim U^I \operatorname{Tor}_{n-1}^{\Lambda,I}(B,K) \\
& \downarrow \cong & & \downarrow & & \downarrow \\
\cdots \longrightarrow & U \operatorname{Tor}_n^{\Lambda}(\prod \Lambda, F) & \longrightarrow & U \operatorname{Tor}_n^{\Lambda}(\prod \Lambda, A) & \longrightarrow & U \operatorname{Tor}_{n-1}^{\Lambda}(\prod \Lambda, K)
\end{array}$$

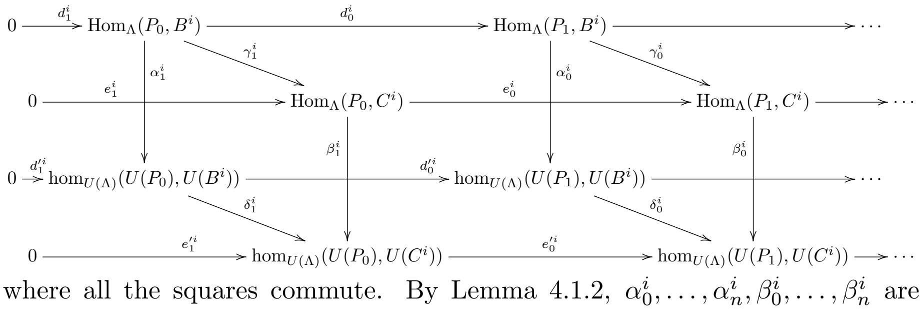
$$\begin{array}{cccccc}
\longrightarrow & \varinjlim U^I \operatorname{Tor}_{n-1}^{\Lambda,I}(B,F) & \longrightarrow & \varinjlim U^I \operatorname{Tor}_{n-1}^{\Lambda,I}(B,A) & \longrightarrow & \cdots \\
& \downarrow \cong & & \downarrow \cong & & \\
\longrightarrow & U \operatorname{Tor}_{n-1}^{\Lambda}(\prod \Lambda, F) & \longrightarrow & U \operatorname{Tor}_{n-1}^{\Lambda}(\prod \Lambda, A) & \longrightarrow & \cdots
\end{array}$$

whose squares commute; it follows by the five lemma that the map

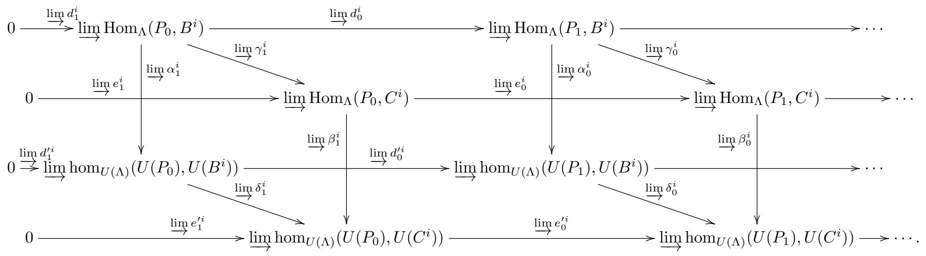
$$\varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(B,K) \to U \operatorname{Tor}_m^{\Lambda}(\prod \Lambda, K)$$

is an isomorphism for $m < n-1$, and an epimorphism for $m = n-1$, for all direct products of copies of Λ. So by hypothesis K is of type FP_{n-1}, so A is of type FP_n.

(i) $\Rightarrow$ (iv): Take a projective resolution P_* of each A with $P_0, \ldots P_n$ finitely generated. Then for each $i \in I$ we get a diagram



where all the squares commute. By Lemma 4.1.2, $\alpha_0^i, \ldots, \alpha_n^i, \beta_0^i, \ldots, \beta_n^i$ are isomorphisms. Now apply $\varinjlim$. Since $\mathrm{hom}_{U(\Lambda)}$ commutes with direct limits in the second argument when the first argument is finitely generated and projective (by [1, Proposition 1.2]), we have a commutative diagram



By hypothesis $\varinjlim U(B^i) = \varinjlim U(C^i)$, so $\varinjlim \alpha_0^i, \dots, \varinjlim \alpha_n^i$, $\varinjlim \beta_0^i, \dots, \varinjlim \beta_n^i$ and $\varinjlim \delta_0^i, \dots, \varinjlim \delta_n^i$ are all isomorphisms. It follows that $\varinjlim \gamma_0^i, \dots, \varinjlim \gamma_n^i$ are, and the result follows after taking cohomology.

(iv) $\Rightarrow$ (v) trivial.

(v) $\Rightarrow$ (i): Induction on n. First suppose $n = 0$: we want to show $A \in P(\Lambda)_0$. Consider the case where B is the direct system $\{A/A'\}$, where A' ranges over the finitely generated submodules of A, with the natural projection maps between them. We claim that $\varinjlim A/A' = 0$. For this, we need to show that for all $x \in A$, there is some A' such that the image of x under the projection $A \xrightarrow{\pi} A/A'$ is 0. So take A' to be the submodule of A generated by x, and we are done. Hence

$$\varinjlim \mathrm{Ext}^0_\Lambda(A, A/A') = \varinjlim \mathrm{Hom}_\Lambda(A, A/A') = 0;$$

in particular, there is some A' for which the projection

$$A \xrightarrow{\pi} A/A'$$

is 0. So $A = A'$ is finitely generated.

For $n > 0$, suppose (v) $\Rightarrow$ (i) holds for $n-1$. We get A finitely generated as before, and an exact sequence

$$0 \to K \to F \to A \to 0,$$

with F free and finitely generated. Then, using our long exact sequence in the first variable, it follows that

$$\varinjlim \mathrm{Ext}^m_\Lambda(K, B^i) = 0$$

for $m \leq n-1$, whenever $\varinjlim B^i = 0$. So by hypothesis K is of type FP_{n-1}, so A is of type FP_n. □

In fact the proof shows slightly more. Given $A \in P(\Lambda)_{n-1}$, $n \geq 0$, pick an exact sequence

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to A \to 0$$

with $P_0, \dots, P_{n-1}$ finitely generated and projective, and let X be a set such that there is an injection $\iota : M \to X$.

Lemma 4.2.3. *Let I be a directed poset, let $C \in P(\Lambda^{op})^I$ have $\prod_X \Lambda$ for all its components with identity maps between them, let $B \in P(\Lambda^{op})^I_1$ such that*

$$\varinjlim U^I(B) \to U(\prod_X \Lambda)$$

is an isomorphism, with $B \to C$ given by the canonical map on each component. Then $A \in P(\Lambda)_n$ if and only if

$$\varinjlim U^I \operatorname{Tor}_{n-1}^{\Lambda,I}(B,A) \to U \operatorname{Tor}_{n-1}^{\Lambda}(\prod_X \Lambda, A)$$

is an isomorphism and

$$\varinjlim U^I \operatorname{Tor}_{n}^{\Lambda,I}(B,A) \to U \operatorname{Tor}_{n}^{\Lambda}(\prod_X \Lambda, A)$$

is an epimorphism.

Corollary 4.2.4. *Suppose $A \in P(\Lambda)$. Then*

$$A \text{ is of type } \mathrm{FP}_1 \Leftrightarrow U(C) \otimes_{U(\Lambda)} U(A) \cong U(C \hat{\otimes}_\Lambda A)$$

for all $C \in P(\Lambda^{op})$.

Proof. $\Rightarrow$: Lemma 4.1.1. $\Leftarrow$: Let C be any product of copies of Λ, $\prod \Lambda$, which is projective, so

$$\operatorname{Tor}_m^{\Lambda}(\prod \Lambda, A) = 0$$

for $m \geq 1$. Hence for any direct system B of modules in $P(\Lambda^{op})$ and any map

$$B \to C = (\prod \Lambda)_{i \in I}$$

such that

$$\varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism,

$$\varinjlim U^I \operatorname{Tor}_m^{\Lambda}(B,A) \to \varinjlim U^I \operatorname{Tor}_m^{\Lambda}(C,A)$$

must be an epimorphism. Then our hypothesis gives that

$$\varinjlim U^I \operatorname{Tor}_0^{\Lambda}(B,A) \to \varinjlim U^I \operatorname{Tor}_0^{\Lambda}(C,A)$$

is an isomorphism, so A is of type FP_1 by (iii) $\Rightarrow$ (i) of the theorem. □

Remark 4.2.5. (a) Ribes-Zalesskii claim in [23, Proposition 5.5.3] that A being finitely generated is enough for

$$U(B) \otimes_{U(\Lambda)} U(A) \to U(B \hat{\otimes}_\Lambda A)$$

to be an isomorphism for all B. (Their notation is slightly different.) If this were the case, then by Corollary 4.2.4 every finitely generated A would be of type FP_1, and hence by an inductive argument would be of type FP_∞ (see Lemma 4.2.9 below). In other words Λ would be noetherian, in the sense of [31], for all profinite Λ. But this isn't true: we will see in Remark 4.3.5(a) that for a group G in certain classes of profinite groups, including prosoluble groups, if G is infinitely generated then $\hat{\mathbb{Z}}$ is of type FP_0 but not FP_1 considered as a $\hat{\mathbb{Z}}[\![G]\!]$-module with trivial G-action, giving a contradiction.

(b) A similar claim to the one in (a) is made by Brumer in [5, Lemma 2.1(ii)], where 'profinite' is replaced by 'pseudocompact'. Since profinite rings and modules are pseudocompact, the argument of (a) shows that Brumer's claim fails too.

Corollary 4.2.6. *If $1 \leq n < \infty$, the following are equivalent for $A \in P(\Lambda)$:*

(i) $A \in P(\Lambda)_n$.

(ii) If I is a directed poset, $B, C \in P(\Lambda^{op})^I$, with a morphism $f : B \to C$ such that

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism, and each component of C is a product of copies of Λ with identity maps between them, then

$$\varinjlim U^I(B \hat{\otimes}_{\Lambda}^{I} A) \to U(\prod \Lambda \hat{\otimes}_{\Lambda} A) = U(\prod A)$$

is an isomorphism and

$$\varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(B, A) = 0$$

for $1 \leq m \leq n-1$.

(iii) $A \in P(\Lambda)_1$ *and*

$$\varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(B, A) = 0$$

for $1 \leq m \leq n-1$.

Proof. Use (i) ⇔ (iii) from Theorem 4.2.2. Then (iii) from the theorem ⇔ (ii) because

$$U \operatorname{Tor}_m^{\Lambda}(\prod \Lambda, A) = 0,$$

for all $m > 0$, and (ii) ⇔ (iii) by Corollary 4.2.4. □

As in Lemma 4.2.3, suppose we have $A \in P(\Lambda)_{n-1}$, $n \geq 0$, pick an exact sequence

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to A \to 0$$

with $P_0, \ldots, P_{n-1}$ finitely generated and projective, and let X be a set such that there is an injection $\iota : M \to X$. Let I be a directed poset, let $C \in P(\Lambda^{op})^I$ have $\prod_X \Lambda$ for all its components with identity maps between them, let $B \in P(\Lambda^{op})_1^I$ such that

$$\varinjlim U^I(B) \to U(\prod_X \Lambda)$$

is an isomorphism, with $B \to C$ given by the canonical map on each component.

Corollary 4.2.7. *Assume in addition that $n \geq 1$. Then $A \in P(\Lambda)_n$ if and only if*

$$\varinjlim U^I \operatorname{Tor}_{n-1}^{\Lambda,I}(B, A) \to U \operatorname{Tor}_{n-1}^{\Lambda}(\prod_X \Lambda, A)$$

is an isomorphism. For $n \geq 2$, $A \in P(\Lambda)_n$ if and only if

$$\varinjlim U^I \operatorname{Tor}_{n-1}^{\Lambda,I}(B, A) = 0.$$

Proof. $U \operatorname{Tor}_n^{\Lambda}(\prod_X \Lambda, A) = 0$, for all $n > 0$. □

Now analogues to other results in [1, Chapter 1.1] follow directly from this.

Corollary 4.2.8. *Suppose $A' \rightarrowtail A \twoheadrightarrow A''$ is an exact sequence in $P(\Lambda)$. Then:*

(i) If $A' \in P(\Lambda)_{n-1}$ and $A \in P(\Lambda)_n$, then $A'' \in P(\Lambda)_n$.

(ii) If $A \in P(\Lambda)_{n-1}$ and $A'' \in P(\Lambda)_n$, then A' is of type FP_{n-1}.

(iii) If A' and A'' are $\in P(\Lambda)_n$ then so is A.

Proof. This follows immediately from the long exact sequences in $\operatorname{Tor}_*^{\Lambda,I}$. □

Lemma 4.2.9. *Let $A \in P(\Lambda)$ be of type FP_n, $n < \infty$, and let*

$$P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

be a partial projective resolution with $P_0, \ldots, P_{n-1}$ finitely generated. Then the kernel $\ker(P_{n-1} \to P_{n-2})$ is finitely generated, so one can extend the resolution to

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0,$$

with P_n finitely generated as well.

Proof. See [1, Proposition 1.5]. □

Corollary 4.2.10. *Suppose $A \in P(\Lambda)$. The following are equivalent:*

(i) $A \in P(\Lambda)_\infty$.

(ii) If I is a directed poset, $B, C \in P(\Lambda^{op})^I$, with a morphism $f : B \to C$ such that

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism, and each component of C is a product of copies of Λ with identity maps between them, then

$$\varinjlim U^I(B \hat{\otimes}_\Lambda^I A) \to U(\prod \Lambda \hat{\otimes}_\Lambda A) = U(\prod A)$$

is an isomorphism and

$$\varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(B, A) = 0$$

for all $m \geq 1$.

(iii) $A \in P(\Lambda)_1$ and

$$\varinjlim U^I \operatorname{Tor}_m^{\Lambda,I}(B, A) = 0$$

for all $m \geq 1$.

(iv) If I is a directed poset, and $B \in P(\Lambda)^I$ such that $\varinjlim U^I(B) = 0$, then

$$\varinjlim U^I \operatorname{Ext}_\Lambda^{m,I}(A, B) = 0$$

for all m.

Proof. (i) ⇒ (ii) ⇒ (iii) follows immediately from Corollary 4.2.6; for (iii) ⇒ (i), Corollary 4.2.6 shows that $A \in P(\Lambda)_n$, for all $n < \infty$, and then Lemma 4.2.9 allows us to construct the required projective resolution of A. (i) ⇒ (iv) follows from Theorem 4.2.2, which also shows that (iv) ⇒ $A \in P(\Lambda)_n$, for all $n < \infty$, and then Lemma 4.2.9 tells us that this implies (i). □

4.3 Group Homology and Cohomology over Direct Systems

Let R be a commutative profinite ring and G a profinite group. Then for I a small category, $A \in P(R[\![G]\!]^{op})^I$, $B \in P(R[\![G]\!])^I$, we define the *homology groups of* G *over* R *with coefficients in* A by

$$H_n^{R,I}(G, A) = \mathrm{Tor}_n^{R[\![G]\!],I}(A, R),$$

and the *cohomology groups with coefficients in* B by

$$H_R^{n,I}(G, B) = \mathrm{Ext}_{R[\![G]\!]}^{n,I}(R, B),$$

where R is a left $R[\![G]\!]$-module via the trivial G-action.

If R is of type FP_n as an $R[\![G]\!]$-module, we say G is *of type* FP_n *over* R. Note that R is finitely generated as an $R[\![G]\!]$-module, so all groups are of type FP_0 over all R. Note also that since $R[\![\{e\}]\!] = R$, R is free as an $R[\![\{e\}]\!]$-module, so the trivial group is of type FP_∞.

Now Theorem 4.2.2 and Corollary 4.2.6 translate to:

Proposition 4.3.1. *Let I be a directed poset. The following are equivalent for $n \geq 1$:*

(i) *G is of type FP_n over R.*

(ii) *Whenever we have $B, C \in P(R[\![G]\!]^{op})^I$, with a morphism $f : B \to C$ such that*

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism, then

$$\varinjlim U^I H_m^{R,I}(G, B) \to \varinjlim U^I H_m^{R,I}(G, C)$$

are isomorphisms for $m < n$ and an epimorphism for $m = n$.

(iii) *G is of type FP_1, and for all products $\prod \Lambda$ of copies of $R[\![G]\!]$, when C has as each of its components $\prod \Lambda$, with identity maps between them, for some B with each component finitely presented,*

$$\varinjlim U^I H_m^{R,I}(G, B) = 0$$

for all $1 \leq m \leq n - 1$.

(iv) *Whenever we have $B, C \in P(R[\![G]\!])^I$, with a morphism $f : B \to C$ such that*

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism, then

$$\varinjlim H_R^{m,I}(G, B) \to \varinjlim H_R^{m,I}(G, C)$$

are isomorphisms for $m < n$ and a monomorphism for $m = n$.

(v) *When C has 0 as each of its components, for some B with each component finitely presented,*
$$\varinjlim H_R^{m,I}(G,B) = 0$$
for $m \leq n$.

Similar results hold for $n = \infty$, by Lemma 4.2.9.

Corollary 4.2.7 translates to:

Lemma 4.3.2. *Suppose G is of type FP_{n-1}, $n \geq 1$, and we have an exact sequence*
$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$
of profinite left $R[\![G]\!]$-modules with $P_0, \ldots, P_{n-1}$ finitely generated and projective. Let I be a directed poset, let $C \in P(R[\![G]\!]^{op})^I$ have $\prod_X R[\![G]\!]$ for all its components with identity maps between them, for a set X such that there is an injection $\iota : M \to X$, let $B \in P(R[\![G]\!]^{op})_1^I$ such that
$$\varinjlim U^I(B) \to U(\prod_X R[\![G]\!])$$
is an isomorphism, with $B \to C$ given by the canonical map on each component. Then G is of type FP_n if and only if
$$\varinjlim U^I H_{n-1}^{R,I}(G,B) \to UH_{n-1}^R(G, \prod_X R[\![G]\!])$$
is an isomorphism.

For $n \geq 2$, G is of type FP_n if and only if
$$\varinjlim U^I H_{n-1}^{R,I}(G,B) = 0.$$

Lemma 4.3.3. *Suppose H is an open subgroup of G. Then H is of type FP_n over R, $n \leq \infty$, if and only if G is. In particular, if G is finite, it is of type FP_∞ over R.*

Proof. H open $\Rightarrow$ H is of finite index in G. It follows from [23, Proposition 5.7.1] that $R[\![G]\!]$ is free and finitely generated as an $R[\![H]\!]$-module, and hence that a finitely generated projective $R[\![G]\!]$-module is also a finitely generated projective $R[\![H]\!]$-module (because projective modules are summands of free ones). So an $R[\![G]\!]$-projective resolution of R, finitely generated up to the nth step, shows that H is of type FP_n.

For the converse, suppose H is of type FP_n, and suppose we have a finitely generated partial $R[\![G]\!]$-projective resolution
$$P_k \to \cdots \to P_0 \to R \to 0, \tag{$*$}$$
for $k < n$. Then since $(*)$ is also a finitely generated partial $R[\![H]\!]$-projective resolution, $\ker(P_k \to P_{k-1})$ is finitely generated as an $R[\![H]\!]$-module, by Lemma 4.2.9. So it is finitely generated as an $R[\![G]\!]$-module too. So we can extend the $R[\![G]\!]$-projective resolution to
$$P_{k+1} \to P_k \to \cdots \to P_0 \to R \to 0,$$
with P_{k+1} finitely generated. Iterate this argument to get that G is of type FP_n. □

We now observe that if a group G is of type FP_n over $\hat{\mathbb{Z}}$, it is of type FP_n over all profinite R (see [23, Lemma 6.3.5]). Indeed, given a partial projective resolution

$$P_n \to P_{n-1} \to \cdots \to P_0 \to \hat{\mathbb{Z}} \to 0$$

of $\hat{\mathbb{Z}}$ as a $\hat{\mathbb{Z}}[\![G]\!]$-module with each P_k finitely generated, apply $-\hat{\otimes}_{\hat{\mathbb{Z}}}R$: this is exact because the resolution is $\hat{\mathbb{Z}}$-split. Trivially $\hat{\mathbb{Z}}\hat{\otimes}_{\hat{\mathbb{Z}}}R \cong R$. Now $\hat{\mathbb{Z}}[\![G]\!]\hat{\otimes}_{\hat{\mathbb{Z}}}R = R[\![G]\!]$ by considering inverse limits of finite quotients, and it follows by additivity that each $P_k\hat{\otimes}_{\hat{\mathbb{Z}}}R$ is a finitely generated projective $R[\![G]\!]$-module, as required.

For a profinite group G, we write $d(G)$ for the minimal cardinality of a set of generators of G. For a profinite $\hat{\mathbb{Z}}[\![G]\!]$-module A, $d_{\hat{\mathbb{Z}}[\![G]\!]}(A)$ is the minimal cardinality of a set of generators of A as a $\hat{\mathbb{Z}}[\![G]\!]$-module. Similarly for abstract groups – except that we count abstract generators instead of topological generators.

