Multiplicative Properties of Integers in Short Intervals

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Submitted for the degree of Ph.D.

Declaration

These doctoral studies were conducted under the supervision of Professor Glyn Harman and Dr Rainer Dietmann.

The work presented in this thesis is the result of original research carried out by myself whilst enrolled in the Department of Mathematics as a candidate for the degree of Doctor of Philosophy. This work has not been submitted for any other degree or award in any other university or educational establishment.

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Abstract

In this thesis we consider multiplicative properties of integers in short intervals using techniques involving exponential sums, sieve methods and a wide variety of other principles from analytic number theory.

The existence of products of three pairwise coprime integers are investigated in short intervals of the form $(x, x + x^{\frac{1}{2}}]$. A general theorem is proved which shows that such integer products exist provided there is a bound on the product of any two of them. The author's result has been published in a Journal of the London Mathematical Society [22].

A particular case of relevance to elliptic curve cryptography, when all three integers are of order $x^{\frac{1}{3}}$, is then presented as a corollary to this result. The techniques used in the proof include Fourier series for fractional parts and bounds for an exponential sum.

We investigate the sum of differences between consecutive primes where the gap between these consecutive primes is greater than $x^{1/2-\Delta}$ for some fixed number $0 < \Delta < 1/48$ and show by using Dirichlet polynomials and the sieve of Harman that

$$\sum_{\substack{p_{n+1}-p_n > x^{1/2-\Delta} \\ x \leq p_n \leq 2x}} p_{n+1} - p_n \ll x^{2/3+5\Delta}$$

and thereby generalise an existing result corresponding to $\Delta = 0$. We show this bound provides significant improvements to several existing results for constant $0 < \Delta \leq -3 + \frac{1}{6}\sqrt{327} = 0.01385....$
We establish a corollary which further improves the currently established bound on the sum of squared differences between consecutive primes in certain intervals.

By applying the result on sums of differences between consecutive primes we prove the existence of a significantly improved form of a prime-representing function. We show that there exists $\alpha > 2$ and $\beta = 1/(\frac{1}{2} + \Delta)$ for $0 < \Delta \leq -3 + \frac{1}{6}\sqrt{327}$ such that the sequence

$$[\alpha^{\beta^n}] \text{ for } \alpha > 2 \text{ is prime for all } n \in \mathbb{N}$$

thereby reducing best known present result $\beta = 2$ in the exponent to $1/(\frac{1}{2} + \Delta) = 1.946067...$.

We also establish the existence of a prime-representing function which only takes values which are primes in Beatty sequences $[m\xi + \eta]$ for irrational $\xi > 1$ and $\eta \in \mathbb{R}$.

Acknowledgements

I am deeply indebted to my supervisors Professor Glyn Harman and Dr Rainer Dietmann. My intention to explore analytic number theory could not have been more amply fulfilled with their immense patience and willingness to give me their time and expertise in guiding me throughout my research. I owe a further debt of gratitude to Professor Harman for initially opening the door of opportunity to pursue research at Royal Holloway University of London, for selecting the problems which form the basis of this thesis, for providing a series of lectures in my first year and for supervising my work with thoroughness. I have gained immensely from supervisions, not least of all a full appreciation of the level of perseverance and tenacity required in this exciting area of mathematical research. I was also delighted to have been invited by Professor Harman to attend the London Paris number theory seminar in the summer of 2012 at University College London and to have been introduced to other leading number theorists such as Ben Green and Gérald Tenenbaum. I wish to express my wholehearted sincere gratitude to Dr Dietmann for being hugely supportive and positive and for offering extremely helpful guidance, suggestions, advice and comments on my thesis and countless other aspects of my doctoral studies. I feel honoured to have been supervised by genuine world authorities in this field and feel privileged to have experienced the excitement of exploring and discovering new results in analytic number theory.

Contents

Chapter 1

Introduction

In this thesis we consider multiplicative properties of integers in short intervals using techniques involving both exponential sums and sieve methods. The underlying theme is rooted in the property of primes and the methods used involve many applications of fundamental results from analytic number theory.