We define the *augmentation ideal* $I_{\hat{\mathbb{Z}}}[\![G]\!]$ to be the kernel of the *evaluation map*

$$\varepsilon : \hat{\mathbb{Z}}[\![G]\!] \to \hat{\mathbb{Z}}, g \mapsto 1,$$

and $I_{\mathbb{Z}}[G]$ similarly for abstract groups. In the abstract case, $d(G)$ is finite if and only if $d_{\mathbb{Z}[G]}(I_{\mathbb{Z}}[G])$ is, and more generally groups are of type FP_1 over any ring if and only if they are finitely generated, by [1, Proposition 2.1]. Similarly pro-p groups are of type FP_1 over $\mathbb{Z}_p$ if and only if they are finitely generated, by [23, Theorem 7.8.1] and [31, Proposition 4.2.3]. The following proposition shows this is no longer the case for profinite groups.

Proposition 4.3.4. *Let G be a profinite group. Then the following are equivalent.*

(i) G is finitely generated.

(ii) There exists some d such that for all open normal subgroups K of G,

$$d(G/K) \leq d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K]) + d,$$

and G is of type FP_1 over $\hat{\mathbb{Z}}$.

Proof. We start by noting:

(a) $d(G) = \sup_K d(G/K)$ by [23, Lemma 2.5.3];

(b) $d_{\hat{\mathbb{Z}}[\![G]\!]}(I_{\hat{\mathbb{Z}}}[\![G]\!]) = \sup_K d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K])$ by [10, Theorem 2.3].

(i) $\Rightarrow$ (ii): For a finitely generated abstract group G,

$$d(G) \geq d_{\mathbb{Z}[G]}(I_{\mathbb{Z}}[G]). \qquad (*)$$

Indeed, if G is generated by $x_1, \dots, x_k$, then one can check that $I_{\mathbb{Z}}[G]$ is generated as a $\mathbb{Z}[G]$-module by $x_1 - 1, \dots, x_k - 1$. Write G as the inverse limit of $\{G/K\}$, where K ranges over the open normal subgroups of G. Then applying $(*)$, for each K

$$d(G/K) \geq d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K]);$$

hence

$$d(G) = \sup_K d(G/K) \geq \sup_K d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K]) = d_{\hat{\mathbb{Z}}[\![G]\!]}(I_{\hat{\mathbb{Z}}}[\![G]\!]),$$

and hence G is of type FP_1 over $\hat{\mathbb{Z}}$. Now set $d = d(G)$: for each K,

$$d(G/K) \leq d \leq d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K]) + d.$$

(ii) $\Rightarrow$ (i): First note that by Lemma 4.2.9, since G is of type FP_1, $d_{\hat{\mathbb{Z}}[\![G]\!]}(I_{\hat{\mathbb{Z}}}[\![G]\!])$ is finite. By (a) and (b),

$$d(G) \leq d_{\hat{\mathbb{Z}}[\![G]\!]}(I_{\hat{\mathbb{Z}}}[\![G]\!]) + d,$$

and the result follows. □

Remark 4.3.5. (a) When, for example, G is prosoluble or 2-generated, it is known that the condition

$$d(G/K) \leq d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K]) + d$$

for all open normal K holds with $d = 0$ – see [14, Proposition 6.2, Theorem 6.9]. Since pro-p groups are pronilpotent, this holds for all pro-p groups. By the Feit-Thompson theorem, it holds for all profinite groups of order coprime to 2.

(b) There are profinite groups G for which the difference between $d(G/K)$ and $d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K])$ is unbounded as K varies. The existence of a group of type FP_1 over $\hat{\mathbb{Z}}$ that is not finitely generated is shown in [10, Example 2.6].

(c) Let π be a set of primes. In fact the proof of [10, Theorem 2.3] that

$$d_{\hat{\mathbb{Z}}[\![G]\!]}(I_{\hat{\mathbb{Z}}}[\![G]\!]) = \sup_K d_{\mathbb{Z}[G/K]}(I_{\mathbb{Z}}[G/K]),$$

and hence the proof of Proposition 4.3.4, go through unchanged if G is a pro-π group and we replace $\hat{\mathbb{Z}}$ with $\mathbb{Z}_{\hat{\pi}}$, or more particularly if G is pro-p and we use $\mathbb{Z}_p$. Thus, applying (a), we recover in a new way the fact that pro-p groups are finitely generated if and only if they are of type FP_1 over $\mathbb{Z}_p$.

Corollary 4.3.6. *Suppose G is prosoluble or 2-generated profinite. Then G is of type FP_∞ over $\hat{\mathbb{Z}}$ if and only if it is finitely generated and whenever $B, C \in P(\hat{\mathbb{Z}}[\![G]\!]^{op})^I$, with a morphism $f : B \to C$ such that*

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism, and each component of C is a product of copies of $\hat{\mathbb{Z}}[\![G]\!]$ with identity maps between them,

$$\varinjlim U^I H_n^{R,I}(G, B) = 0$$

for all $n \geq 1$.

Proof. Proposition 4.3.1 and Proposition 4.3.4. □

We have, for $H_*^{R,I}$, a Lyndon-Hochschild-Serre spectral sequence for profinite groups.

Theorem 4.3.7. *Let G be a profinite group, K a closed normal subgroup and suppose $B \in P(R[\![G]\!]^{op})^I$. Then there exists a spectral sequence $(E^t_{r,s})$ with the property that*

$$E^2_{r,s} \cong H^{R,I}_r(G/K, H^{R,I}_s(K,B))$$

and

$$E^2_{r,s} \Rightarrow H^{R,I}_{r+s}(G,B).$$

Proof. [23, Theorem 7.2.4] and Proposition 1.2.5. □

Theorem 4.3.8. *Let G be a profinite group and K a closed normal subgroup. Suppose K is of type FP_m over R, $m \leq \infty$. Suppose $n \leq \infty$, and let $s = \min\{m,n\}$.*

(i) If G is of type FP_n over R then G/K is of type FP_s over R.

(ii) If G/K is of type FP_n over R then G is of type FP_s over R.

Proof. For simplicity we prove the case $m = \infty$. The proof for m finite is similar.

Since K is of type FP_∞, by Proposition 4.3.1 we have that, whenever $B, C \in P(R[\![G]\!]^{op})^I$, with a morphism $f : B \to C$ such that

$$\varinjlim U^I(f) : \varinjlim U^I(B) \to \varinjlim U^I(C)$$

is an isomorphism,

$$\varinjlim U^I H^{R,I}_s(K,B) \to \varinjlim U^I H^{R,I}_s(K,C)$$

is an isomorphism for all s; hence, when the components of C are products of copies of $R[\![G]\!]$ with identity maps between them,

$$\varinjlim U^I H^{R,I}_s(K,B) \to \varinjlim U^I H^{R,I}_s(K, \prod R[\![G]\!]) = U(\prod H^R_s(K, R[\![G]\!]))$$

is an isomorphism for all s; $R[\![G]\!]$ is a free $R[\![K]\!]$-module by [23, Corollary 5.7.2], so this is 0 for $s \geq 1$, and for $s = 0$ it gives

$$\varinjlim U^I(B \hat{\otimes}^I_{R[\![K]\!]} R) = U(\prod R[\![G]\!] \hat{\otimes}_{R[\![K]\!]} R) = U(\prod R[\![G/K]\!])$$

by [23, Proposition 5.8.1]. So the spectral sequence from Theorem 4.3.7 collapses to give an isomorphism

$$H^{R,I}_r(G/K, B\hat{\otimes}^I_{R[\![K]\!]} R) \cong H^{R,I}_r(G,B). \qquad (*)$$

By Lemma 4.2.9, it is enough to prove the theorem for $n < \infty$. We use induction on n. Note that G and G/K are always both of type FP_0, so we may assume the theorem holds for $n-1$. Suppose G and G/K are of type FP_{n-1}, and that we have exact sequences

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

of profinite left $R[\![G]\!]$-modules with $P_0, \ldots, P_{n-1}$ finitely generated and projective, and

$$0 \to M' \to P'_{n-1} \to \cdots \to P'_0 \to R \to 0$$

of profinite left $R[\![G/K]\!]$-modules with $P'_0, \dots, P'_{n-1}$ finitely generated and projective. Choose a set X such that there are injections $\iota : M \to X$ and $\iota' : M' \to X$. Let I be a directed poset, let $C \in P(R[\![G]\!]^{op})^I$ have $\prod_X R[\![G]\!]$ for all its components with identity maps between them, let $B \in P(R[\![G]\!]^{op})^I_1$ such that

$$\varinjlim U^I(B) \to U(\prod_X R[\![G]\!])$$

is an isomorphism, with $B \to C$ given by the canonical map on each component. Finally, note that

$$B\hat{\otimes}^I_{R[\![K]\!]} R \in P(R[\![G/K]\!]^{op})^I_1 :$$

for each B^i, there is an exact sequence

$$F_1 \to F_0 \to B^i \to 0$$

with F_0 and F_1 free and finitely generated $R[\![G]\!]$-modules, so by the right exactness of $-\hat{\otimes}^I_{R[\![K]\!]} R$ there is an exact sequence

$$F_1\hat{\otimes}^I_{R[\![K]\!]} R \to F_0\hat{\otimes}^I_{R[\![K]\!]} R \to B^i\hat{\otimes}^I_{R[\![K]\!]} R \to 0$$

with $F_0\hat{\otimes}^I_{R[\![K]\!]} R$ and $F_1\hat{\otimes}^I_{R[\![K]\!]} R$ free and finitely generated $R[\![G/K]\!]$-modules by [23, Proposition 5.8.1]. Therefore G is of type FP_n if and only if

$$\varinjlim U^I H^{R,I}_{n-1}(G, B) \to U H^R_{n-1}(G, \prod_X R[\![G]\!])$$

is an isomorphism (by Lemma 4.3.2) if and only if

$$\varinjlim U^I H^{R,I}_{n-1}(G/K, B\hat{\otimes}^I_{R[\![K]\!]} R) \to U H^R_{n-1}(G/K, \prod_X R[\![G/K]\!])$$

is an isomorphism (by (∗)) if and only if G/K is of type FP_n (by Lemma 4.3.2). □

Let $\mathcal{C}$ be a non-empty *class* of finite groups. Being of type FP_n over R as a pro-$\mathcal{C}$ group is exactly the same as being of type FP_n over R as a profinite group, so working in the pro-$\mathcal{C}$ universe instead of the profinite one gives nothing new. On the other hand, amalgamated free pro-$\mathcal{C}$ products of pro-$\mathcal{C}$ groups are not the same as amalgamated free profinite products of pro-$\mathcal{C}$ groups, and pro-$\mathcal{C}$ HNN-extensions of pro-$\mathcal{C}$ groups are not the same as profinite HNN-extensions of pro-$\mathcal{C}$ groups – essentially because, in the pro-$\mathcal{C}$ case, we take a pro-$\mathcal{C}$ completion of the abstract amalgamated free product or abstract HNN-extension, rather than taking a profinite completion of them. Thus, by working over a class $\mathcal{C}$, we can achieve more general results.

In the abstract case, Bieri uses his analogous results to give conditions on the FP-type of amalgamated free products and HNN-extensions of groups using the Mayer-Vietoris sequence on their homology. His approach does not entirely translate to the pro-$\mathcal{C}$ setting, but we obtain some partial results.

See [23, 9.2] for the definition of amalgamated free products in the pro-$\mathcal{C}$ case, and [23, 9.4] for HNN-extensions. We say that an amalgamated free pro-$\mathcal{C}$ product $G = G_1 \amalg_H G_2$ is *proper* if the canonical homomorphisms $G_1 \to G$ and

$G_2 \to G$ are monomorphisms. Similarly, we say that a pro-$\mathcal{C}$ HNN-extension $G = \mathrm{HNN}(H, A, f)$ is *proper* if the canonical homomorphism $H \to G$ is a monomorphism.

Suppose, for the rest of the section, that $\mathcal{C}$ is closed under taking subgroups, quotients and extensions. For example, $\mathcal{C}$ could be all finite groups, or all finite p-groups – or, for example, all finite soluble π-groups, where π is a set of primes. Suppose R is a pro-$\mathcal{C}$ ring.

Proposition 4.3.9. *Let $G = G_1 \amalg_H G_2$ be a proper amalgamated free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups. Suppose B is a profinite right $R[\![G]\!]$-module. Then there is a long exact sequence of profinite R-modules*

$$\begin{aligned}\cdots &\to H_{n+1}^R(G, B) \to H_n^R(H, B) \to H_n^R(G_1, B) \oplus H_n^R(G_2, B) \\ &\to H_n^R(G, B) \to \cdots \to H_0^R(G, B) \to 0,\end{aligned}$$

which is natural in B.

Proof. See [23, Proposition 9.2.13] for the long exact sequence. Naturality follows by examining the maps involved. □

Proposition 4.3.10. *Let $G = G_1 \amalg_H G_2$ be a proper amalgamated free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups. Suppose $B \in P(R[\![G]\!]^{op})^I$. Then there is a long exact sequence in $P(R)^I$*

$$\begin{aligned}\cdots &\to H_{n+1}^{R,I}(G, B) \to H_n^{R,I}(H, B) \to H_n^{R,I}(G_1, B) \oplus H_n^{R,I}(G_2, B) \\ &\to H_n^{R,I}(G, B) \to \cdots \to H_0^{R,I}(G, B) \to 0,\end{aligned}$$

which is natural in B.

Proof. This follows immediately from the naturality of the long exact sequence in Proposition 4.3.9. □

We can now give a result analogous to the first part of [1, Proposition 2.13 (1)].

Proposition 4.3.11. *Let $G = G_1 \amalg_H G_2$ be a proper amalgamated free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups. If G_1 and G_2 are of type FP_n over R and H is of type FP_{n-1} over R then G is of type FP_n over R.*

Proof. Take C as in Proposition 4.3.1 to have as each component a product of copies of $R[\![G]\!]$, with identity maps between the components. Apply Proposition 4.3.1 to the long exact sequence in Proposition 4.3.10. Then the Five Lemma gives the result, by Proposition 4.3.1. □

See [23, Chapter 3.3] for the construction and properties of free pro-$\mathcal{C}$ groups.

Corollary 4.3.12. *Finitely generated free pro-$\mathcal{C}$ groups are of type FP_∞ over all pro-$\mathcal{C}$ rings R.*

Proof. Unamalgamated free pro-$\mathcal{C}$ products are always proper by [23, Corollary 9.1.4]. □

For proper profinite HNN-extensions of profinite groups, we also have a Mayer-Vietoris sequence which is natural in the second variable – see [23, Proposition 9.4.2]. It immediately follows in the same way as for Proposition 4.3.10 that we get a long exact sequence over a functor category.

Proposition 4.3.13. *Let $G = \mathrm{HNN}(H, A, f)$ be a proper pro-$\mathcal{C}$ HNN-extension of pro-$\mathcal{C}$ groups. Suppose $B \in P(R[\![G]\!]^{op})^I$. Then there is a long exact sequence in $P(R)^I$*

$$\begin{aligned}\cdots \to H_{n+1}^{R,I}(G,B) \to H_n^{R,I}(A,B) \to H_n^{R,I}(H,B)\\ \to H_n^{R,I}(G,B) \to \cdots \to H_0^{R,I}(G,B) \to 0,\end{aligned}$$

which is natural in B.

From this, we can get in exactly the same way as for free products with amalgamation a result for HNN-extensions, corresponding to the first part of [1, Proposition 2.13 (2)].

Proposition 4.3.14. *Let $G = \mathrm{HNN}(H, A, f)$ be a proper pro-$\mathcal{C}$ HNN-extension of pro-$\mathcal{C}$ groups. If H is of type FP_n over R and A is of type FP_{n-1} over R then G is of type FP_n over R.*

4.4 Applications

Example 4.4.1. We show that torsion-free procyclic groups are of type FP_∞ over R. See [23, Chapter 2.7] for the results on procyclic groups that will be needed in this paper. Any procyclic group G is finitely generated, so of type FP_1 by Proposition 4.3.4. If G is torsion-free, its Sylow p-subgroups are all either 0 or isomorphic to $\mathbb{Z}_p$, so (assuming $G \neq 1$) it is well-known that G has cohomological dimension 1 – see [23, Theorem 7.3.1, Theorem 7.7.4]. Consider the short exact sequence

$$0 \to \ker\varepsilon \to R[\![G]\!] \xrightarrow{\varepsilon} R \to 0,$$

where ε is the evaluation map defined earlier. The kernel $\ker\varepsilon$ is finitely generated by Lemma 4.2.9. We claim that it is projective – and hence that our exact sequence is a finitely generated projective resolution of R, showing G is of type FP_∞. To see this, let A be any profinite right $R[\![G]\!]$-module, and consider the long exact sequence

$$\begin{aligned}\cdots \to \mathrm{Tor}_1^{R[\![G]\!]}(A,R) \to \mathrm{Tor}_0^{R[\![G]\!]}(A,\ker\varepsilon)\\ \to \mathrm{Tor}_0^{R[\![G]\!]}(A,R[\![G]\!]) \to \mathrm{Tor}_0^{R[\![G]\!]}(A,R) \to 0.\end{aligned}$$

Since $R[\![G]\!]$ is free as an $R[\![G]\!]$-module, we get

$$\mathrm{Tor}_n^{R[\![G]\!]}(A,R[\![G]\!]) = 0$$

for all $n \geq 1$, and so

$$\mathrm{Tor}_n^{R[\![G]\!]}(A,\ker\varepsilon) \cong \mathrm{Tor}_{n+1}^{R[\![G]\!]}(A,R)$$

for all $n \geq 1$. Now G has cohomological dimension 1, so

$$\mathrm{Tor}_{n+1}^{R[\![G]\!]}(A, R) = H_{n+1}^{R}(G, A) = 0$$

for $n \geq 1$, so $\ker \varepsilon$ is projective.

We can now use this example to construct some more groups of type FP_∞.

It is known in the abstract case that polycyclic groups are of type FP_∞ over $\mathbb{Z}$ ([1, Examples 2.6]). In the profinite case, it has not been known whether poly-procyclic groups are of type FP_∞ over $\hat{\mathbb{Z}}$. A result was known for pro-p groups: poly-(pro-p-cyclic) groups are shown to be of type FP_∞ over $\mathbb{Z}_p$ in [31, Corollary 4.2.5]. This proof uses that, for a pro-p group G, $H^n_{\mathbb{Z}_p}(G, A)$ is finite for all finite $A \in D(\mathbb{Z}_p[\![G]\!])$ if and only if G is of type FP_∞ over $\mathbb{Z}_p$. Indeed, one might expect a similar result to be true for profinite G which only have finitely many primes in their order, but an obstruction to using this method for general profinite groups is that there are infinitely many primes, so one cannot build up to these groups from pro-p ones using the spectral sequence finitely many times. Similarly, although we showed directly that torsion-free procyclic groups are of type FP_∞ over R, there are procyclic groups which are not even virtually torsion-free, in contrast to the pro-p case, as for example the group $\prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$.

We now define a class of profinite groups: the elementary amenable profinite groups. The definition is entirely analogous to the hereditary definition of elementary amenable abstract groups given in [15]. Let $\mathscr{X}_0$ be the class containing only the trivial group, and let $\mathscr{X}_1$ be the class of profinite groups which are (finitely generated abelian)-by-finite. Now define $\mathscr{X}_\alpha$ to be the class of groups G which have a normal subgroup K such that $G/K \in \mathscr{X}_1$ and every finitely generated subgroup of K is in $\mathscr{X}_{\alpha-1}$ for α a successor ordinal. Finally, for α a limit define $\mathscr{X}_\alpha = \bigcup_{\beta<\alpha} \mathscr{X}_\beta$. Then $\mathscr{X} = \bigcup_\alpha \mathscr{X}_\alpha$ is the class of elementary amenable profinite groups. For $G \in \mathscr{X}$ we define the *class* of G to be the least α with $G \in \mathscr{X}_\alpha$.

Note that soluble profinite groups are clearly elementary amenable.

A profinite group G is said to have *finite rank* if there is some r such that every subgroup H of G is generated by r elements.

Proposition 4.4.2. *Suppose G is an elementary amenable profinite group of finite rank. Then G is of type FP_∞ over any profinite ring R.*

Proof. By [23, Theorem 2.7.2], every procyclic group is a quotient of $\hat{\mathbb{Z}}$ by a torsion-free procyclic group; $\hat{\mathbb{Z}}$ and torsion-free procyclic groups are of type FP_∞ by Example 4.4.1. Therefore procyclic groups are of type FP_∞ by Theorem 4.3.8, and finitely generated abelian groups are a finite direct sum of procyclic groups by [33, Proposition 8.2.1(iii)], so we get that finitely generated abelian groups are of type FP_∞ by applying Theorem 4.3.8 finitely many times.

Now use induction on the class of G. If $G \in \mathscr{X}_1$, take a finite index abelian $H \leq G$. We have shown H is of type FP_∞, so G is too by Lemma 4.3.3. The case where G has class α is trivial for α a limit, so suppose α is a successor. Choose some $K \lhd G$ such that every finitely generated subgroup of K is in $\mathscr{X}_{\alpha-1}$ and G/K is in $\mathscr{X}_1$. Since G is of finite rank, K is finitely generated, so it is in $\mathscr{X}_{\alpha-1}$. By the inductive hypothesis we get that K is of type FP_∞, and G/K is too so G is by Theorem 4.3.8. □

We spend the rest of Section 4.4 constructing pro-$\mathcal{C}$ groups of type FP_n but not of type FP_{n+1} over $\mathbb{Z}_{\hat{\mathcal{C}}}$ for every $n < \infty$, for C closed under subgroups, quotients and extensions, as before. King in [17, Theorem F] gives pro-p groups of type FP_n but not of type FP_{n+1} over $\mathbb{Z}_p$, but as far as we know the case with $\mathbb{Z}_{\hat{\mathcal{C}}}$ has not been done before for any other class $\mathcal{C}$. Our construction is analogous to [1, Proposition 2.14].

Given a profinite space X, we can define the *free pro-$\mathcal{C}$ group on X*, $F_{\mathcal{C}}(X)$, together with a canonical continuous map $\iota : X \to F_{\mathcal{C}}(X)$, by the following universal property: for any pro-$\mathcal{C}$ group G and continuous $\phi : X \to G$, there is a unique continuous homomorphism $\bar{\phi} : F_{\mathcal{C}}(X) \to G$ such that $\phi = \bar{\phi}\iota$. For a class C closed under subgroups, quotients and extensions, $F_{\mathcal{C}}(X)$ exists for all X by [23, Proposition 3.3.2].

Fix $n \geq 0$. Let $\langle x_k, y_k \rangle$ be the free pro-$\mathcal{C}$ group on the two generators x_k, y_k, for $1 \leq k \leq n$, and write D_n for their direct product (D_0 is the empty product, i.e. the trivial group). Let $F_{\mathbb{Z}_{\hat{\mathcal{C}}}}$ be the free pro-$\mathcal{C}$ group on generators $\{a_l : l \in \mathbb{Z}_{\hat{\mathcal{C}}}\}$, given the usual pro-$\mathcal{C}$ topology. We define a continuous left D_n-action on $F_{\mathbb{Z}_{\hat{\mathcal{C}}}}$ in the following way. For each k, we have a continuous homomorphism $\langle x_k, y_k \rangle \to \mathbb{Z}_{\hat{\mathcal{C}}}$ defined by $x_k, y_k \mapsto 1$. Now this gives a continuous $D_n \to \mathbb{Z}_{\hat{\mathcal{C}}}^n$. Composing this with n-fold addition

$$\mathbb{Z}_{\hat{\mathcal{C}}}^n \to \mathbb{Z}_{\hat{\mathcal{C}}}, (a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$$

gives a continuous homomorphism

$$f : D_n \to \mathbb{Z}_{\hat{\mathcal{C}}}.$$

Now we can define a continuous action of D_n on $\mathbb{Z}_{\hat{\mathcal{C}}}$ by

$$D_n \times \mathbb{Z}_{\hat{\mathcal{C}}} \xrightarrow{f \times id} \mathbb{Z}_{\hat{\mathcal{C}}} \times \mathbb{Z}_{\hat{\mathcal{C}}} \xrightarrow{+} \mathbb{Z}_{\hat{\mathcal{C}}}.$$

Finally, by [23, Exercise 5.6.2(d)], this action extends uniquely to a continuous action on $F_{\mathbb{Z}_{\hat{\mathcal{C}}}}$.

Now we can form the semi-direct product $A_n = F_{\mathbb{Z}_{\hat{\mathcal{C}}}} \rtimes D_n$, and by [23, Exercise 5.6.2(b),(c)] it is a pro-$\mathcal{C}$ group. We record here the universal property of semi-direct products of pro-$\mathcal{C}$ groups; it is a direct translation of the universal property of semi-direct products of abstract groups from [3, III.2.10, Proposition 27 (2)], which we will need later.

Lemma 4.4.3. *Suppose we have pro-$\mathcal{C}$ groups N, H and K, with continuous homomorphisms $\sigma : H \to Aut(N)$ (with the compact-open topology), $f : N \to K$ and $g : H \to K$ such that, for all $x \in N$ and $y \in H$,*

$$g(y)f(x)g(y^{-1}) = f(\sigma(y)(x)).$$

Then there is a unique continuous homomorphism

$$h : N \rtimes H \to K$$

such that

$$f = h \circ (N \to N \rtimes H)$$

and

$$g = h \circ (H \to N \rtimes H).$$

Proof. By [3, III.2.10, Proposition 27 (2)] we know there is a unique homomorphism $h : N \rtimes H \to K$ satisfying these conditions, except that we need to check h is continuous. The proof in [3] constructs h as the map $(x, y) \mapsto f(x)g(y)$; this is the composite of the continuous maps (of spaces)

$$N \times H \to K \times K \to K,$$

where the second map is just multiplication in K. □

We need two more results about the A_n before we can prove the main proposition. Let $\langle x_n \rangle$ be the free pro-$\mathcal{C}$ group generated by x_n.

Lemma 4.4.4. *For each $n > 0$, $F_{\mathbb{Z}_{\hat{\mathcal{C}}}} \rtimes \langle x_n \rangle$ is the free pro-$\mathcal{C}$ group on two generators.*

Proof. We will show that this group satisfies the requisite universal property. We claim that it is generated by a_0 and x_n. Clearly allowing x_n (and x_n^{-1}) to act on a_0 gives a_l, for each $l \in \mathbb{Z}$. Now $\{a_l : l \in \mathbb{Z}_{\hat{\mathcal{C}}}\}$ (abstractly) generates a dense subgroup H of $F_{\mathbb{Z}_{\hat{\mathcal{C}}}}$; $\{a_l : l \in \mathbb{Z}\}$ is dense in $\{a_l : l \in \mathbb{Z}_{\hat{\mathcal{C}}}\}$, so it (abstractly) generates a dense subgroup K of H; by transitivity of denseness, K is dense in $F_{\mathbb{Z}_{\hat{\mathcal{C}}}}$, so $\{a_l : l \in \mathbb{Z}\}$ topologically generates $F_{\mathbb{Z}_{\hat{\mathcal{C}}}}$.

It remains to show that given a pro-$\mathcal{C}$ K and a map

$$f : \{a_0, x_n\} \to K$$

there is a continuous homomorphism

$$g : F_{\mathbb{Z}_{\hat{\mathcal{C}}}} \rtimes \langle x_n \rangle \to K$$

such that $f = g\iota$, where ι is the inclusion $\{a_0, x_n\} \to F_{\mathbb{Z}_{\hat{\mathcal{C}}}} \rtimes \langle x_n \rangle$. Observe, as in [23, p.91], that by the universal property of inverse limits it suffices to check the existence of g when K is finite.

To construct g, we first note that $f|_{x_n}$ extends uniquely to a continuous homomorphism

$$g' : \langle x_n \rangle \to K.$$

Now we define a continuous map of sets

$$f' : \{a_l : l \in \mathbb{Z}_{\hat{\mathcal{C}}}\} \to K$$

by

$$f'(a_l) = g'(l \cdot x_n) f(a_0) g'(l \cdot x_n)^{-1},$$

where we write $l \cdot x_n$ for the image of l under the obvious isomorphism $\mathbb{Z}_{\hat{\mathcal{C}}} \cong \langle x_n \rangle$; f' extends uniquely to a continuous homomorphism

$$g'' : F_{\mathbb{Z}_{\hat{\mathcal{C}}}} \to K.$$

Finally, by the universal property of semi-direct products, Lemma 4.4.3, we will have the existence of a continuous homomorphism g satisfying the required property as long as

$$g'(y) g''(x) g'(y)^{-1} = g''(\sigma(y)(x)),$$

for all $x \in F_{\mathbb{Z}_{\hat{\mathcal{C}}}}$ and $y \in \langle x_n \rangle$, where σ is the continuous homomorphism $\langle x_n \rangle \to Aut(F_{\mathbb{Z}_{\hat{\mathcal{C}}}})$. This is clear by construction. □

By Corollary 4.3.12, $F_{\mathbb{Z}_{\hat{\mathcal{C}}}} \rtimes \langle x_n \rangle$ is now of type FP_∞ over $\mathbb{Z}_{\hat{\mathcal{C}}}$; hence, by Theorem 4.3.8,

$$A_{n-1} \rtimes \langle x_n \rangle = (F_{\mathbb{Z}_{\hat{\mathcal{C}}}} \rtimes \langle x_n \rangle) \rtimes D_{n-1}$$

is too.

The next lemma is entirely analogous to [1, Proposition 2.15].

Lemma 4.4.5. *If a pro-$\mathcal{C}$ group G is of type* FP_n *over* $\mathbb{Z}_{\hat{\mathcal{C}}}$ *then* $H_m^{\mathbb{Z}_{\hat{\mathcal{C}}}}(G, \mathbb{Z}_{\hat{\mathcal{C}}})$ *is a finitely generated profinite abelian group for* $0 \leq m \leq n$.

Proof. Take a projective resolution P_* of $\mathbb{Z}_{\hat{\mathcal{C}}}$ as a $\mathbb{Z}_{\hat{\mathcal{C}}}[\![G]\!]$-module with trivial G-action, with $P_0, \ldots, P_n$ finitely generated. Then $H_*^{\mathbb{Z}_{\hat{\mathcal{C}}}}(G, \mathbb{Z}_{\hat{\mathcal{C}}})$ is the homology of the complex $\mathbb{Z}_{\hat{\mathcal{C}}} \hat{\otimes}_{\mathbb{Z}_{\hat{\mathcal{C}}}[\![G]\!]} P_*$, for which

$$\mathbb{Z}_{\hat{\mathcal{C}}} \hat{\otimes}_{\mathbb{Z}_{\hat{\mathcal{C}}}[\![G]\!]} P_0, \ldots, \mathbb{Z}_{\hat{\mathcal{C}}} \hat{\otimes}_{\mathbb{Z}_{\hat{\mathcal{C}}}[\![G]\!]} P_n$$

are finitely generated $\mathbb{Z}_{\hat{\mathcal{C}}}$-modules. Now $\mathbb{Z}_{\hat{\mathcal{C}}}$ is procyclic, hence a principal ideal domain, which implies by standard arguments that $\mathbb{Z}_{\hat{\mathcal{C}}}$ is noetherian in the sense that submodules of finitely generated $\hat{\mathbb{Z}}$-modules are finitely generated, and the result follows: finitely generated pro-$\mathcal{C}$ abelian groups are exactly the finitely generated pro-$\mathcal{C}$ $\mathbb{Z}_{\hat{\mathcal{C}}}$-modules. □

Proposition 4.4.6. *(i) A_n is of type* FP_n *over* $\mathbb{Z}_{\hat{\mathcal{C}}}$.

(ii) A_n is not of type FP_{n+1} *over* $\mathbb{Z}_{\hat{\mathcal{C}}}$.

Proof. (i) $n = 0$ is trivial. Next, we observe that A_n can be thought of as the pro-$\mathcal{C}$ HNN-extension of $A_{n-1} \rtimes \langle x_n \rangle$ with associated subgroup A_{n-1} and stable letter y_n, since the universal properties are the same in this case. It is clear that this extension is proper.

We can now use induction: assume A_{n-1} is of type FP_{n-1} over $\mathbb{Z}_{\hat{\mathcal{C}}}$ (which we already have for $n = 0$). Then $A_{n-1} \rtimes \langle x_n \rangle$ is of type FP_∞, so the result follows from Proposition 4.3.14.

(ii) By Lemma 4.4.5, it is enough to show that, for each n, $H_{n+1}^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_n, \mathbb{Z}_{\hat{\mathcal{C}}})$ is not finitely generated. We prove this by induction once more. Exactly as in [23, Lemma 6.8.6],

$$H_1^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_0, \mathbb{Z}_{\hat{\mathcal{C}}}) = F_{\mathbb{Z}_{\hat{\mathcal{C}}}} / \overline{[F_{\mathbb{Z}_{\hat{\mathcal{C}}}}, F_{\mathbb{Z}_{\hat{\mathcal{C}}}}]},$$

i.e. the pro-$\mathcal{C}$ free abelian group on the space $\mathbb{Z}_{\hat{\mathcal{C}}}$, not finitely generated. As before, A_n is the pro-$\mathcal{C}$ HNN-extension of $A_{n-1} \rtimes \langle x_n \rangle$ with associated subgroup A_{n-1} and stable letter y_n, and we get the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to H_{n+1}^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_{n-1} \rtimes \langle x_n \rangle, \mathbb{Z}_{\hat{\mathcal{C}}}) &\to H_{n+1}^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_n, \mathbb{Z}_{\hat{\mathcal{C}}}) \to H_n^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_{n-1}, \mathbb{Z}_{\hat{\mathcal{C}}}) \\ &\to H_n^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_{n-1} \rtimes \langle x_n \rangle, \mathbb{Z}_{\hat{\mathcal{C}}}) \to \cdots \end{aligned}$$

By Lemma 4.4.5 $H_{n+1}^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_{n-1} \rtimes \langle x_n \rangle, \mathbb{Z}_{\hat{\mathcal{C}}})$ and $H_n^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_{n-1} \rtimes \langle x_n \rangle, \mathbb{Z}_{\hat{\mathcal{C}}})$ are finitely generated; by hypothesis $H_n^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_{n-1}, \mathbb{Z}_{\hat{\mathcal{C}}})$ is not finitely generated. Hence $H_{n+1}^{\mathbb{Z}_{\hat{\mathcal{C}}}}(A_n, \mathbb{Z}_{\hat{\mathcal{C}}})$ is not finitely generated, as required.

□

Chapter 5

Profinite Groups of Type FP_∞

5.1 Signed Permutation Modules

Suppose G is a profinite group. Write G-Top for the category of topological G-spaces and G-Pro for the category of profinite G-spaces. We write elements of G-Pro as (X, α), where X is the underlying space and $\alpha : G \times X \to X$ is the action; where this is clear we may just write X. Now pick $X \in G$-Pro. Then the action of G on X induces an action of $R[\![G]\!]$ on $R[\![X]\!]$, by the universal property of group rings, making $R[\![X]\!]$ an $R[\![G]\!]$-module. We call modules with this form permutation modules, and we call the orbits and stabilisers of G acting on X the orbits and stabilisers of $R[\![X]\!]$. Permutation modules satisfy the following universal property: given an $R[\![G]\!]$ permutation module $R[\![X]\!]$, any continuous G-map from X to a profinite $R[\![G]\!]$-module M factors as $X \to R[\![X]\!] \to M$ for a unique continuous $R[\![G]\!]$-homomorphism $R[\![X]\!] \to M$, where $X \to R[\![X]\!]$ is the canonical G-map. This can be seen by first restricting $R[\![X]\!]$ and M to $P(R)$, making $R[\![X]\!]$ a free R-module, and then noting that continuous R-homomorphisms $R[\![X]\!] \to M$ are continuous $R[\![G]\!]$-homomorphisms if and only if they are compatible with the G-action. For later, we note that this universal property can be expressed by the formula in the following lemma.

Lemma 5.1.1. *Write $C_G(X, M)$ for the set of continuous G-maps $X \to M$. Make $C_G(X, M)$ into a $U(R)$-module by the map $r \cdot f = rf$; in other words, $(r \cdot f)(x) = r \cdot (f(x))$. Then (as $U(R)$-modules) $\mathrm{Hom}_{R[\![G]\!]}(R[\![X]\!], M) \cong C_G(X, M)$.*

Proof. That $\mathrm{Hom}_{R[\![G]\!]}(R[\![X]\!], M)$ and $C_G(X, M)$ are isomorphic as sets is simply a restatement of the universal property. That they have the same $U(R)$-module structure is clear from the definition of multiplication by r. □

Signed permutation modules are $R[\![G]\!]$-modules which as R-modules are free with basis X, and whose G-action comes from a continuous action of G on $X \cup -X \subset R[\![X]\!]$ such that $g \cdot -x = -(g \cdot x)$ for all $g \in G, x \in X \cup -X$; the terminology appears in [30, Definition 5.1], though in fact the definitions are slightly different: both this definition and that of [30] are attempts to deal with the 'twist' by a sign that appears in the tensor-induced complexes of [25, 7.3].

The reason for the change is that our definition seems to be needed for Lemma 5.1.4.

In the same way as for permutation modules, one can see that signed permutation modules satisfy the property that any continuous G-map f from $X \cup -X$ to an $R[\![G]\!]$-module M such that $f(-x) = -f(x)$ for all x extends uniquely to a continuous $R[\![G]\!]$-homomorphism $R[\![X]\!] \to M$.

For this paragraph, assume $\operatorname{char} R \neq 2$. Now suppose P is a signed permutation module of the form $R[\![X]\!]$. Write $\overline{X}$ for the quotient G-space of $X \sqcup -X$ given by $x \sim -x$, and $\sim$ for the map $X \sqcup -X \to \overline{X}$. Then we make the convention that when we talk about the G-stabilisers of P, we will always mean the G-stabilisers of $\overline{X}$, and the G-orbits of P will always mean the preimages in X of the G-orbits of $\overline{X}$.

If on the other hand $\operatorname{char} R = 2$, signed permutation modules are just permutation modules. So here the G-stabilisers of $R[\![X]\!]$ will be the G-stabilisers of X and the G-orbits will be the G-orbits of X. We also define, for $\operatorname{char} R = 2$, $\overline{X} = X$ and $\sim = \mathrm{id}_X$. We use the notation that G acts on $X \cup -X$ to cover both cases.

We now need to establish some basic properties of signed permutation modules. The following lemma is an adaptation of [23, Lemma 5.6.4(a)].

Lemma 5.1.2. *Suppose $R[\![X]\!]$ is a signed permutation module. Then $X \cup -X = \varprojlim (X_i \cup -X_i)$, where the $X_i \cup -X_i$ are finite quotients of $X \cup -X$ as G-spaces such that the map $X \cup -X \to X_i \cup -X_i$ sends X to X_i and $-X$ to $-X_i$. Thus*

$$R[\![X]\!] = \varprojlim_{P(R[\![G]\!])} R[X_i] = \varprojlim_{P(R[\![G]\!])} R_j[X_i],$$

where the R_j are the finite quotients of R. We say that such quotients $R_j[X_i]$ of $R[\![X]\!]$ preserve the algebraic structure.

Proof. If $\operatorname{char} R = 2$, we are done. Assume $\operatorname{char} R \neq 2$.

Consider the set S of clopen equivalence relations $\mathcal{R}$ on $X \cup -X$ such that, considered as a subset of $(X \cup -X) \times (X \cup -X)$,

$$\mathcal{R} \subseteq (X \times X) \cup (-X \times -X)$$

and

$$(x, y) \in \mathcal{R} \Leftrightarrow (-x, -y) \in \mathcal{R}.$$

In other words, an equivalence relation $\mathcal{R} \in S$ is one which does not identify anything in X with anything in $-X$ and identifies two elements in $-X$ whenever it identifies the corresponding two elements of X; then $\mathcal{R} \in S$ if and only if $(X \cup -X)/\mathcal{R}$ has the form $X_i \cup -X_i$ for some finite quotient X_i of X (as profinite spaces, not profinite G-spaces). Since $X = \varprojlim_{Pro} X_i$,

$$X \cup -X = \varprojlim_{Pro} X_i \cup -X_i = \varprojlim_{Pro,S} (X \cup -X)/\mathcal{R}.$$

We want to show that for every $\mathcal{R} \in S$ there is some $\mathcal{R}' \subseteq \mathcal{R}$ which is G-invariant: then it follows that

$$X \cup -X = \varprojlim_{G\text{-}Pro,\{\mathcal{R} \in S : \mathcal{R} \text{ is } G\text{-invariant}\}} (X \cup -X)/\mathcal{R}$$

by [23, Lemma 1.1.9], because $\{\mathcal{R} \in S : \mathcal{R} \text{ is } G\text{-invariant}\}$ is cofinal in S, and all these quotients are quotients as G-spaces.