The existence of products of three pairwise coprime integers are investigated in short intervals of the form $(x, x + x^{\frac{1}{2}}]$. A general theorem is proved which shows that such integer products exist provided there is a bound on the product of any two of them [22]. A particular case of relevance to elliptic curve cryptography, when all three integers are of order $x^{\frac{1}{3}}$, is then presented as a corollary to this result. The techniques used in the proof include Fourier series for fractional parts and bounds for an exponential sum.

We also investigate sums of differences between consecutive primes where the gap between these consecutive primes is greater than $x^{1/2-\Delta}$ for some fixed number $0 < \Delta < 1/48$ and show by using Dirichlet polynomials and the sieve of Harman that

$$\sum_{\substack{p_{n+1}-p_n>x^{1/2-\Delta}\\ x\le p_n\le 2x}} p_{n+1}-p_n \ll x^{2/3+5\Delta}$$

and thereby generalise an existing result corresponding to $\Delta = 0$. We show this bound provides significant improvements to several existing results for constant $0 < \Delta \le -3 + \frac{1}{6}\sqrt{327} = 0.01385....$

By applying the result on sums of differences between consecutive primes we prove the existence of a significantly improved form of a prime-representing function. We show that there exists $\alpha > 2$ and $\beta = 1/(\frac{1}{2}+\Delta)$ for $0 < \Delta \le -3 + \frac{1}{6}\sqrt{327}$ such that the sequence

$$[\alpha^{\beta^n}] \text{ for } \alpha > 2 \text{ is prime for all } n \in \mathbb{N}$$

thereby reducing best known present result $\beta = 2$ in the exponent to $1/(\frac{1}{2}+\Delta) = 1.946067...$ A second corollary further improves the sum of squared differences between consecutive primes in certain intervals.

We explore prime-representing functions further in the last chapter and we establish the existence of a prime-representing function which only takes values which are primes in Beatty sequences $[m\xi+\eta]$ for irrational $\xi > 1$ and $\eta \in \mathbb{R}$.

THE SIEVE METHOD

We introduce some standard notation and identities which lie at the heart of the sieve of Harman, in particular the Buchstab's identity and its iterations. We also provide an outline of the sieve method to explain the strategy adopted in establishing the main theorem.

Definition: The expression $a \sim A$ is used when $A < a \leq 2A$.

Let $\mathcal{E}$ be a finite subset of $\mathbb{N}$. We denote the cardinality of $\mathcal{E}$ by $|\mathcal{E}|$. Define

$$\mathcal{E}_d = \{m | dm \in \mathcal{E}\}$$

and

$$S(\mathcal{E}, z) = |\{m \in \mathcal{E} | (m, P(z)) = 1\}|$$

where

$$P(z) = \prod_{p<z} p.$$

The *Buchstab identity* is

$$S(\mathcal{E}, z) = S(\mathcal{E}, w) - \sum_{w \leq p < z} S(\mathcal{E}_p, p),$$

when $z > w \geq 2$.

Let $\mathcal{A} = (y, y+y\delta_{\mathcal{A}}] \cap \mathbb{N}$ and $\mathcal{B} = (y, y+y\delta_{\mathcal{B}}] \cap \mathbb{N}$ where $\delta_{\mathcal{B}} = \exp(-\frac{1}{2}(\log x)^{1/2})$.

$\mathcal{A}$ is the set to be sieved and $\mathcal{B}$ is called the *comparison set*.

The choice of the interval for the comparison set $\mathcal{B}$ arises from the need to ensure this is a known set of integers which is required for the sieve method. More specifically the set will always be larger than the set $\mathcal{A}$ which it will contain and such that it is large enough for there to be asymptotically the right number of primes in $(y, y + y\delta_{\mathcal{B}}]$ by the prime number theorem with a good error term.

Note that the choice of value for $\delta_{\mathcal{B}}$ is for technical reasons as it enables a clean $y(\log x)^{-1}$ main term without the need for $\mathrm{Li}(x)$ (see [14] p339-340 for details).