So suppose $\mathcal{R} \in S$ and define $\mathcal{R}' = \bigcap_{g \in G} g\mathcal{R}$, where

$$g\mathcal{R} = \{(gx, gy) \in (X \cup -X) \times (X \cup -X) : (x, y) \in \mathcal{R}\}.$$

Now we see in exactly the same way as the proof of [23, Lemma 5.6.4(a)] that $\mathcal{R}'$ is clopen; clearly $\mathcal{R}'$ is G-invariant, and $\mathcal{R}' \in S$ because

$$\mathcal{R}' \subseteq \mathcal{R} \subseteq (X \times X) \cup (-X \times -X)$$

and

$$\begin{aligned}
(x, y) \in \mathcal{R}' &\Leftrightarrow (x, y) \in g\mathcal{R}, \forall g \\
&\Leftrightarrow (g^{-1}x, g^{-1}y) \in \mathcal{R}, \forall g \\
&\Leftrightarrow (-g^{-1}x, -g^{-1}y) \in \mathcal{R}, \forall g \\
&\Leftrightarrow (-x, -y) \in g\mathcal{R}, \forall g \\
&\Leftrightarrow (-x, -y) \in \mathcal{R}'.
\end{aligned}$$

It follows that $R[\![X]\!] = \varprojlim_{P(R[\![G]\!])} R_j[X_i]$ because every continuous G-map f from $X \cup -X$ to a finite $R[\![G]\!]$-module M such that $f(-x) = -f(x)$ for all x factors through some quotient G-space $X_i \cup -X_i$, and clearly the induced map $f' : X_i \cup -X_i \to M$ satisfies $f'(-x) = -f'(x)$, so every morphism $R[\![X]\!] \to M$ factors through some $R[X_i]$ by the universal property of signed permutation modules, and hence through some $R_j[X_i]$. □

Lemma 5.1.3. *Suppose $R[\![X]\!]$ is a signed $R[\![G]\!]$ permutation module, and that G acts freely on $\overline{X}$. Then $R[\![X]\!]$ is free.*

Proof. If $\operatorname{char} R = 2$, we are done. Assume $\operatorname{char} R \neq 2$.

As profinite G-spaces, $\overline{X} = G \times \overline{Y}$ for some $\overline{Y}$ on which G acts trivially by [23, Corollary 5.6.6]; take the preimage Y of $\overline{Y}$ in X. Then we want to show $R[\![X]\!]$ is a free $R[\![G]\!]$-module with basis Y. Now G acts freely on $G \times Y$, so by the universal property of free R-modules it is enough to show that $X \cup -X = G \times Y \cup -(G \times Y)$ as topological spaces. The inclusion $Y \to X \cup -X$ gives a continuous map

$$G \times Y \to G \times (X \cup -X) \dot{\to} X \cup -X$$

and similarly for $-(G \times Y)$, after multiplying by -1. Hence we get a continuous map $G \times Y \cup -(G \times Y) \to X \cup -X$ which is bijective by the choice of Y, so the two are homeomorphic because they are compact and Hausdorff. □

Permutation modules behave nicely with respect to induction of modules; we want to show the same is true of signed permutation modules.

We first recall the definition of induction: on H-spaces, for $H \leq G$, we define Ind_H^G by the universal property that, given $X \in H\text{-}Pro$, $X' \in G\text{-}Pro$ and a continuous map $f : X \to X'$ as H-spaces, f factors uniquely through a map $f' : \operatorname{Ind}_H^G X \to X'$ of G-spaces. Clearly $\operatorname{Ind}_H^G X$ is unique up to isomorphism. In addition this property makes Ind_H^G a functor in the obvious way. Analogously, given $A \in P(R[\![H]\!]), B \in P(R[\![G]\!])$, Ind_H^G is defined by the universal property that a morphism $f : A \to B$ in $P(R[\![H]\!])$ factors uniquely through $f' : \operatorname{Ind}_H^G A \to B$ in $P(R[\![G]\!])$.

Recall also that, given $H \leq G$, it is possible to choose a closed left transversal T of H by [33, Proposition 1.3.2]. In other words, T is a closed subset of G containing exactly one element of each left coset of H in G. By [33, Proposition 1.3.4] we then have a homeomorphism $G \cong T \times H$ as spaces.

Lemma 5.1.4. *Let $H \leq G$, and suppose $R[\![X]\!] \in P(R[\![H]\!])$ is a signed permutation module. Then $\operatorname{Ind}_H^G R[\![X]\!]$ is a signed permutation module in $P(R[\![G]\!])$.*

Proof. Assume $\operatorname{char} R \neq 2$; otherwise we are done.

We know that $X \cup -X \in H\text{-}Pro$, and any composite map $f : X \cup -X \to R[\![X]\!] \to M$ for $M \in P(R[\![G]\!])$ satisfies $f(-x) = -f(x)$ for all $x \in X$.

Now $\operatorname{Ind}_H^G(X \cup -X)$ can be constructed in the following way: choose a closed left transversal T of H in G and take the space $T \times (X \cup -X)$ with the product topology. Every element of G can be written uniquely in the form th with $t \in T, h \in H$. So given $g \in G, t \in T$, write gt in the form $t'h$, $t' \in T, h \in H$ and define $g \cdot (t, x) = (t', h \cdot x)$. This gives an abstract group action on $T \times (X \cup -X)$ because, if $g_2 t = t' h_2$ and $g_1 t' = t'' h_1$, for $g_1, g_2 \in G, t, t', t'' \in T, h_1, h_2 \in H$, $g_1 g_2 t = t'' h_1 h_2$ and hence

$$g_1 \cdot (g_2 \cdot (t, x)) = g_1 \cdot (t', h_2 \cdot x) = (t'', h_1 \cdot (h_2 \cdot x)) = (t'', (h_1 h_2) \cdot x) = (g_1 g_2) \cdot (t, x);$$

to see the action is continuous, note that we can write the action as the following composite of continuous maps:

$$\begin{aligned} G \times T \times (X \cup -X) &\xrightarrow{m \times \mathrm{id}} G \times (X \cup -X) \\ &\xrightarrow{\theta \times \mathrm{id}} T \times H \times (X \cup -X) \\ &\xrightarrow{\mathrm{id} \times \alpha} T \times (X \cup -X). \end{aligned}$$

Here m is multiplication in G, θ is the homeomorphism $G \to T \times H$, and α is the H-action on $X \cup -X$. We claim that the space $T \times (X \cup -X)$, with this G-action, satisfies the universal property to be $\operatorname{Ind}_H^G(X \cup -X)$, where the canonical map $X \cup -X \to \operatorname{Ind}_H^G(X \cup -X)$ is given by $x \mapsto (1, x)$. Indeed, given $M \in P(R[\![G]\!])$ and a continuous map

$$f : X \cup -X \to M$$

of H-spaces such that $f(-x) = -f(x)$ for all $x \in X$, define

$$f' : T \times (X \cup -X) \to M, f' : (t, x) \mapsto t \cdot f(x) :$$

this is a G-map because, for $gt = t'h$, $g \in G, t, t' \in T, h \in H$,

$$f'(g \cdot (t, x)) = f'(t', h \cdot x) = t' \cdot f(h \cdot x) = (t'h) \cdot f(x) = g \cdot (t \cdot f(x)).$$

The uniqueness of this choice of map is clear. Moreover, we have

$$f'(t, -x) = t \cdot f(-x) = t \cdot (-f(x)) = -(t \cdot f(x)) = -f'(t, x),$$

and hence by the universal property of signed permutation modules f' extends uniquely to a morphism $R[\![T \times X]\!] \to M$, where $R[\![T \times X]\!]$ is the signed permutation module with the G-action on $T \times X \cup -(T \times X)$ given by the G-action on $T \times (X \cup -X)$. By the universal property of induced modules this $R[\![T \times X]\!]$ is $\operatorname{Ind}_H^G R[\![X]\!]$. □

If $R[\![X]\!]$ is a signed $R[\![G]\!]$ permutation module with $\overline{X} \cong G/H$ as G-spaces, we may write $R[\![X]\!] = R[\![G/H;\sigma]\!]$, where σ is the G-action on $X \cup -X$, with the understanding that G acts on $G/H \cup -G/H$, for each $g \in G, tH \in G/H$, by either $\sigma(g,tH) = gtH$ or $\sigma(g,tH) = -gtH$ (and similarly for $-tH \in -G/H$). When there is no ambiguity we may simply write $R[\![G/H]\!]$ for this. In particular, each element of H acts on the cosets $1H \cup -1H$ by multiplication by ± 1, giving a continuous homomorphism $\varepsilon : H \to \{\pm 1\}$, which we will refer to as the twist homomorphism of $R[\![G/H;\sigma]\!]$.

Lemma 5.1.5. *Write R' for a copy of R on which H acts by $h \cdot r = \varepsilon(h)r$. Then we have $\mathrm{Ind}_H^G R' = R[\![G/H;\sigma]\!]$.*

Proof. Assume $\operatorname{char} R \neq 2$; otherwise we are done.

By Lemma 5.1.4 we have that $\mathrm{Ind}_H^G R' = R[\![\mathrm{Ind}_H^G\{\pm 1\}]\!]$. We will show that, as G-spaces,

$$\mathrm{Ind}_H^G(\{\pm 1\}, \varepsilon) \cong (G/H \cup -G/H, \sigma).$$

Now by the choice of ε we have a continuous map

$$f : \{\pm 1\} \to G/H \cup -G/H, \pm 1 \mapsto \pm 1H$$

of H-spaces satisfying $f(-x) = -f(x)$, and the proof of Lemma 5.1.4 gives us a continuous map $f' : \mathrm{Ind}_H^G\{\pm 1\} \to G/H \cup -G/H$ of G-spaces extending this, such that $f'(-x) = -f'(x)$. Explicitly, choosing a closed left transversal T of H as before, $\mathrm{Ind}_H^G\{\pm 1\} = T \cup -T$, and $f'(t) = \sigma(t, 1H)$ for $t \in T$, $f'(t) = \sigma(t, -1H)$ for $t \in -T$. Now f' is bijective because every element of $G/H \cup -G/H$ can be written uniquely in the form $\sigma(t, 1H)$ or $\sigma(t, -1H)$ for some $t \in T$. Therefore f' is a homeomorphism, and we are done. □

Finally, we justify our introduction of signed permutation modules, instead of permutation modules. As stated at the beginning of the section, they are an attempt to deal with the tensor-induced complexes of [25, 7.3]. We sketch the construction of these complexes.

To fix notation, we start by defining wreath products. Given $G \in PGrp$, let G^n be the direct product in $PGrp$ of n copies of G. Let S_n be the symmetric group on n letters, acting on the right. Then the wreath product of G by S_n, written $G \wr S_n$, is the semidirect product of G^n and S_n, where S_n acts by permuting the copies of G. More explicitly, we can write $G \wr S_n$ as $G^n \times S_n$ as a space, with group operation

$$(h_1, \dots, h_n, \pi) \cdot (h'_1, \dots, h'_n, \pi') = (h_1 h'_{1\pi}, \dots, h_n h'_{n\pi}, \pi\pi').$$

Since the action of S_n on G^n is continuous, this makes $G \wr S_n$ a topological group, which is then profinite because it is compact, Hausdorff and totally disconnected.

Suppose $G \in PGrp$. Let P_* be a non-negative complex of profinite $R[\![G]\!]$-modules. Then one can take the n-fold tensor power of P_*, $P_*^{\hat{\otimes} n}$, over R by defining

$$P_k^{\hat{\otimes} n} = \bigoplus_{i_1 + \cdots + i_n = k} P_{i_1} \hat{\otimes}_R \cdots \hat{\otimes}_R P_{i_n},$$

with the differential maps coming from repeated use of the sign trick in [32, 1.2.5]: this gives a non-negative complex of profinite R-modules. Moreover, by [25, 7.3], it can be made into a complex of $R[\![G\wr S_n]\!]$-modules by the $G\wr S_n$-action

$$(h_1,\dots,h_n,\pi)\cdot(q_1\hat{\otimes}\cdots\hat{\otimes}q_n)=(-1)^\nu\cdot h_1q_{1\pi}\hat{\otimes}\cdots\hat{\otimes}h_nq_{n\pi}$$

where the $q_i\in P_*$ are homogeneous elements and ν is the integer

$$\nu=\sum_{i<j,i\pi>j\pi}\deg(q_{i\pi})\deg(q_{j\pi}).$$

We can now generalise [25, 7.4] slightly – the proof is largely the same.

Proposition 5.1.6. *Suppose*

$$\cdots\to R[\![X_1]\!]\to R[\![X_0]\!]\to 0$$

is an exact sequence in $P(R[\![G]\!])$ of signed permutation modules, and write P_* for this chain complex. Then $P_*^{\hat{\otimes}n}$ is an exact sequence of signed permutation modules in $P(R[\![G\wr S_n]\!])$.

Proof. Assume $\operatorname{char}R\neq 2$; the proof for $\operatorname{char}R=2$ is similar.

First note that each module in P_* is free as an R-module, so, for each i, $R[\![X_i]\!]\hat{\otimes}_R-$ is an exact functor on R-modules, and hence $P_*^{\hat{\otimes}n}$ is exact by $n-1$ applications of [32, Lemma 2.7.3]. Now as R-modules one has

$$P_k^{\hat{\otimes}n}=\bigoplus_{i_1+\cdots+i_n=k}R[\![X_{i_1}]\!]\hat{\otimes}_R\cdots\hat{\otimes}_RR[\![X_{i_n}]\!]=R[\![\bigsqcup_{i_1+\cdots+i_n=k}X_{i_1}\times\cdots\times X_{i_n}]\!]$$

by [23, Exercise 5.5.5(a)], so we simply need to show that

$$\bigsqcup_{i_1+\cdots+i_n=k}X_{i_1}\times\cdots\times X_{i_n}\cup-(\bigsqcup_{i_1+\cdots+i_n=k}X_{i_1}\times\cdots\times X_{i_n})$$

is a $G\wr S_n$-subspace of $R[\![\bigsqcup_{i_1+\cdots+i_n=k}X_{i_1}\times\cdots\times X_{i_n}]\!]$. If $x_1\hat{\otimes}\cdots\hat{\otimes}x_n$ is an element of this subspace, so that $x_j\in X_{i_j}\cup -X_{i_j}$ for each j, we have

$$(h_1,\dots,h_n,\pi)\cdot(x_1\hat{\otimes}\cdots\hat{\otimes}x_n)=(-1)^\nu\cdot h_1x_{1\pi}\hat{\otimes}\cdots\hat{\otimes}h_nx_{n\pi},$$

and then $i_{1\pi}+\cdots+i_{n\pi}=i_1+\cdots i_n=k$. Moreover, for each j we have

$$x_{j\pi}\in X_{i_{j\pi}}\cup -X_{i_{j\pi}}\Rightarrow h_jx_{j\pi}\in X_{i_{j\pi}}\cup -X_{i_{j\pi}},$$

as required. □

5.2 A Hierarchy of Profinite Groups

We define classes of groups and closure operations on them as in [19], except that all our groups are required to be profinite and all maps continuous. Thus, for example, all our subgroups will be assumed to be closed unless stated otherwise. As there, for a class of profinite groups $\mathfrak{X}$, we let $\mathbf{s}\mathfrak{X}$ be the class of closed subgroups of groups in $\mathfrak{X}$, and $\mathbf{L}\mathfrak{X}$ be those profinite groups G such that every finite subset of G is contained in some subgroup $H\leq G$ with $H\in\mathfrak{X}$. We also define a more general version $\mathbf{L}'$ of $\mathbf{L}$: $\mathbf{L}'\mathfrak{X}$ is the class of profinite groups G which have a direct system of subgroups $\{G_i\}$, ordered by inclusion, whose union is dense in G, such that $G_i \in \mathfrak{X}$ for every i. Given two classes $\mathfrak{X}$ and $\mathfrak{Y}$, we write $\mathfrak{X}\mathfrak{Y}$ for extensions of a group in $\mathfrak{X}$ by a group in $\mathfrak{Y}$.

Lastly, we define $\mathbf{H}_R\mathfrak{X}$ to be the profinite groups G for which there is an exact sequence $0 \to P_n \to \cdots \to P_0 \to R \to 0$ of $R[\![G]\!]$-modules, where, for each i, P_i is a signed permutation module, all of whose stabilisers are in $\mathfrak{X}$. We will refer to this as a finite length signed permutation resolution of G.

Note that $\mathbf{H}_R$ is not a closure operation. Instead, we use it to define inductively the class of groups $(\mathbf{H}_R)_\alpha\mathfrak{X}$ for each ordinal α: $(\mathbf{H}_R)_0\mathfrak{X} = \mathfrak{X}$, $(\mathbf{H}_R)_\alpha\mathfrak{X} = \mathbf{H}_R((\mathbf{H}_R)_{\alpha-1}\mathfrak{X})$ for α a successor, and $(\mathbf{H}_R)_\alpha\mathfrak{X} = \bigcup_{\beta<\alpha}(\mathbf{H}_R)_\beta\mathfrak{X}$ for α a limit. Finally, we write $\widehat{\mathbf{H}_R}\mathfrak{X} = \bigcup_\alpha(\mathbf{H}_R)_\alpha\mathfrak{X}$. It is easy to check that $\widehat{\mathbf{H}_R}$ is a closure operation.

Similarly we can define $(\mathbf{LH}_R)_0\mathfrak{X} = \mathfrak{X}$ and $(\mathbf{L}'\mathbf{H}_R)_0\mathfrak{X} = \mathfrak{X}$, then $(\mathbf{LH}_R)_\alpha\mathfrak{X} = \mathbf{LH}_R((\mathbf{LH}_R)_{\alpha-1}\mathfrak{X})$ and $(\mathbf{L}'\mathbf{H}_R)_\alpha\mathfrak{X} = \mathbf{L}'\mathbf{H}_R((\mathbf{L}'\mathbf{H}_R)_{\alpha-1}\mathfrak{X})$ for α a successor, and finally $(\mathbf{LH}_R)_\alpha\mathfrak{X} = \bigcup_{\beta<\alpha}(\mathbf{LH}_R)_\beta\mathfrak{X}$ and $(\mathbf{L}'\mathbf{H}_R)_\alpha\mathfrak{X} = \bigcup_{\beta<\alpha}(\mathbf{L}'\mathbf{H}_R)_\beta\mathfrak{X}$ for α a limit. Then let $\widehat{\mathbf{LH}_R}\mathfrak{X} = \bigcup_\alpha(\mathbf{LH}_R)_\alpha\mathfrak{X}$ and $\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X} = \bigcup_\alpha(\mathbf{L}'\mathbf{H}_R)_\alpha\mathfrak{X}$: this gives two more closure operations with $\widehat{\mathbf{H}_R}\mathfrak{X} \leq \widehat{\mathbf{LH}_R}\mathfrak{X} \leq \mathbf{L}'\widehat{\mathbf{LH}_R}\mathfrak{X} \leq \widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X}$ for all $\mathfrak{X}$. The final inequality holds because $\mathbf{L}'\widehat{\mathbf{LH}_R}\mathfrak{X} \leq \mathbf{L}'\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X}$ and

$$\mathbf{L}'(\mathbf{L}'\mathbf{H}_R)_\alpha\mathfrak{X} \leq \mathbf{L}'\mathbf{H}(\mathbf{L}'\mathbf{H}_R)_\alpha\mathfrak{X} = (\mathbf{L}'\mathbf{H}_R)_{\alpha+1}\mathfrak{X}, \forall\alpha \Rightarrow \mathbf{L}'\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X} = \widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X}.$$

Remark 5.2.1. In the abstract case, [19, 2.2] shows that any countable $\widehat{\mathbf{LH}_R}\mathfrak{X}$-group is actually in $\widehat{\mathbf{H}_R}\mathfrak{X}$, greatly diminishing the importance of $\mathbf{L}$, inasfar as the hierarchy is used to study finitely generated groups. The same argument does not work for profinite groups.

From now on, $\mathfrak{F}$ will mean the class of finite groups, and $\mathfrak{I}$ the class of the trivial group.

Proposition 5.2.2. *Let $\mathfrak{X}$ be a class of profinite groups.*

(i) $\mathbf{S}\widehat{\mathbf{H}_R}\mathfrak{X} \leq \widehat{\mathbf{H}_R}\mathbf{S}\mathfrak{X}$.

(ii) $(\widehat{\mathbf{H}_R}\mathbf{S}\mathfrak{X})\mathfrak{F} \leq \widehat{\mathbf{H}_R}\mathbf{S}(\mathfrak{X}\mathfrak{F})$.

(iii) $(\widehat{\mathbf{H}_R}\mathfrak{F})(\widehat{\mathbf{H}_R}\mathfrak{F}) = \widehat{\mathbf{H}_R}\mathfrak{F}$.

Proof. (i) Use induction on α. We will show that $\mathbf{S}(\mathbf{H}_R)_\alpha\mathfrak{X} \leq (\mathbf{H}_R)_\alpha\mathbf{S}\mathfrak{X}$ for each α. The case when α is 0 or a limit ordinal is trivial. Suppose $G \in \mathbf{S}(\mathbf{H}_R)_{\alpha+1}\mathfrak{X}$ and pick $H \in (\mathbf{H}_R)_{\alpha+1}\mathfrak{X}$ with $G \leq H$. Take a finite length signed permutation resolution of H with stabilisers in $(\mathbf{H}_R)_\alpha\mathfrak{X}$. Restricting this resolution to G gives a finite length signed permutation resolution whose stabilisers are subgroups of the stabilisers in the original resolution of H, so the stabilisers are in $\mathbf{S}(\mathbf{H}_R)_\alpha\mathfrak{X} \leq (\mathbf{H}_R)_\alpha\mathbf{S}\mathfrak{X}$, where the inequality holds by our inductive hypothesis, and hence $G \in (\mathbf{H}_R)_{\alpha+1}\mathbf{S}\mathfrak{X}$.

(ii) Use induction on α. We will show that $((\mathbf{H}_R)_\alpha\mathbf{S}\mathfrak{X})\mathfrak{F} \leq (\mathbf{H}_R)_\alpha\mathbf{S}(\mathfrak{X}\mathfrak{F})$ for each α. The case when α is 0 or a limit ordinal is trivial. So suppose $G \in ((\mathbf{H}_R)_{\alpha+1}\mathbf{S}\mathfrak{X})\mathfrak{F}$, and suppose $H \trianglelefteq_{\text{open}} G$, $H \in (\mathbf{H}_R)_{\alpha+1}\mathbf{S}\mathfrak{X}$. Take a finite length signed permutation resolution of H with stabilisers in $(\mathbf{H}_R)_\alpha\mathbf{S}\mathfrak{X}$. Then we get a finite length signed permutation resolution of $H \wr S_{|G/H|}$

by Proposition 5.1.6. Moreover, G embeds in $H \wr S_{|G/H|}$ by [25, 7.1], so by restriction this is also a finite length signed permutation resolution of G. Finally, it is clear from the construction that the stabilisers under the G-action are all finite extensions of subgroups of stabilisers in the original resolution of H, which are in $(\mathbf{H}_R)_\alpha \mathbf{S}\mathfrak{X}$ by (i) and our inductive hypothesis. Therefore this tensor-induced complex shows that $G \in (\mathbf{H}_R)_{\alpha+1}\mathbf{S}\mathfrak{X}\mathfrak{F}$.

(iii) Use induction on α. We will show that $(\widehat{\mathbf{H}_R}\mathfrak{F})((\mathbf{H}_R)_\alpha\mathfrak{F}) \leq \widehat{\mathbf{H}_R}\mathfrak{F}$ for each α. The other inequality is clear. The case when α is a limit ordinal is trivial; the case $\alpha = 0$ holds by (ii). Suppose $G \in (\widehat{\mathbf{H}_R}\mathfrak{F})((\mathbf{H}_R)_{\alpha+1}\mathfrak{F})$ and pick $H \trianglelefteq G$ such that $H \in \widehat{\mathbf{H}_R}\mathfrak{F}$ and $G/H \in (\mathbf{H}_R)_{\alpha+1}\mathfrak{F}$. Take a finite length signed permutation resolution of G/H with stabilisers in $(\mathbf{H}_R)_\alpha\mathfrak{F}$. Restricting this resolution to G gives a finite length signed permutation resolution whose stabilisers are extensions of H by stabilisers in the original resolution of G/H, so the stabilisers are in $(\widehat{\mathbf{H}_R}\mathfrak{F})((\mathbf{H}_R)_\alpha\mathfrak{F}) \leq \widehat{\mathbf{H}_R}\mathfrak{F}$, where the inequality holds by our inductive hypothesis, and hence $G \in \widehat{\mathbf{H}_R}\mathfrak{F}$.

□

Proposition 5.2.3. *Let $\mathfrak{X}$ be a class of profinite groups.*

(i) $\mathbf{S}\widehat{\mathbf{LH}_R}\mathfrak{X} \leq \widehat{\mathbf{LH}_R}\mathbf{S}\mathfrak{X}$.

(ii) $(\widehat{\mathbf{LH}_R}\mathbf{S}\mathfrak{X})\mathfrak{F} \leq \widehat{\mathbf{LH}_R}\mathbf{S}(\mathfrak{X}\mathfrak{F})$.

(iii) $(\widehat{\mathbf{LH}_R}\mathfrak{F})(\widehat{\mathbf{LH}_R}\mathfrak{F}) = \widehat{\mathbf{LH}_R}\mathfrak{F}$.

Proof. (i) Use induction on α. We will show $\mathbf{SH}_R(\mathbf{LH}_R)_\alpha\mathfrak{X} \leq \mathbf{H}_R(\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X}$ and hence that $\mathbf{S}(\mathbf{LH}_R)_{\alpha+1}\mathfrak{X} \leq (\mathbf{LH}_R)_{\alpha+1}\mathbf{S}\mathfrak{X}$ for each α. The case when α is 0 or a limit ordinal is trivial. Suppose first that $G_1 \in \mathbf{SH}_R(\mathbf{LH}_R)_\alpha\mathfrak{X}$ and pick $H_1 \in \mathbf{H}_R(\mathbf{LH}_R)_\alpha\mathfrak{X}$ with $G_1 \leq H_1$. Take a finite length signed permutation resolution of H_1 with stabilisers in $(\mathbf{LH}_R)_\alpha\mathfrak{X}$. Restricting this resolution to G_1 gives a finite length signed permutation resolution whose stabilisers are subgroups of the stabilisers in the original resolution of H_1, so the stabilisers are in $\mathbf{S}(\mathbf{LH}_R)_\alpha\mathfrak{X} \leq (\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X}$, where the inequality holds by our inductive hypothesis, and hence $G_1 \in \mathbf{H}_R(\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X}$. Suppose next that $G_2 \in \mathbf{S}(\mathbf{LH}_R)_{\alpha+1}\mathfrak{X}$ and pick $H_2 \in (\mathbf{LH}_R)_{\alpha+1}\mathfrak{X}$ with $G_2 \leq H_2$. Every finitely generated subgroup of H_2 is contained in some $K \leq H_2$ with $K \in \mathbf{H}_R(\mathbf{LH}_R)_\alpha\mathfrak{X}$, so every finitely generated subgroup of H_2 is in $\mathbf{H}_R(\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X}$ by our inductive hypothesis. In particular this is true for the finitely generated subgroups of G_2, and hence $G_2 \in (\mathbf{LH}_R)_{\alpha+1}\mathbf{S}\mathfrak{X}$.

(ii) Use induction on α. The case when α is 0 or a limit ordinal is trivial. We will show that $((\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X})\mathfrak{F} \leq (\mathbf{LH}_R)_\alpha\mathbf{S}(\mathfrak{X}\mathfrak{F})$ for each α. So suppose $G \in ((\mathbf{LH}_R)_{\alpha+1}\mathbf{S}\mathfrak{X})\mathfrak{F}$, and suppose $H \trianglelefteq_{\text{open}} G$, $H \in (\mathbf{LH}_R)_{\alpha+1}\mathbf{S}\mathfrak{X}$. It suffices to prove that every finitely generated subgroup of G belongs to $\mathbf{H}_R(\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X}$, and so we may assume that G is finitely generated. This implies H is finitely generated, by [23, Proposition 2.5.5], so $H \in \mathbf{H}_R(\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X}$. Take a finite length signed permutation resolution of H with stabilisers in $(\mathbf{LH}_R)_\alpha\mathbf{S}\mathfrak{X}$. Then we get a finite length signed permutation resolution of $H \wr S_{|G/H|}$ by Proposition 5.1.6. Moreover, G embeds in $H \wr S_{|G/H|}$ by [25, 7.1], so by restriction this is also a finite length signed permutation resolution of G. Finally, it is clear from the construction that the stabilisers under the G-action are all finite extensions of subgroups of stabilisers in the signed permutation resolution of H, which are in $(\mathbf{LH}_R)_\alpha \mathbf{S}\mathfrak{X}$ by (i) and our inductive hypothesis. Therefore this tensor-induced complex shows that $G \in \mathbf{H}_R(\mathbf{LH}_R)_\alpha \mathbf{S}\mathfrak{X}$.

(iii) Use induction on α. We will show that $(\widehat{\mathbf{LH}_R}\mathfrak{F})((\mathbf{LH}_R)_\alpha \mathfrak{F}) \leq \widehat{\mathbf{LH}_R}\mathfrak{F}$ for each α. The other inequality is clear. The case when α is a limit ordinal is trivial; the case $\alpha = 0$ holds by (ii). Suppose $G \in (\widehat{\mathbf{LH}_R}\mathfrak{F})((\mathbf{LH}_R)_{\alpha+1}\mathfrak{F})$ and pick $H \trianglelefteq G$ such that $H \in \widehat{\mathbf{LH}_R}\mathfrak{F}$ and $G/H \in (\mathbf{LH}_R)_{\alpha+1}\mathfrak{F}$. It suffices to prove that every finitely generated subgroup of G belongs to $\widehat{\mathbf{LH}_R}\mathfrak{F}$, and so we may assume that G is finitely generated. This implies G/H is finitely generated, so $G/H \in \mathbf{H}_R(\mathbf{LH}_R)_\alpha \mathfrak{F}$. Take a finite length signed permutation resolution of G/H with stabilisers in $(\mathbf{LH}_R)_\alpha \mathfrak{F}$. Restricting this resolution to G gives a finite length signed permutation resolution whose stabilisers are extensions of H by stabilisers in the original resolution of G/H, so the stabilisers are in $(\widehat{\mathbf{LH}_R}\mathfrak{F})((\mathbf{LH}_R)_\alpha \mathfrak{F}) \leq \widehat{\mathbf{LH}_R}\mathfrak{F}$, where the inequality holds by our inductive hypothesis, and hence $G \in \widehat{\mathbf{LH}_R}\mathfrak{F}$.

□

Remark 5.2.4. The reason we sometimes use $\mathbf{L}$ rather than $\mathbf{L}'$ is that $\mathbf{L}$ is closed under extensions; if one could show the same was true for $\mathbf{L}'$ then one could construct a class containing all elementary amenable groups (see below) for which the main result would hold. However, we can still recover 'most' elementary amenable groups using a combination of $\widehat{\mathbf{LH}_R}\mathfrak{F}$ and $\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{F}$.

We can also compare the classes produced by using different base rings.

Lemma 5.2.5. *Suppose S is a commutative profinite R-algebra. Then $\widehat{\mathbf{H}_R}\mathfrak{X} \leq \widehat{\mathbf{H}_S}\mathfrak{X}$, $\widehat{\mathbf{LH}_R}\mathfrak{X} \leq \widehat{\mathbf{LH}_S}\mathfrak{X}$ and $\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X} \leq \widehat{\mathbf{L}'\mathbf{H}_S}\mathfrak{X}$.*

Proof. Clearly $\mathfrak{X} \leq \mathfrak{Y} \Rightarrow \mathbf{L}\mathfrak{X} \leq \mathbf{L}\mathfrak{Y}$ and $\mathbf{L}'\mathfrak{X} \leq \mathbf{L}'\mathfrak{Y}$, so we just need to show $\mathfrak{X} \leq \mathfrak{Y} \Rightarrow \mathbf{H}_R\mathfrak{X} \leq \mathbf{H}_S\mathfrak{Y}$: then it will follow by induction that for each α that $(\mathbf{H}_R)_\alpha\mathfrak{X} \leq (\mathbf{H}_S)_\alpha\mathfrak{X}$, $(\mathbf{LH}_R)_\alpha\mathfrak{X} \leq (\mathbf{LH}_S)_\alpha\mathfrak{X}$ and $(\mathbf{L}'\mathbf{H}_R)_\alpha\mathfrak{X} \leq (\mathbf{L}'\mathbf{H}_S)_\alpha\mathfrak{X}$, as required. Given $G \in \mathbf{H}_R\mathfrak{X}$ and a finite length signed permutation resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0 \tag{$*$}$$

with stabilisers in $\mathfrak{X}$, note that, since every module in the sequence is R-free, the sequence is R-split, so the sequence

$$0 \to S\hat{\otimes}_R P_n \to S\hat{\otimes}_R P_{n-1} \to \cdots \to S\hat{\otimes}_R P_0 \to S\hat{\otimes}_R R \cong S \to 0 \tag{$**$}$$

is exact – here each module is made into an $S[\![G]\!]$-module by taking the S-action on S and the G-action on P_i.

Now, for a signed $R[\![G]\!]$ permutation module $R[\![X]\!]$, $S\hat{\otimes}_R R[\![X]\!] = S[\![X]\!]$ as S-modules by [33, Proposition 7.7.8], and then clearly the G-action makes this into a signed $S[\![G]\!]$ permutation module. So, applying this to $(**)$, we have a finite length signed permutation resolution of S as an $S[\![G]\!]$-module, and the stabilisers are all in $\mathfrak{X}$ because the stabilisers in $(*)$ are, so we are done. □

The next lemma gives a profinite analogue of the Eilenberg swindle; it is very similar to [33, Exercise 11.7.3(a)], though using a slightly different definition of free modules. Recall that projective modules are summands of free ones.

Lemma 5.2.6. *Suppose $P \in P(\Lambda)$ is projective, where Λ is a profinite R-algebra. Then there is a free $F \in P(\Lambda)$ such that $P \oplus F$ is free.*

Proof. Take $Q \in P(\Lambda)$ projective such that $P \oplus Q$ is free on some space X. Take F to be a countably infinite coproduct of copies of $Q \oplus P$ in $P(\Lambda)$: by the universal properties of coproducts and free modules, F is free on (the profinite completion of) a countably infinite disjoint union of copies of X. So is

$$P \oplus F = P \oplus Q \oplus P \oplus Q \oplus \cdots,$$

for the same reason. □

Note that, in the same way, for a summand P of a signed permutation module in $P(R[\![G]\!])$ there is a signed permutation module F such that $P \oplus F$ is a signed permutation module. It is this trick that allows us to define $\mathbf{H}_R$ using finite length resolutions of signed permutation modules, rather than resolutions of summands of signed permutation modules, without losing anything: we can always replace a resolution of the latter kind with one of the former. In particular we get the following corollary.

We define the cohomological dimension of a profinite group G over R, $\mathrm{cd}_R\, G$, to be $\mathrm{pd}_{R[\![G]\!]}\, R$, the minimal length of a projective resolution of R in $R[\![G]\!]$, where R has trivial G-action.

Corollary 5.2.7. *Groups of finite cohomological dimension over R are in $\mathbf{H}_R\mathfrak{I}$.*

Proof. Put $\Lambda = R[\![G]\!]$. Given a finite length projective resolution of R,

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0,$$

we can assume $P_0, \ldots, P_{n-1}$ are free. Indeed, one can see this inductively: if $P_0, \ldots, P_{i-1}$ are free, $i \leq n-1$, take some Q such that $P_i \oplus Q$ is free, and replace P_i, P_{i+1} with $P_i \oplus Q, P_{i+1} \oplus Q$, with the map between them given by $(P_{i+1} \to P_i) \oplus \mathrm{id}_Q$. Then take a free module F such that $P_n \oplus F$ is free:

$$0 \to P_n \oplus F \to P_{n-1} \oplus F \to P_{n-2} \to \cdots \to P_0 \to R \to 0$$

gives the required resolution. □

Let π be a finite set of primes.

Proposition 5.2.8. *Elementary amenable pro-π groups are in $\widehat{\mathbf{LH}_{\hat{\mathbb{Z}}}}\mathfrak{F}$.*

Proof. We use induction on the elementary amenable class α; the case $\alpha = 0$ is trivial. The case of limit ordinals is also trivial. So suppose α is a successor, and suppose $\mathscr{X}_{\alpha-1} \leq \widehat{\mathbf{LH}_{\hat{\mathbb{Z}}}}\mathfrak{F}$. Then $\mathbf{L}\mathscr{X}_{\alpha-1} \leq \mathbf{L}\widehat{\mathbf{LH}_{\hat{\mathbb{Z}}}}\mathfrak{F} \leq \mathbf{LH}_{\hat{\mathbb{Z}}}\widehat{\mathbf{LH}_{\hat{\mathbb{Z}}}}\mathfrak{F} = \widehat{\mathbf{LH}_{\hat{\mathbb{Z}}}}\mathfrak{F}$. Suppose $G \in \mathscr{X}_\alpha$, and take a normal subgroup $G_1 \in \mathbf{L}\mathscr{X}_{\alpha-1}$ such that G/G_1 is in $\mathscr{X}_1$. Now G/G_1 is virtually torsion-free finitely generated abelian, so it has a finite index subgroup which is torsion-free abelian and hence this subgroup has finite cohomological dimension by [33, Proposition 8.2.1, Theorem 11.6.9]. Therefore by Corollary 5.2.7 it is in $\mathbf{H}_{\hat{\mathbb{Z}}}\mathfrak{I} \leq \mathbf{H}_{\hat{\mathbb{Z}}}\mathfrak{F}$, and hence G/G_1 is in $\mathbf{H}_{\hat{\mathbb{Z}}}\mathfrak{F}$ too by Proposition 5.2.2(ii). Therefore $G \in \widehat{\mathbf{LH}_{\hat{\mathbb{Z}}}}\mathfrak{F}$ by Proposition 5.2.3(iii). □

Now we note that many elementary amenable profinite groups are prosoluble: these include soluble profinite groups, and by the Feit-Thompson theorem they include all elementary amenable pro-$2'$ groups, where $2'$ is the set of all primes but 2.

Corollary 5.2.9. *Elementary amenable prosoluble groups are in* $\widehat{\mathbf{L}'\mathbf{H}_{\hat{\mathbb{Z}}}}\mathfrak{F}$.

Proof. We show that these groups are in $\mathbf{L}'\widehat{\mathbf{L}\mathbf{H}_{\hat{\mathbb{Z}}}}\mathfrak{F}$. By [23, Proposition 2.3.9], prosoluble groups G have a Sylow basis; that is, a choice $\{S_p : p \text{ prime}\}$ of one Sylow subgroup for each p such that $S_pS_q = S_qS_p$ for each p, q. Therefore, writing p_n for the nth prime, we have a subgroup $G_n = S_{p_1} \cdots S_{p_n}$ for each n, and hence a direct system $\{G_n\}$ of subgroups of G whose union is dense in G. By Proposition 5.2.8 each G_n is in $\widehat{\mathbf{L}\mathbf{H}_{\hat{\mathbb{Z}}}}\mathfrak{F}$, so we are done. □

Note that in fact this shows that, for any prosoluble group G – and hence any profinite group of odd order – if each G_n is in some $\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X}$, in the same notation as above, then G is too.

Profinite groups acting on profinite trees with well-behaved stabilisers give further examples of groups in our class, in the spirit of [19, 2.2(iii)], though the profinite case seems to be rather harder to control here than the abstract one. See [23, Chapter 9.2] for the definitions of (proper) pro-$\mathcal{C}$ free products with amalgamation.

Lemma 5.2.10. *Suppose G_1, G_2, H are pro-$\mathcal{C}$ groups, where $\mathcal{C}$ is a class of finite groups closed under taking subgroups, quotients and extensions. Suppose we have $G_1, G_2, H \in \widehat{\mathbf{L}\mathbf{H}_R}\mathfrak{X}$ (or $\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X}$). Write $G_1 *_H G_2$ for the free pro-$\mathcal{C}$ product of G_1 and G_2 with amalgamation by H, and suppose it is proper. Then $G_1 *_H G_2 \in \widehat{\mathbf{L}\mathbf{H}_R}\mathfrak{X}$ (or $\widehat{\mathbf{L}'\mathbf{H}_R}\mathfrak{X}$).*

Proof. We get a finite length permutation resolution from [13, Theorem 2.1]. □

We finish this section by listing, for convenience, some groups in $\widehat{\mathbf{L}'\mathbf{H}_{\hat{\mathbb{Z}}}}\mathfrak{F}$.

- Finite groups (with the discrete topology) are in $\mathfrak{F}$.
- Profinite groups of finite virtual cohomological dimension over $\hat{\mathbb{Z}}$ are in $\mathbf{H}_{\hat{\mathbb{Z}}}\mathfrak{F}$, by Corollary 5.2.7 and Proposition 5.2.3(ii). Hence:
- Free profinite groups are in $\mathbf{H}_{\hat{\mathbb{Z}}}\mathfrak{F}$.
- Soluble profinite groups are in $(\mathbf{L}'\mathbf{H}_{\hat{\mathbb{Z}}})_\omega\mathfrak{F}$, by Corollary 5.2.9.
- Elementary amenable pro-p groups are in $\widehat{\mathbf{L}'\mathbf{H}_{\hat{\mathbb{Z}}}}\mathfrak{F}$ for all p, by Proposition 5.2.8, and elementary amenable profinite groups of odd order are too, by Corollary 5.2.9.

Finally, for G a compact p-adic analytic group, G is a virtual Poincaré duality group at the prime p by [31, Theorem 5.1.9], and hence by definition G has finite virtual cohomological dimension over $\mathbb{Z}_p$, and so $G \in \mathbf{H}_{\mathbb{Z}_p}\mathfrak{F}$. In particular this includes $\mathbb{Z}_p$-linear groups by [33, Proposition 8.5.1].

5.3 Type L Systems

To be able to use the hierarchy of groups defined in the last section, we want to relate the construction of a group within the hierarchy to its cohomology, and so gain results about the structure of the group, analogously to [19]. Specifically, this section will deal with the interaction of cohomology and the closure operation $\mathbf{L}$, and the next one with the interaction of cohomology and $\mathbf{H}_R$.

For the rest of the chapter, we will be studying profinite groups of type FP_∞, so unless stated otherwise we will write $\mathbf{Hom}_\Lambda(-,-)$ for $\mathbf{Hom}_\Lambda^{(P_0,P)}(-,-)$, $\mathrm{Ext}_\Lambda^n(-,-)$ for $\mathrm{Ext}_\Lambda^{(P_\infty,P),n}(-,-)$ and $H_R^n(G,-)$ for $H_R^{(P_\infty,P),n}(G,-)$.

We call a direct system $\{A^i : i \in I\}$ of Λ-modules a Type L system if there is some $i_0 \in I$ such that the maps $f^{i_0 i} : A^{i_0} \to A^i$ for each $i \geq i_0$ are all epimorphisms. Then, considering $\{A^i\}$ as a direct system in $T(\Lambda)$,

$$U(\varinjlim_{T(\Lambda)} A^i) = \varinjlim_{Mod(U(\Lambda))} U(A^i) = U(A^{i_0})/\bigcup_{i \geq i_0} \ker(Uf^{i_0 i})$$

by [32, Lemma 2.6.14] and the remark after Proposition 2.1.3. We can see from this, and from the proof of Proposition 2.1.3, that $\varinjlim_{T(\Lambda)} A^i$ has as its underlying module $U(A^{i_0})/\bigcup_{i\geq i_0} \ker(Uf^{i_0 i})$, with the strongest topology making each map

$$f^i : A^i \to U(A^{i_0})/\bigcup_{i \geq i_0} \ker(Uf^{i_0 i})$$

continuous, such that $U(A^{i_0})/\bigcup_{i\geq i_0} \ker(Uf^{i_0 i})$ is made into a topological Λ-module. But the quotient topology induced by the map f^{i_0} satisfies these conditions: it makes $U(A^{i_0})/\bigcup_{i\geq i_0} \ker(Uf^{i_0 i})$ into a topological Λ-module by [3, III.6.6]; it makes f^{i_0} continuous; it makes each f^i, $i \geq i_0$ continuous because, given an open set U in $A^{i_0}/\bigcup ker(f^{i_0 i})$,

$$(f^{i_0})^{-1}(U) = (f^{i_0 i})^{-1}(f^i)^{-1}(U)$$

is open in A^{i_0}, and A^i has the quotient topology coming from $f^{i_0 i}$ (because all the modules are compact and Hausdorff), and by the definition of the quotient topology this means that $(f^i)^{-1}(U)$ is open in A^i, as required. Hence

$$\varinjlim_{T(\Lambda)} A^i \cong A^{i_0}/\bigcup_{i\geq i_0} \ker(f^{i_0 i})$$

as topological modules. Note that this quotient is compact, as the continuous image of A^{i_0}.

Recall from Proposition 2.2.4 that we know $\varinjlim_{P(\Lambda)} A^i$ is the profinite completion of $\varinjlim_{T(\Lambda)} A^i$. Hence there is a canonical homomorphism

$$\phi : \varinjlim_{T(\Lambda)} A^i \to \varinjlim_{P(\Lambda)} A^i.$$

Since $\varinjlim_{P(\Lambda)} A^i$ is Hausdorff, $\ker(\phi) = \phi^{-1}(0)$ is closed in $\varinjlim_{T(\Lambda)} A^i$; in particular, $\ker(\phi)$ contains the closure of $\{0\}$ in $\varinjlim_{T(\Lambda)} A^i$. Hence ϕ factors (uniquely) as

$$\varinjlim_{T(\Lambda)} A^i \cong A^{i_0}/\bigcup_{i\geq i_0} \ker(f^{i_0 i}) \xrightarrow{\psi} A^{i_0}/\overline{\bigcup_{i\geq i_0} \ker(f^{i_0 i})} \to \varinjlim_{P(\Lambda)} A^i.$$

Now $A^{i_0}/\overline{\bigcup_{i\geq i_0} \ker(f^{i_0 i})}$ is a quotient of a profinite Λ-module by a closed submodule, so it is profinite; hence, by the universal property of profinite completions, ψ factors uniquely through ϕ. It follows that

$$\varinjlim_{P(\Lambda)} A^i \cong A^{i_0}/\overline{\bigcup_{i\geq i_0} \ker(f^{i_0 i})}.$$

Given $A \in P(\Lambda)$, we can think of A as an object of $P(R)$ by restriction. This functor is representable in the sense that it is given by $\mathbf{Hom}_\Lambda(\Lambda, -) : P(\Lambda) \to P(R)$.

Lemma 5.3.1. *Direct limits of Type L systems commute with restriction. Explicitly, let $\{A^i\}$ be a Type L system in $P(\Lambda)$. Then*

$$\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(\Lambda, A^i) = \mathbf{Hom}_\Lambda(\Lambda, \varinjlim_{P(\Lambda)} A^i).$$

Proof. By our construction of direct limits in $P(\Lambda)$, both sides are just the restriction to $P(R)$ of $A^{i_0}/\overline{\bigcup_{i \geq i_0} \ker(f^{i_0 i})}$, given the quotient topology. □

It follows by additivity that

$$\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P, A^i) = \mathbf{Hom}_\Lambda(P, \varinjlim_{P(\Lambda)} A^i)$$

for all finitely generated projective $P \in P(\Lambda)$.

Proposition 5.3.2. *Suppose $M \in P(\Lambda)$ is of type* FP_∞*, and let $\{A^i\}$ be a Type L system in $P(\Lambda)$. Then for each n we have an epimorphism*

$$\varinjlim_{P(R)} \mathrm{Ext}^n_\Lambda(M, A^i) \to \mathrm{Ext}^n_\Lambda(M, \varinjlim_{P(\Lambda)} A^i).$$

Proof. We show this in two stages. Take a projective resolution

$$\cdots \to P_2 \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \to 0$$

of M with each P_n finitely generated. We will show first that

$$H^n(\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_*, A^i)) = \mathrm{Ext}^n_\Lambda(M, \varinjlim_{P(\Lambda)} A^i).$$

To see this, consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_0, A^i) & \longrightarrow & \varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_1, A^i) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Hom}_\Lambda(P_0, \varinjlim_{P(\Lambda)} A^i) & \longrightarrow & \mathbf{Hom}_\Lambda(P_1, \varinjlim_{P(\Lambda)} A^i) & \longrightarrow & \cdots
\end{array}$$

in $P(R)$. The homology of the top row is

$$H^n(\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_*, A^i)),$$

the homology of the bottom row is $\mathrm{Ext}^n_\Lambda(M, \varinjlim_{P(\Lambda)} A^i)$, and the previous lemma shows that the vertical maps are all isomorphisms.

The second stage is to give epimorphisms

$$\varinjlim_{P(R)} \mathrm{Ext}^n_\Lambda(M, A^i) \to H^n(\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_*, A^i)).$$

Recall that $\varinjlim_{P(R)}$ is right-exact, so that we get an exact sequence

$$\begin{aligned}\varinjlim_{P(R)} \ker(\mathbf{Hom}_\Lambda(f_n, A^i)) \to & \varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_n, A^i) \\ \to & \varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_{n+1}, A^i),\end{aligned}$$

and hence an epimorphism

$$\varinjlim_{P(R)} \ker(\mathbf{Hom}_\Lambda(f_n, A^i)) \to \ker(\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(f_n, A^i)).$$

Now consider the commutative diagram

$$\begin{array}{ccccccc}
\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_{n-1}, A^i) & \longrightarrow & \varinjlim_{P(R)} \ker(\mathbf{Hom}_\Lambda(f_n, A^i)) & \longrightarrow & \varinjlim_{P(R)} \mathrm{Ext}^n_\Lambda(M, A^i) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow & & \downarrow & & \\
\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_{n-1}, A^i) & \longrightarrow & \ker(\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(f_n, A^i)) & \longrightarrow & H^n(\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_*, A^i)) & \longrightarrow & 0
\end{array}$$

whose top row is exact because $\varinjlim_{P(R)}$ is right-exact, and whose bottom row is exact by definition of homology. It follows by the Five Lemma that

$$\varinjlim_{P(R)} \mathrm{Ext}^n_\Lambda(M, A^i) \to H^n(\varinjlim_{P(R)} \mathbf{Hom}_\Lambda(P_*, A^i))$$

is an epimorphism, as required. $\square$

The next lemma will allow us to make new Type L systems from old ones.

Lemma 5.3.3. *Suppose $G \in PGrp$. Suppose $M \in P(R[\![G]\!])$ is projective as an R-module, by restriction, and let $\{A^i\}$ be a Type L system in $P(R[\![G]\!])$. Then $\{M \hat{\otimes}_R A^i\}$ is a Type L system in $P(R[\![G]\!])$, where each $M \hat{\otimes}_R A^i$ is given the diagonal G-action.*

Proof. Because $M\hat{\otimes}_R -$ preserves epimorphisms, we just need to show that it commutes with direct limits of Type L systems; that is, we have to show that if $\{A^i, f^{ij}\}$ is a Type L system, then

$$M\hat{\otimes}_R \varinjlim_{P(R)} A^i \cong \varinjlim_{P(R)} (M\hat{\otimes}_R A^i).$$

We have a canonical homomorphism

$$g : \varinjlim_{P(R)} (M\hat{\otimes}_R A^i) \to M\hat{\otimes}_R \varinjlim_{P(R)} A^i;$$

it is an epimorphism because the epimorphism

$$g^{i_0} : M\hat{\otimes}_R A^{i_0} \to M\hat{\otimes}_R \varinjlim_{P(R)} A^i$$

factors as

$$M\hat{\otimes}_R A^{i_0} \xrightarrow{h} \varinjlim_{P(R)} (M\hat{\otimes}_R A^i) \xrightarrow{g} M\hat{\otimes}_R \varinjlim_{P(R)} A^i.$$

In fact we will show that $\ker g^{i_0} \subseteq \ker h$; this implies that g is injective, as required.

Now

$$\ker g^{i_0} = M\hat{\otimes}_R \ker f^{i_0} = M\hat{\otimes}_R \overline{\bigcup_{i \geq i_0} \ker f^{i_0 i}}$$

by the exactness of $M\hat{\otimes}_R-$ (because M is R-projective) and the construction of direct limits of Type L systems; moreover, writing $g^{i_0 i}$ for $M\hat{\otimes}_R A^{i_0} \to M\hat{\otimes}_R A^i$, we have by exactness of $M\hat{\otimes}_R-$ again that $\ker g^{i_0 i} = M\hat{\otimes}_R \ker f^{i_0 i}$. Hence, by the construction of direct limits of Type L systems,

$$\ker h = \overline{\bigcup_{i \geq i_0} \ker g^{i_0 i}} = \overline{\bigcup_{i \geq i_0} M\hat{\otimes}_R \ker f^{i_0 i}}.$$

Thus we are reduced to showing that the subspace

$$\bigcup_{i \geq i_0} M\hat{\otimes}_R \ker f^{i_0 i} \subseteq M\hat{\otimes}_R \overline{\bigcup_{i \geq i_0} \ker f^{i_0 i}}$$

is dense. This can be seen by considering inverse limits: if $M = \varprojlim M_j$, then $\overline{\bigcup_{i \geq i_0} \ker f^{i_0 i}} = \varprojlim N_k$, with all the M_j, N_k finite, then

$$M\hat{\otimes}_R \overline{\bigcup_{i \geq i_0} \ker f^{i_0 i}} = \varprojlim M_j \hat{\otimes}_R N_k,$$

and by the denseness of $\bigcup_{i \geq i_0} \ker f^{i_0 i}$ in $\overline{\bigcup_{i \geq i_0} \ker f^{i_0 i}}$, for each k there is some i such that $\ker f^{i_0 i} \to N_k$ is surjective, so $M_j \hat{\otimes}_R \ker f^{i_0 i} \to M_j \hat{\otimes}_R N_k$ is too. Denseness follows by [23, Lemma 1.1.7]. □

Finally, we need one more result to apply this to the problem of getting information about group structure. Suppose $G \in PGrp$, let H be a subgroup of G, and let $\{H_i\}$ be a direct system of (closed) subgroups of H, with inclusion maps between them, whose union H' is dense in H – note that H' is an (abstract) subgroup of H, because the system is direct. Thus we get a corresponding direct system $\{R[\![G/H_i]\!]\}$ of $R[\![G]\!]$ permutation modules whose maps come from quotients $G/H_i \to G/H_j$. Note that this system is Type L, because the maps $R[\![G/H_i]\!] \to R[\![G/H_j]\!]$ are all epimorphisms.

Lemma 5.3.4. $\varinjlim_{P(R[\![G]\!])} R[\![G/H_i]\!] = R[\![G/H]\!]$. *Hence* $\{R[\![G/H_i]\!]\}$ *is Type L.*

Proof. Recall that, for $X \in G\text{-}Pro$ and $M \in P(R[\![G]\!])$, we let $C_G(X, M)$ be the $U(R)$-module of continuous G-maps $X \to M$. We have

$$\begin{aligned}
\mathrm{Hom}_{R[\![G]\!]}(\varinjlim_{P(R[\![G]\!])} R[\![G/H_i]\!], M) &= \varprojlim_{Mod(U(R))} \mathrm{Hom}_{R[\![G]\!]}(R[\![G/H_i]\!], M) \\
&= \varprojlim_{Mod(U(R))} C_G(G/H_i, M) \\
&= C_G(\varinjlim_{G\text{-}Pro} G/H_i, M) \\
&= \mathrm{Hom}_{R[\![G]\!]}(R[\![\varinjlim_{G\text{-}Pro} G/H_i]\!], M) :
\end{aligned}$$

the first and third equalities hold by the universal property of colimits; the second and fourth hold by the universal property of permutation modules, Lemma 5.1.1. Since this holds for all M, we have $\varinjlim_{P(R[\![G]\!])} R[\![G/H_i]\!] = R[\![\varinjlim_{G\text{-}Pro} G/H_i]\!]$, so we just need to show that $\varinjlim_{G\text{-}Pro} G/H_i = G/H$.

To see this, we will show first that $\varinjlim_{G\text{-}Top} G/H_i = G/H'$. Note that we have compatible epimorphisms $G/H_i \to G/H'$, and hence an epimorphism $f : \varinjlim_{G\text{-}Top} G/H_i \to G/H'$. Note also that the maps $G/H_i \to \varinjlim_{G\text{-}Top} G/H_i$ are surjective. Suppose $f(x) = f(y)$ for $x, y \in \varinjlim_{G\text{-}Top} G/H_i$. Take a representative x' of x in some G/H_{i_1}, and a representative y' of y in some G/H_{i_2}. Now the images of x' and y' are in the same left coset of H' in G, i.e. $x'h_1 = y'h_2$ for some $h_1, h_2 \in H'$. Write H_j for the subgroup of H' generated by h_1, h_2, x and y. Thus the images of x' and y' are in the same left coset of H_j in G, i.e. x' and y' have the same image in G/H_j and hence in $\varinjlim_{G\text{-}Top} G/H_i$, so $x = y$, and f is injective. Finally, note that, in exactly the same way as the construction of Type L direct limits, $\varinjlim_{G\text{-}Top} G/H_i$ has the quotient topology coming from G, which is the same as the one on G/H'. Thus, by Proposition 2.2.4, $\varinjlim_{G\text{-}Pro} G/H_i$ is the profinite completion of G/H', which is just $G/\overline{H'} = G/H$ by the same argument as for Type L systems. □

By Lemma 5.3.1, $\varinjlim_{P(R)} R[\![G/H_i]\!] = R[\![G/H]\!]$ as well; indeed, by the same lemma, any compatible collection of G-actions on these modules gives a direct limit whose underlying R-module is $R[\![G/H]\!]$, and whose G-action is just the one coming from any of the quotient maps

$$f_i : R[\![G/H_i]\!] \to R[\![G/H]\!].$$

So if $R[\![G/H; \sigma]\!]$ is a signed $R[\![G]\!]$ permutation module, define $R[\![G/H_i; \sigma_i]\!]$ for each i to be a signed $R[\![G]\!]$ permutation module by the G-action $\sigma_i(g, x) = gx$ if $\sigma(g, f_i(x)) = gf_i(x)$ and $\sigma_i(g, x) = -gx$ if $\sigma(g, f_i(x)) = -gf_i(x)$, for all $g \in G, x \in G/H_i \cup -G/H_i$. Clearly these G-actions are all compatible, and they have as their direct limit (in $P(\Lambda)$) $R[\![G/H; \sigma]\!]$. In particular, we get the following result.

Corollary 5.3.5. *Given a signed $R[\![G]\!]$ permutation module $R[\![G/H; \sigma]\!]$, and a direct system $\{H_i\}$ of subgroups of H whose union is dense in H, there is a Type L system of signed permutation modules of the form $R[\![G/H_i; \sigma_i]\!]$ whose direct limit is $R[\![G/H; \sigma]\!]$.*

5.4 Type H Systems

Suppose $A \in P(\Lambda)$ has the form $\varprojlim_{j\in J} A_j$, where each $A_j \in P(\Lambda)$ is finite. Suppose in addition that each A_j is a direct sum $A_{j,1} \oplus \cdots \oplus A_{j,n_j}$ of Λ-modules such that, whenever $j_1 \geq j_2$, the morphism $\phi_{j_1 j_2} : A_{j_1} \to A_{j_2}$ has the property that, for each k, $\phi_{j_1 j_2}(A_{j_1,k})$ is contained in some $A_{j_2,k'}$. Then we say A has the structure of a Type H system.

In the same notation, write I_j for the set $\{1, \ldots, n_j\}$. Then the structure of the Type H system induces a map $\psi_{j_1 j_2} : I_{j_1} \to I_{j_2}$ for each $j_1 \geq j_2$ in J, giving an inverse system $\{I_j : j \in J\}$: if $\phi_{j_1 j_2}(A_{j_1,k}) \subseteq A_{j_2,k'}$, define $\psi_{j_1 j_2}(k) = k'$. Write I for the inverse limit and ι_j for the map $I \to I_j$. I is clearly profinite, because it is the inverse limit of a system of finite sets; also I is non-empty by [23, Proposition 1.1.4]. Now pick $i \in I$. We call $A^i = \varprojlim_j A_{j,\iota_j(i)}$ the ith component of A.

Proposition 5.4.1. *Suppose $M \in P(\Lambda)$ is of type FP_∞. Suppose $A \in P(\Lambda)$ has the structure of a Type H system. Suppose $\{A^i : i \in I\}$ are the components of A. Then for each n we have an epimorphism*

$$\bigoplus_{P(R),i} \mathrm{Ext}^n_\Lambda(M, A^i) \to \mathrm{Ext}^n_\Lambda(M, A).$$

Proof. Many aspects of the Type H structure carry over to $\mathrm{Ext}^n_\Lambda(M, A)$.

$$\mathrm{Ext}^n_\Lambda(M, A) = \varprojlim_{P(R),j} \mathrm{Ext}^n_\Lambda(M, A_j)$$

by [31, Theorem 3.7.2], and similarly

$$\mathrm{Ext}^n_\Lambda(M, A^i) = \varprojlim_{P(R),j} \mathrm{Ext}^n_\Lambda(M, A_{j,\iota_j(i)}) \qquad (*)$$

for each $i \in I$. By additivity,

$$\mathrm{Ext}^n_\Lambda(M, A_j) = \mathrm{Ext}^n_\Lambda(M, A_{j,1}) \oplus \cdots \oplus \mathrm{Ext}^n_\Lambda(M, A_{j,n_j}).$$

Note that each $\mathrm{Ext}^n_\Lambda(M, A_j)$ and $\mathrm{Ext}^n_\Lambda(M, A_{j,k})$ is finite, because M is of type FP_∞, and there are only finitely many homomorphisms from a finitely generated Λ-module to a finite one.

Write C_j for the image of

$$g_j : \mathrm{Ext}^n_\Lambda(M, A) \to \mathrm{Ext}^n_\Lambda(M, A_j) :$$

then by [23, Corollary 1.1.8(a)] we know $\mathrm{Ext}^n_\Lambda(M, A) = \varprojlim_{P(R),j} C_j$. We claim that

$$f_j : \bigoplus_{P(R),i} \mathrm{Ext}^n_\Lambda(M, A^i) \to C_j$$

is an epimorphism for each j, and then the proposition will follow by [23, Corollary 1.1.6]. To see this claim, fix some j, and suppose that the image of f_j is some submodule $C'_j \neq C_j$. We will obtain a contradiction by showing that the image of f_j is strictly larger than C'_j.

Now define, for $j' \geq j$, $I'_{j'} \subseteq I_{j'}$ to be those elements k of $I_{j'}$ for which the image of $\mathrm{Ext}^n_\Lambda(M, A_{j',k})$ in $\mathrm{Ext}^n_\Lambda(M, A_j)$ is not contained in C'_j. For each $j' \geq j$ the map g_j factors as

$$\mathrm{Ext}^n_\Lambda(M, A) \xrightarrow{g_{j'}} \mathrm{Ext}^n_\Lambda(M, A_{j'}) \xrightarrow{g_{j'j}} \mathrm{Ext}^n_\Lambda(M, A_j),$$

so $\mathrm{im}(g_{j'j}) \supseteq \mathrm{im}(g_{j'}) = C_j$; hence $I'_{j'} \neq \varnothing$, and so $I' = \varprojlim_{j' \geq j} I'_{j'} \neq \varnothing$ by [23, Proposition 1.1.4].

Pick $i \in I'$. By definition of I', $\mathrm{Ext}^n_\Lambda(M, A_{j,\iota_j(i)})$ is not contained in C'_j, so $\mathrm{Ext}^n_\Lambda(M, A_{j,\iota_j(i)}) \setminus C'_j \neq \varnothing$. Suppose that, for each $x \in \mathrm{Ext}^n_\Lambda(M, A_{j,\iota_j(i)}) \setminus C'_j$, there is some $j_x \geq j$ such that

$$x \notin \mathrm{im}(f_{j_x,\iota_{j_x}(i)} : \mathrm{Ext}^n_\Lambda(M, A_{j_x,\iota_{j_x}(i)}) \to \mathrm{Ext}^n_\Lambda(M, A_j)).$$

Since J is directed, there is some $j_0 \in J$ such that $j_0 \geq j_x$ for all x in the finite set $\mathrm{Ext}^n_\Lambda(M, A_{j,\iota_j(i)}) \setminus C'_j$. For each such x, $\mathrm{im}(f_{j_0,\iota_{j_0}(i)}) \subseteq \mathrm{im}(f_{j_x,\iota_{j_x}(i)})$ and hence $x \notin \mathrm{im}(f_{j_0,\iota_{j_0}(i)})$, so that $\mathrm{im}(f_{j_0,\iota_{j_0}(i)}) \subseteq C'_j$. But we chose $\iota_{j_0}(i)$ to be in I'_{j_0}, so $\mathrm{im}(f_{j_0,\iota_{j_0}(i)}) \nsubseteq C'_j$, contradicting our supposition. Therefore there must be some $x \in \mathrm{Ext}^n_\Lambda(M, A_{j,\iota_j(i)}) \setminus C'_j$ such that, for every $j' \geq j$, $x \in \mathrm{im}(f_{j',\iota_{j'}})$.

Write f^i_j for the map $\mathrm{Ext}^n_\Lambda(M, A^i) \to \mathrm{Ext}^n_\Lambda(M, A_j)$, so that by $(*)$ we have $f^i_j = \varprojlim_{j'} f_{j',\iota_{j'}}$. For every $j' \geq j$ we have $f^{-1}_{j',\iota_{j'}}(x) \neq \varnothing$, and hence, taking inverse limits over j', we get $(f^i_j)^{-1}(x) \neq \varnothing$ by [23, Proposition 1.1.4], so that $x \in \mathrm{im}(f^i_j) \setminus C'_j$. Finally, it is clear from the definitions that $\mathrm{im}(f_j) \supseteq \mathrm{im}(f^i_j)$, so $x \in \mathrm{im}(f_j) \setminus C'_j$, proving our claim and giving the result. □

As in the last section, we want to be able to make new Type H systems from old ones.

Lemma 5.4.2. *Suppose $G \in PGrp$. Suppose $M, A \in P(R[\![G]\!])$, $M = \varprojlim_k M_k$, and let $A = \varprojlim_j A_{j,1} \oplus \cdots \oplus A_{j,n_j}$ have the structure of a Type H system in $P(\Lambda)$. Suppose $\{A^i : i \in I\}$ are the components of A. Then $M \hat{\otimes}_R A \in P(R[\![G]\!])$, with the diagonal G-action, has the structure of a Type H system given by $\varprojlim_{j,k}(M_k \hat{\otimes}_R A_{j,1}) \oplus \cdots \oplus (M_k \hat{\otimes}_R A_{j,n_j})$ with components $\{M \hat{\otimes}_R A^i\}$.*

Proof. This is immediate, since $\hat{\otimes}_R$ commutes with $\varprojlim$ and finite direct sums commute with both. □

Again, this section finishes with a couple of lemmas allowing us to get information about group structure.

Lemma 5.4.3. *Suppose $G \in PGrp$, and suppose $\{X_j\}$ is an inverse system in G-Pro with $X = \varprojlim_j X_j$. If each X_j has a single G-orbit, so does X.*

Proof. Write ϕ_j for the map $X \to X_j$. For $Y \subseteq X$, define $G \cdot Y = \{gy : g \in G, y \in Y\}$. Pick $x \in X$. Then $\phi_j(G \cdot \{x\}) = G \cdot \phi_j(\{x\}) = X_j$ for each j, because each X_j has a single G-orbit, so $G \cdot \{x\} = \varprojlim_j X_j = X$ by [23, Corollary 1.1.8(a)]. Hence the orbit of x is the whole of X. □

Lemma 5.4.4. *Suppose $G \in PGrp$, and suppose $R[\![X]\!] \in P(R[\![G]\!])$ is a signed permutation module. Then $R[\![X]\!]$ has the structure of a Type H system whose components are signed permutation modules $R[\![X^i]\!]$, where the X^i are the G-orbits of $R[\![X]\!]$.*

Proof. By Lemma 5.1.2 we can write $X \cup -X = \varprojlim X_j \cup -X_j$, where the $X_j \cup -X_j$ are finite quotients of $X \cup -X$ preserving the algebraic structure. If $R = \varprojlim_l R_l$, $R[\![X]\!] = \varprojlim_{j,l} R_l[X_j]$, and each $R_l[X_j]$ is a signed $R_l[\![G]\!]$ permutation module. Now as a G-space $\overline{X_j} = (X_j \cup -X_j)/\sim$ is the disjoint union of its orbits $\overline{X_{j,1}}, \ldots, \overline{X_{j,n_j}}$, where $\sim$ is the relation $x \sim -x$, so $X_j \cup -X_j$ is the disjoint union of G-spaces $X_{j,1} \cup -X_{j,1}, \ldots, X_{j,n_j} \cup -X_{j,n_j}$. Therefore we get

$$R_l[X_j] = R_l[X_{j,1}] \oplus \cdots \oplus R_l[X_{j,n_j}].$$

For $l_1 \geq l_2$ and $j_1 \geq j_2$, write $\phi_{(l_1,j_1)(l_2,j_2)}$ for the map $R_{l_1}[X_{j_1}] \to R_{l_2}[X_{j_2}]$. For each orbit $\overline{X_{j_1,k_1}}$,

$$\phi_{(l_1,j_1)(l_2,j_2)}(X_{j_1,k_1} \cup -X_{j_1,k_1}) = X_{j_2,k_2} \cup -X_{j_2,k_2},$$

where $\overline{X_{j_2,k_2}}$ is the image of $\overline{X_{j_1,k_1}}$ in $\overline{X_{j_2}}$ (since $\overline{X_{j_1,k_1}}$ has only one orbit, so any G-map image of it has one orbit too). Therefore

$$\phi_{(l_1,j_1)(l_2,j_2)}(R_{l_1}[X_{j_1,k_1}]) = R_{l_2}[X_{j_2,k_2}],$$

and hence we have the structure of a Type H system.

Now write $I_j = \{1, \ldots, n_j\}$, define the maps $I_{j'} \to I_j$ for $j' \geq j$ coming from the Type H structure, and let $I = \varprojlim_j I_j$, $\iota_j : I \to I_j$. We give a bijection between I and the set of orbits of $\overline{R[\![X]\!]}$. Given a G-orbit X' of $R[\![X]\!]$, any G-map image of it has one orbit too, so for each j, l the image of

$$R[\![X']\!] \to R[\![X]\!] \to R_l[X_j]$$

must be contained in some $R_l[X_{j,i_j}]$ with one orbit. Define the element $i \in I$ to be the inverse limit over j of i_j and define the map $b : \{\text{orbits of } R[\![X]\!]\} \to I$ by $b(X') = i$.

Conversely, for $i \in I$, each $\overline{X_{j,\iota_j(i)}}$ has a single G-orbit, so $\overline{X^i} = \varprojlim_j \overline{X_{j,\iota_j(i)}}$ does too, by Lemma 5.4.3. It is easy to see the map $i \mapsto X^i$ is inverse to b, giving the result. □

5.5 The Main Result

We can now use these results to get information about groups of type FP_∞ in $\widehat{\mathbf{L'H}_R\mathfrak{F}}$. Given abelian categories C, D, define a $(-\infty, \infty)$ cohomological functor from C to D to be a sequence of additive functors $T^i : C \to D$, $i \in \mathbb{Z}$, with natural connecting homomorphisms such that for every short exact sequence $0 \to L \to M \to N \to 0$ in C we get a long exact sequence

$$\cdots \to T^{n-1}(N) \to T^n(L) \to T^n(M) \to T^n(N) \to \cdots.$$

We start by giving a (slight) generalisation of [19, 3.1], which holds for all $(-\infty, \infty)$ cohomological functors. The proof is a dimension-shifting argument which goes through entirely unchanged.

Lemma 5.5.1. *Let T^* be a $(-\infty, \infty)$ cohomological functor from C to D. Let*

$$0 \to M_r \to M_{r-1} \to \cdots \to M_0 \to L \to 0$$

be an exact sequence in D. If $T^i(L) \neq 0$ for some i then $T^{i+j}(M_j) \neq 0$ for some $0 \leq j \leq r$.

Define $H^n_R(G,-) = 0$ for $n < 0$. The functors $H^*_R(G,-)$ thus defined form a $(-\infty,\infty)$ cohomological functor from $P(R[\![G]\!])$ to $P(R)$.

The following theorem corresponds roughly to [19, 3.2].

Theorem 5.5.2. *Suppose $G \in \widehat{\mathbf{L'H}_R}\mathfrak{X}$ is of type* FP_∞. *Then there is some subgroup $H \leq G$ which is in $\mathfrak{X}$, some signed $R[\![G]\!]$ permutation module $R[\![G/H;\sigma]\!]$ and some n such that $H^n_R(G, R[\![G/H;\sigma]\!]) \neq 0$.*

Proof. Note first that $H^0_R(G,R) = R \neq 0$.

Consider the collection $\mathcal{O}$ of ordinals β for which there exists $i \geq 0$ and $H \leq G$ such that $H \in (\mathbf{L'H}_R)_\beta\mathfrak{X}$ and $H^i_R(G, R[\![G/H;\tau]\!]) \neq 0$, for some signed $R[\![G]\!]$ permutation module $R[\![G/H;\tau]\!]$. It suffices to prove $0 \in \mathcal{O}$. Observe first that $\mathcal{O}$ is non-empty, because $G \in (\mathbf{L'H}_R)_\alpha\mathfrak{X}$ for some α, and then $\alpha \in \mathcal{O}$ by hypothesis. So we need to show that if $0 \neq \beta \in \mathcal{O}$, there is some $\gamma < \beta$ such that $\gamma \in \mathcal{O}$.

So suppose $H \in (\mathbf{L'H}_R)_\beta\mathfrak{X}$ and $H^i_R(G, R[\![G/H;\tau]\!]) \neq 0$. If β is a limit, H is in $(\mathbf{L'H}_R)_\gamma\mathfrak{X}$ for some $\gamma < \beta$, and we are done; so assume β is a successor ordinal. Now pick a direct system $\{H_j\}$ of subgroups of H whose union is dense in H, with $H_j \in \mathbf{H}_R(\mathbf{L'H}_R)_{\beta-1}\mathfrak{X}$ for each j. Then we have a Type L system $\{R[\![G/H_j;\tau_j]\!]\}$ whose direct limit is $R[\![G/H;\tau]\!]$ by Corollary 5.3.5, so we have an epimorphism

$$\varinjlim_{P(R),j} H^i_R(G, R[\![G/H_j;\tau_j]\!]) \to H^i_R(G, R[\![G/H;\tau]\!])$$

by Proposition 5.3.2: thus there is some j such that $H^i_R(G, R[\![G/H_j;\tau_j]\!]) \neq 0$ too.

Suppose $R[\![G/H_j;\tau_j]\!]$ has twist homomorphism $\delta : H_j \to \{\pm 1\}$, and write R' for a copy of R on which H_j acts by $h \cdot r = \delta(h)r$. Recall that $H_j \in \mathbf{H}_R(\mathbf{L'H}_R)_{\beta-1}\mathfrak{X}$; take a finite length signed permutation resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

of R as a trivial $R[\![H_j]\!]$-module with stabilisers in $(\mathbf{L'H}_R)_{\beta-1}\mathfrak{X}$, and apply induction $\mathrm{Ind}^G_{H_j}(-\hat{\otimes}_R R')$, where $-\hat{\otimes}_R R'$ is given the diagonal H_j action, to get a sequence

$$\begin{aligned} 0 \to \mathrm{Ind}^G_{H_j}(P_n\hat{\otimes}_R R') &\to \mathrm{Ind}^G_{H_j}(P_{n-1}\hat{\otimes}_R R') \to \cdots \\ &\to \mathrm{Ind}^G_{H_j}(P_0\hat{\otimes}_R R') \to \mathrm{Ind}^G_{H_j}(R\hat{\otimes}_R R') \to 0 \end{aligned}$$

which is exact by [23, Theorem 6.10.8(c)]. Now

$$\mathrm{Ind}^G_{H_j}(R\hat{\otimes}_R R') = R[\![G/H_j;\tau_j]\!]$$

by Lemma 5.1.5; hence, by Lemma 5.5.1,

$$H^{i+r}_R(G, \mathrm{Ind}^G_{H_j}(P_r\hat{\otimes}_R R')) \neq 0$$

for some $0 \leq r \leq n$. Now $\mathrm{Ind}^G_{H_j}(P_r\hat{\otimes}_R R')$ is a signed permutation module P by Lemma 5.1.4, so it has the structure of a Type H system, by Lemma 5.4.4, with components of the form $R[\![G/K;\tau']\!]$, some $K \in (\mathbf{L'H}_R)_{\beta-1}\mathfrak{X}$. By Proposition 5.4.1,

$$\bigoplus_{P(R),K} H^{i+r}_R(G, R[\![G/K;\tau']\!]) \to H^{i+r}_R(G,P)$$

is an epimorphism, so there is some K such that $H_R^{i+r}(G, R[\![G/K;\tau']\!]) \neq 0$. Since $K \in (\mathbf{LH}_R)_{\beta-1}\mathfrak{X}$, this completes the inductive step of the proof. □

In particular, if the only $\mathfrak{X}$-subgroup of G is the trivial one, by Lemma 5.1.3 there is some n such that $H_R^n(G, R[\![G]\!]) \neq 0$. If $\mathfrak{X} = \mathfrak{F}$, we can say slightly more.

Corollary 5.5.3. *Suppose $G \in \widehat{\mathbf{L'H}_R}\mathfrak{F}$ is of type* FP_∞. *Then there is some n such that $H_R^n(G, R[\![G]\!]) \neq 0$.*

Proof. From Theorem 5.5.2 we know there is a finite $H \leq G$, a σ and an n such that $H_R^n(G, R[\![G/H;\sigma]\!]) \neq 0$. Pick an open normal subgroup $U \leq G$ such that $U \cap H = 1$: such a subgroup exists because the open normal subgroups of G form a fundamental system of neighbourhoods of the identity by [23, Lemma 2.1.1]. Then the Lyndon-Hochschild-Serre spectral sequence [31, Theorem 4.2.6] gives that $H_R^i(U, R[\![G/H;\sigma]\!]) \neq 0$, for some $i \leq n$. As a U-space, the stabiliser of the coset $gH, g \in G$, has the form

$$U \cap H^{g^{-1}} = U^{g^{-1}} \cap H^{g^{-1}} = (U \cap H)^{g^{-1}} = 1;$$

hence G/H is free as a U-space, so as an $R[\![U]\!]$-module $R[\![G/H;\sigma]\!]$ is finitely generated and free by Lemma 5.1.3, and additivity gives $H_R^i(U, R[\![U]\!]) \neq 0$. Now

$$\begin{aligned} H_R^i(G, R[\![G]\!]) = &= H_R^i(G, Ind_U^G R[\![U]\!]) \\ &= H_R^i(G, Coind_U^G R[\![U]\!]) && \text{by [31, (3.3.7)]} \\ &= H_R^i(U, R[\![U]\!]) \neq 0 && \text{by [23, Theorem 10.6.5]}, \end{aligned}$$

as required. □

As in [19, Theorem A], there is no particular reason to restrict from Ext-functors to group cohomology: all we need to know is that the first variable of these functors is of type FP_∞ over $R[\![G]\!]$, and that it is projective on restriction to R. We sketch the proof of the theorem which follows from this observation; it is almost exactly the same as the proof of Theorem 5.5.2.

Theorem 5.5.4. *Suppose $G \in \widehat{\mathbf{L'H}_R}\mathfrak{X}$. Suppose $M \in P(R[\![G]\!])$ is projective as an R-module, by restriction, and is of type* FP_∞ *over $R[\![G]\!]$. Then there is some subgroup $H \leq G$ which is in $\mathfrak{X}$, some signed $R[\![G]\!]$ permutation module $R[\![G/H;\sigma]\!]$ and some n such that* $\mathrm{Ext}^n_{R[\![G]\!]}(M, M\hat{\otimes}_R R[\![G/H;\sigma]\!]) \neq 0$.

Proof. Replace $H_R^i(G, -)$ with $\mathrm{Ext}^i_{R[\![G]\!]}(M, -)$. Replace the signed permutation module coefficients $R[\![X]\!]$ of these functors with $M\hat{\otimes}_R R[\![X]\!]$, with the diagonal G-action. Then the proof goes through as before, after noting three things: that $M\hat{\otimes}_R-$ preserves Type L structures by Lemma 5.3.3, that it preserves Type H structures by Lemma 5.4.2, and that it preserves exactness of finite length signed permutation resolutions because finite length signed permutation resolutions of R are R-split. □

Once again, if the only $\mathfrak{X}$-subgroup of G is the trivial one, by Lemma 5.1.3 there is some n such that $\mathrm{Ext}^n_{R[\![G]\!]}(M, M\hat{\otimes}_R R[\![G]\!]) \neq 0$. There is also a result corresponding to Corollary 5.5.3.

5.6 Soluble Groups

We now establish some properties of nilpotent profinite groups; here we take nilpotent to mean that a group's (abstract) upper central series becomes the whole group after finitely many steps. All these results correspond closely to known ones in the abstract case, but there doesn't seem to be a good profinite reference, so they are included here.

Lemma 5.6.1. *Each term in the upper central series of a profinite group is closed.*

Proof. We show first that $Z_1(G)$ is closed. For each $g \in G$, the centraliser $C_G(g)$ of g is the inverse image of 1 in the continuous map $G \to G, x \mapsto [g, x]$, so it is closed. Then $Z_1(G) = \bigcap_{g \in G} C_G(g)$ is closed.

Now we use induction: suppose $Z_{i-1}(G)$ is closed. We know the centre of $G/Z_{i-1}(G)$ is closed, and $Z_i(G)$ is the preimage of $Z(G/Z_{i-1}(G))$ under the projection $G \to G/Z_{i-1}(G)$; hence $Z_i(G)$ is closed too. □

Thus nilpotent profinite groups are exactly the profinite groups which are nilpotent as abstract groups.

Since G is nilpotent as an abstract group, write

$$G = C^1_{abs}(G) \rhd C^2_{abs}(G) \rhd \cdots$$

for the terms of the abstract lower central series of G, and define the profinite upper central series by $C^n(G) = \overline{C^n_{abs}(G)}$. Each $C^n(G)$ is normal, as the closure of a normal subgroup. Moreover, since $[G, C^n_{abs}(G)] = C^{n+1}_{abs}(G)$, we have $\overline{[G, C^n(G)]} = \overline{[G, C^n_{abs}(G)]} = C^{n+1}(G)$. If G has nilpotency class k, $C^{k+1}_{abs}(G) = 1 \Rightarrow C^{k+1}(G) = 1$. In particular $C^n(G)$ has nilpotency class $k + 1 - n$, for $n \leq k$.

Lemma 5.6.2. *Suppose G is a finitely generated nilpotent profinite group of nilpotency class k. Then every subgroup $H \leq G$ is finitely generated.*

Proof. Let X be a finite generating set for G. Write G^{abs} for the dense subgroup of G generated abstractly by X, and $C^n_{abs}(G^{abs})$ for the terms in its (abstract) upper central series. Now

$$\overline{C^2_{abs}(G^{abs})} = \overline{[G^{abs}, G^{abs}]} = \overline{[\overline{G^{abs}}, \overline{G^{abs}}]} = \overline{[G, G]} = C^2(G).$$

By [24, 5.2.17], $C^2_{abs}(G^{abs})$ is abstractly finitely generated, so its closure $C^2(G)$ is topologically finitely generated.

We now prove the lemma by induction on k: when $k = 1$, G is abelian, and we are done by [33, Proposition 8.2.1]. So suppose the result holds for $k - 1$. Since $C^2(G)$ has class $k - 1$ and $G/C^2(G)$ has class 1, by hypothesis $H \cap C^2(G)$ and $H/(H \cap C^2(G))$ are both finitely generated, and hence H is too. □

Lemma 5.6.3. *Suppose G is a finitely generated torsion-free nilpotent profinite group. Then G is poly-(torsion-free procyclic).*

Proof. Suppose G has nilpotency class k. Consider the upper central series of G,

$$1 \lhd Z_1(G) \lhd \cdots \lhd Z_k(G) = G.$$

If we show that every factor $Z_{j+1}(G)/Z_j(G)$ is torsion-free, then it will follow by Lemma 5.6.2 that every $Z_{i+1}(G)/Z_i(G)$ is finitely generated torsion-free abelian, hence poly-(torsion-free procyclic), and we will be done. Both these facts are known in the abstract case.

Clearly $Z_1(G)$ is torsion-free, and we use induction on k, on the hypothesis that $Z_{j+1}(G)/Z_j(G)$ is torsion-free whenever G of nilpotency class k has $Z_1(G)$ torsion-free. $k = 1$ is trivial. For $k > 1$, we show first that $Z_2(G)/Z_1(G)$ is torsion-free by showing, for each $1 \neq x \in Z_2(G)/Z_1(G)$, that there is some $\phi \in Hom(Z_2(G)/Z_1(G), Z_1(G))$ such that $\phi(x) \neq 1$. Then the result follows because $Z_1(G)$ is torsion-free. So pick a preimage x' of x in $Z_2(G)$. $x' \notin Z_1(G)$, so there is some $g \in G$ such that $1 \neq [g, x'] \in Z_1(G)$. Now define

$$\phi' : Z_2(G) \to Z_1(G), y \mapsto [g, y];$$

note that ϕ' is a homomorphism, because

$$[g, y_1y_2] = [g, y_1][y_1, [g, y_2]][g, y_2] = [g, y_1][g, y_2],$$

since $[g, y_2] \in Z_1(G) \Rightarrow [y_1, [g, y_2]] = 1$. $\phi'(x') \neq 1$, and $Z_1(G) \leq ker(\phi')$, so this induces $\phi : Z_2(G)/Z_1(G) \to Z_1(G)$ such that $\phi(x) \neq 1$, as required.

By hypothesis, the centre of $G/Z_1(G)$ being torsion-free implies that

$$Z_{j+1}(G)/Z_j(G) = Z_j(G/Z_1(G))/Z_{j-1}(G/Z_1(G))$$

is torsion-free, for each j. □

By [33, Proposition 8.1.1], the class of profinite groups of finite rank is closed under taking subgroups, quotients and extensions. Procyclic groups have finite rank by [33, Proposition 8.2.1], and hence Lemma 5.6.3 shows that finitely generated torsion-free nilpotent profinite groups have finite rank. Note too that this implies that such groups are of type FP_∞ over any R, by Proposition 4.4.2.

Lemma 5.6.4. *Let G be a finitely generated torsion-free nilpotent pro-p group of nilpotency class k, and let F be a free profinite $\mathbb{Z}_p[\![G]\!]$-module. If $H^n_{\mathbb{Z}_p}(G, F) \neq 0$, then $k \leq n$ and G has rank $\leq n$.*

Proof. Suppose G has Hirsch length m. Note that $k \leq m$ by Lemma 5.6.3. Then Lemma 5.6.3 gives also that $cd_{\mathbb{Z}_p} G = m$, by [31, Proposition 4.3.1], and that G is a Poincaré duality group in dimension m by [31, Theorem 5.1.9]. Hence $H^i_{\mathbb{Z}_p}(G, F) = 0$ for $i \neq m$, and so $m = n \geq k$. G is built, by extensions, out of n groups of rank 1, so G has rank $\leq n$, by repeated applications of [33, Proposition 8.1.1(b)]. □

Lemma 5.6.5. *Let G be a profinite group, and suppose $M \in PD(R[\![G]\!])$ such that $H^{(IP,PD),n}_R(G, M) \neq 0$ for some i. If H is a subnormal subgroup of G, there is some $i \leq n$ such that $H^{(IP,PD),i}_R(H, M) \neq 0$.*

Proof. For H normal, we use the Lyndon-Hochschild-Serre spectral sequence, Theorem 3.3.12. For H subnormal, we have a sequence

$$H = G_m \lhd G_{m-1} \lhd \cdots \lhd G_0 = G,$$

and we use the spectral sequence repeatedly to show that for each $0 \leq k \leq m$ there is some $n_k \leq n$ such that $H^{(IP,PD),n_k}_R(G_k, M) \neq 0$. □

Lemma 5.6.6. *Every subgroup H of a profinite nilpotent group G is subnormal.*

Proof. Consider the upper central series of G,

$$1 \lhd Z_1(G) \lhd \cdots \lhd Z_k(G) = G.$$

Then

$$H \leq HZ_1(G) \leq \cdots \leq HZ_k(G) = G$$

gives a subnormal series for H: to see that $HZ_i(G)$ is normal in $HZ_{i+1}(G)$, note that H clearly normalises $HZ_i(G)$, and $Z_{i+1}(G)$ does because

$$[Z_{i+1}(G), HZ_i(G)] \leq [Z_{i+1}(G), G] \leq Z_i(G) \leq HZ_i(G),$$

so $HZ_{i+1}(G)$ does too. □

For abstract groups, the Fitting subgroup is defined to be the join of the nilpotent normal subgroups. [33, Section 8.4] defines a profinite Fitting subgroup of a profinite group G as the inverse limit of the Fitting subgroups of the finite quotients of G; this is not the definition we will use. Instead we define the *abstract Fitting subgroup* to be the abstract subgroup generated by the nilpotent normal closed subgroups of G.

Proposition 5.6.7. *Let G be a torsion-free pro-p group, N its abstract Fitting subgroup, and $\bar{N} \geq N$ the closure of N in G. If there is some free profinite $\mathbb{Z}_p[\![G]\!]$-module $F \in PD(\mathbb{Z}_p[\![G]\!])$ such that $H^{(IP,PD),n}_{\mathbb{Z}_p}(G, F) \neq 0$, then $\bar{N}$ is nilpotent of nilpotency class and rank $\leq n$.*

Proof. We claim the join of any finite collection $N_1, \ldots, N_m$ of nilpotent normal closed subgroups of G is nilpotent, normal and closed. Consider the abstract join N' of these as subgroups of an abstract group: then it is known that N' is nilpotent and normal. Moreover, because all the subgroups are normal, $N' = N_1 \cdots N_m$, which is closed in G, so N' is the join of $N_1, \ldots, N_m$ as profinite subgroups of G, and we are done.

So we can see N as the directed union of the nilpotent normal subgroups of G. Suppose H is a finitely generated subgroup of G, generated by finitely many elements of N. Then H is contained in a nilpotent normal subgroup of G (and so it is also contained in N); hence it is finitely generated torsion-free nilpotent pro-p, and subnormal by Lemma 5.6.6. So by Lemma 5.6.5 $H^{(IP,PD),i}_{\mathbb{Z}_p}(H, F) \neq 0$ for some $i \leq n$. Also H is of type FP_∞ over $\mathbb{Z}_p$, so $H^i_{\mathbb{Z}_p}(H, F) \neq 0$ by Proposition 3.4.2 and hence H has nilpotency class and rank $\leq n$ by Lemma 5.6.4.

This holds for every finitely generated subgroup of N, so N is nilpotent of class $\leq n$. Thus the continuous map

$$\overbrace{N \times N \times \cdots \times N}^{n+1} \to \overbrace{[N, [N, [\cdots, N] \cdots]]}^{n+1}$$

has image 1, and by continuity its closure

$$\overbrace{\bar{N} \times \bar{N} \times \cdots \times \bar{N}}^{n+1} \to \overbrace{[\bar{N}, [\bar{N}, [\cdots, \bar{N}] \cdots]]}^{n+1}$$

also has image 1. Therefore $\bar{N}$ is nilpotent of class $\leq n$ too, and it is normal because N is, so by definition of N we have $\bar{N} \leq N \Rightarrow \bar{N} = N$. Finally, we have shown that every finitely generated subgroup of $\bar{N}$ has rank $\leq n$, so $\bar{N}$ does too. □

One of the useful properties of the Fitting subgroup for abstract soluble groups is that it contains its own centraliser. The easiest way to show that the same property holds for profinite soluble groups is to show that the two are the same.

Lemma 5.6.8. *Let G be a profinite group and N its abstract Fitting subgroup. Write G^{abs} for G considered as an abstract group, and let N^{abs} be the Fitting subgroup of G^{abs}. Then, as (abstract) subgroups of G, $N = N^{abs}$. Thus, for G soluble, N contains its own centraliser in G.*

Proof. Every nilpotent normal closed subgroup H of G is a nilpotent normal abstract subgroup, so every such H is contained in N^{abs}, and hence so is N, i.e. $N \leq N^{\mathrm{abs}}$.

Suppose instead that H is a nilpotent normal abstract subgroup of nilpotency class i. Then the closure $\bar{H}$ is a normal closed subgroup of G. As before, the continuous map

$$\overbrace{H \times H \times \cdots \times H}^{i+1} \to \overbrace{[H, [H, [\cdots, H] \cdots]]}^{i+1}$$

has image 1, and by continuity its closure

$$\overbrace{\bar{H} \times \bar{H} \times \cdots \times \bar{H}}^{i+1} \to \overbrace{[\bar{H}, [\bar{H}, [\cdots, \bar{H}] \cdots]]}^{i+1}$$

also has image 1, so $\bar{H}$ is nilpotent. Hence $H \leq \bar{H} \leq N$, and therefore $N^{\mathrm{abs}} \leq N$. □

The following result corresponds roughly to [18, Theorem B], and answers [23, Open Question 6.12.1] in the torsion-free case.

Theorem 5.6.9. *Let G be a virtually torsion-free soluble pro-p group of type FP_∞ over $\mathbb{Z}_p$. Then G has finite rank.*

Proof. We can assume G is torsion-free: if it isn't, take a finite index torsion-free subgroup. Write N for the abstract Fitting subgroup of G. By Corollary 5.5.3 there is some n such that $H^n_{\mathbb{Z}_p}(G, \mathbb{Z}_p[\![G]\!]) \neq 0$; by Proposition 3.4.2 we have $H^{(IP,PD),n}_{\mathbb{Z}_p}(G, \mathbb{Z}_p[\![G]\!]) \neq 0$ too, because G is of type FP_∞, and because G is second-countable, by Remark 4.3.5(c), so $\mathbb{Z}_p[\![G]\!]$ is second-countable too and hence in $PD(\mathbb{Z}_p[\![G]\!])$ by Remark 2.3.20(i). Thus Proposition 5.6.7 gives us that N is closed and has finite rank. Then, writing $C_G(N)$ for the centraliser of N in G, we have a monomorphism $G/C_G(N) \to Aut(N)$, and $Aut(N)$ has finite rank by [11, Theorem 5.7], so $G/C_G(N)$ has finite rank too. But by Lemma 5.6.8 $C_G(N) \leq N$ has finite rank, so G does. □

We observe that, by Proposition 4.4.2, Theorem 5.6.9 has the following converse: Suppose G is a soluble pro-p group of finite rank. Then G is virtually torsion-free of type FP_∞ over $\mathbb{Z}_p$.

Note that the proof uses the fact that G is second-countable. The class of second-countable profinite groups includes all finitely generated profinite groups, and hence all pro-p groups of type FP_1 over $\mathbb{Z}_p$ (and all subgroups of such groups), by Remark 4.3.5(c). By Remark 4.3.5(a), in fact all prosoluble groups of type FP_1 over $\hat{\mathbb{Z}}$ are finitely generated. On the other hand, the following example shows that a group in $\widehat{\mathbf{L}'\mathbf{H}_{\hat{\mathbb{Z}}}\mathfrak{F}}$ need not be second-countable even if it is of type FP_1. This example is adapted from [10, Example 2.6]; the approach is the same, but we construct groups which are not second-countable.

Example 5.6.10. Consider a product of copies of A_5, the alternating group on 5 letters, indexed by a set I. Suppose I has cardinality $\aleph_\alpha$ for some ordinal α. Since A_5 is simple, the finite quotients of $\prod_I A_5$ are all $\prod_{i=1}^n A_5$. By [10, Example 2.6], the minimal number of generators of $\prod_{i=1}^n A_5$ tends to ∞ as n does, but the augmentation ideal $\ker(\hat{\mathbb{Z}}[\![\prod_{i=1}^n A_5]\!] \to \hat{\mathbb{Z}})$ is 2-generated for all n. It follows by [10, Theorem 2.3] that $\prod_I A_5$ is of type FP_1 over $\hat{\mathbb{Z}}$.

Since A_5 is discrete, the family F of neighbourhoods of 1 in $\prod_I A_5$ of the form

$$(\prod_{\{i \in I : i \neq i_1, \ldots, i_t\}} A_5) \times \{1\}_{i_1} \times \cdots \times \{1\}_{i_t},$$

for any $i_1, \ldots, i_t \in I$, is a fundamental system of neighbourhoods of 1 in $\prod_I A_5$. Since I has cardinality $\aleph_\alpha$, F does too. Hence by [23, Proposition 2.6.1] $\prod_I A_5$ has weight $\aleph_\alpha$. In particular, for $\alpha > 0$, $\prod_I A_5$ is not second-countable.

Finally, to see that $\prod_I A_5 \in \widehat{\mathbf{L}'\mathbf{H}_{\hat{\mathbb{Z}}}\mathfrak{F}}$, the easiest way is to note that $\bigoplus_I A_5$ is dense in $\prod_I A_5$, and $\bigoplus_I A_5$ is clearly locally finite, so we have $\prod_I A_5 \in \widehat{\mathbf{L}'\mathfrak{F}}$.

Question 5.6.11. *Are there profinite groups of type* FP_2 *which are not finitely generated?*

5.7 An Alternative Finiteness Condition

We change notational convention once again: in this section, $H^n(G, -)$ will mean $H_{\hat{\mathbb{Z}}}^{(P,D),n}(G, -)$.

According to [23, Open Question 6.12.1], Kropholler has posed the question: "Let G be a soluble pro-p group such that $H^n(G, \mathbb{Z}/p\mathbb{Z})$ is finite for every n. Is G poly[-pro]cyclic?". Now, we know by [31, Corollary 4.2.5] that requiring $H^n(G, \mathbb{Z}/p\mathbb{Z})$ to be finite for every n is equivalent to requiring that G be of type p-FP_∞, and by [31, Proposition 4.2.3] equivalent to requiring that $H^n(G, A)$ is finite for every n and every finite $\mathbb{Z}_p[\![G]\!]$-module A. Also, by [33, Proposition 8.2.2], G is poly-procyclic if and only if it has finite rank. So there are two possible profinite analogues of this question, either of which, if the answer were yes, would imply [23, Open Question 6.12.1].

Question 5.7.1. *Let G be a soluble profinite group such that $H^n(G, A)$ is finite for every n and every finite $\hat{\mathbb{Z}}[\![G]\!]$-module A. Is G of finite rank?*

Question 5.7.2. *Let G be a soluble profinite group of type* FP_∞ *over $\hat{\mathbb{Z}}$. Is G of finite rank?*

We will show that the answer to the first of these questions is no. Question 5.7.2 remains open.

In this section, all modules will be left modules.

By analogy to the pro-p case, we define profinite G to be *of type* FP'_n (over $\hat{\mathbb{Z}}$) if, for all finite $\hat{\mathbb{Z}}[\![G]\!]$-modules A, $m \leq n$, $H_{\hat{\mathbb{Z}}}^m(G, A)$ is finite. This definition extends in the obvious way to profinite modules over any profinite ring. Clearly, by the Lyndon-Hochschild-Serre spectral sequence [23, Theorem 7.2.4], being of type FP'_n is closed under extensions. In the same way as [31, Proposition 4.2.2], $\mathrm{FP}_n \Rightarrow \mathrm{FP}'_n$ for all $n \leq \infty$; in this section we will see that the converse is not true.

Lemma 5.7.3. *If G is pronilpotent of type* FP_1*, the minimal number of generators of its p-Sylow subgroups is bounded above.*

Proof. For G pronilpotent, by [23, Proposition 2.3.8], G is the direct product of its (unique for each p) p-Sylow subgroups. If G is finitely generated, pick a set of generators for G; then their images in each p-Sylow subgroup under the canonical projection map generate that subgroup. Hence the minimal number of generators of the p-Sylow subgroups of G is bounded above. We know G is *a fortiori* prosoluble, so by Proposition 4.3.4 and Remark 4.3.5(a) G is of type FP_1 if and only if it is finitely generated, and the result follows. □

Lemma 5.7.4. *Suppose A is a finite G-module whose order is coprime to that of G. Then $H^n_{\hat{\mathbb{Z}}}(G, A)$ is 0 for all $n > 0$.*

Proof. By [23, Corollary 7.3.3], $cd_p(G) = 0$ for $p \nmid |G|$. In particular,

$$H^n_{\hat{\mathbb{Z}}}(G, A)_p = 0 \text{ for all } p \mid |A|, n > 0.$$

On the other hand, by [23, Proposition 7.1.4],

$$H^n_{\hat{\mathbb{Z}}}(G, A) = \bigoplus_{p||A|} H^n_{\hat{\mathbb{Z}}}(G, A_p) = \bigoplus_{p||A|} H^n_{\hat{\mathbb{Z}}}(G, A)_p = 0.$$

□

Proposition 5.7.5. *Let G be pronilpotent. Then G is of type* FP'_n *if and only if every p-Sylow subgroup is of type* FP'_n*.*

Proof. Suppose every p-Sylow subgroup is of type FP'_n. Suppose A is a finite $\hat{\mathbb{Z}}[\![G]\!]$-module. Now A is finite, so only finitely many primes divide the order of A. Suppose $p_1, \ldots, p_m$ are the primes for which $p_i \mid |A|$, and write π for the set of primes without $p_1, \ldots, p_m$. Write G again as the direct product of its p-Sylow subgroups, $G = \prod_p S_p$. By the Lyndon-Hochschild-Serre spectral sequence ([23, Theorem 7.2.4]) $\prod_{i=1}^m S_{p_i}$ is of type FP'_n. Thus, applying the spectral sequence again, $H^{r+s}_{\hat{\mathbb{Z}}}(G, A)$ is a sequence of extensions of the groups $H^r_{\hat{\mathbb{Z}}}(\prod_{i=1}^m S_{p_i}, H^s_{\hat{\mathbb{Z}}}(\prod_{p\in\pi} S_p, A))$, which by Lemma 5.7.4 collapses to give $H^{r+s}_{\hat{\mathbb{Z}}}(\prod_{i=1}^m S_{p_i}, A^{\prod_{p\in\pi} S_p})$, finite.

Conversely, if some S_p is not of type FP'_n, there is some S_p-module A and some $k \leq n$ such that $H^k_{\hat{\mathbb{Z}}}(S_p, A)$ is infinite. All groups are of type FP_0 and hence of type FP'_0, so we have $k > 0$. Then by Lemma 5.7.4 we have that

$$H^k_{\hat{\mathbb{Z}}}(S_p, A) = \bigoplus_{p'||A|} H^k_{\hat{\mathbb{Z}}}(S_p, A_{p'}) = H^k_{\hat{\mathbb{Z}}}(S_p, A_p)$$

is infinite, and so we may assume $A = A_p$. Then we can make A a G-module by having every $S_{p'}$, $p' \neq p$, act trivially on A, and the spectral sequence together with Lemma 5.7.4 gives that

$$H^k_{\hat{\mathbb{Z}}}(G, A) = H^k_{\hat{\mathbb{Z}}}(S_p, A^{\prod_{p'\neq p} S_{p'}}) = H^k_{\hat{\mathbb{Z}}}(S_p, A),$$

which is infinite, and hence G is not of type FP'_n. □

Finally, as promised, we will answer Question 5.7.1 by constructing a soluble (in fact torsion-free abelian) profinite group of type FP'_∞ which is not finitely generated, and hence not of type FP_1 by Proposition 4.3.4 and Remarks 4.3.5(a), and not of finite rank.

Example 5.7.6. Write p_n for the nth prime, and consider the abelian profinite group $G = \prod_n(\prod_{i=1}^n \mathbb{Z}_{p_n})$. By Lemma 5.7.3, G is not of type FP_1. By Example 4.4.1, and the Lyndon-Hochschild-Serre spectral sequence, the p_n-Sylow subgroup $\prod_{i=1}^n \mathbb{Z}_{p_n}$ of G is of type FP_∞ for each n, and hence of type FP'_∞. So by Proposition 5.7.5, G is of type FP'_∞.

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