With this notation we note that $\pi(y+\delta_{\mathcal{A}}y)-\pi(y)$ is just $S(\mathcal{A}, 2x^{1/2})$ which by Buchstab's identity can be decomposed into sums that are more straightforward to handle and a similar decomposition is applied to $S(\mathcal{B}, 2x^{1/2})$. Hence the quantity $S(\mathcal{A}, 2x^{1/2})$ can be compared to $S(\mathcal{B}, 2x^{1/2})$ which can be regarded as known. The decompositions (which are always an even number of iterations) will be of the form which we will write as follows:

$$S(\mathcal{A}, 2x^{1/2}) = \sum_{j=1}^{k} S_j - \sum_{j=k+1}^{l} S_j$$

and

$$S(\mathcal{B}, 2x^{1/2}) = \sum_{j=1}^{k} S_j^* - \sum_{j=k+1}^{l} S_j^*$$

where $S_j, S_j^* \geq 0$ and we can find asymptotic formulae of the form

$$S_j = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} S_j^* + \frac{\delta_{\mathcal{A}} y}{\log y}(A(x,y) + o(1)).$$

We aim to obtain a non-trivial lower bound for $S(\mathcal{A}, 2x^{1/2})$ and we therefore discard those parts of the positive sums in the decomposition for which there do not exist asymptotic formulae. Using this approach and combining the asymptotic formulae above gives a lower bound of the form (for $j \leq t \leq k$)

$$S(\mathcal{A}, 2x^{1/2}) \geq \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} \left(S(\mathcal{B}, 2x^{1/2}) - \sum_{j=t+1}^{k} S_j^* \right) + \frac{\delta_{\mathcal{A}} y}{\log y}(A(x,y) + o(1)).$$

Next $\delta_{\mathcal{B}}$ has been chosen sufficiently large as to use the prime number theorem to obtain an asymptotic formula for S_j^*. By standard methods (see [14] p57-59) after $2n$ iterations of Buchstab's identity we obtain n integrals corresponding to the sums that cannot be further decomposed and for which asymptotic formulae do not exist:

$$\sum_{x^{\nu}<p_n<...<p_1<x^{\lambda}} S(\mathcal{B}_{p_1...p_n}, p_n) =$$

$$\frac{\delta_{\mathcal{B}} y}{\log y}(1+o(1)) \int_{\alpha_1=\nu}^{\lambda} \int_{\alpha_2=\nu}^{\alpha_1} ... \int_{\alpha_n=\nu}^{\alpha_{n-1}} \omega\left(\frac{1-\alpha_1-...-\alpha_n}{\alpha_n}\right) \frac{d\alpha_n...d\alpha_1}{\alpha_1...\alpha_{n-1}\alpha_n^2}$$

where $\omega(u)$ is Buchstab's function (see for example [14] p14).

Notational convention: We define $p_j = x^{\alpha_j}$ so that, for example, the condition $x^{\gamma_j} \leq p_j \leq x^{\beta_j}$ can be replaced by $\gamma_j \leq \alpha_j \leq \beta_j$.

Crucially since $S(\mathcal{B}_{p_1...p_n}, p_n) = \frac{\delta_{\mathcal{B}} y}{\log y}(1+o(1))$, then we obtain the result (3.3) if the sum of all the contributions from the integrals corresponding to S_j^* with $t < j < k$ is strictly less than 1 which will then accomplish a positive lower bound. We therefore use numerical integration in this final step of the proof since the integrals provide a significant contribution to the discarded sums.

We will then often apply Buchstabs' identity twice or a greater even number of iterations. For example after two applications of Buchstab's identity we obtain

$$S(\mathcal{A}, 2x^{1/2}) \geq S(\mathcal{A}, x^{\nu(0)}) - \sum_{x^{\nu(0)} \leq p_1 < 2x^{1/2}} S(\mathcal{A}_{p_1}, x^{\nu(\alpha_1)}) \tag{1.1}$$

$$+ \sum_{\substack{\nu(0)\leq\alpha_1<1/2 \\ \nu(\alpha_1)\leq\alpha_2<\min(\alpha_1,(1-\alpha_1)/2)}} S(\mathcal{A}_{p_1p_2}, p_2).$$

Here $\nu(\alpha)$ is a piece-wise linear positive function of a non-negative variable α.

The corresponding decomposition for $S(\mathcal{B}, 2x^{1/2})$ is

$$S(\mathcal{B}, 2x^{1/2}) = S(\mathcal{B}, x^{\nu(0)}) - \sum_{x^{\nu(0)} \leq p_1 < 2x^{1/2}} S(\mathcal{B}_{p_1}, x^{\nu(\alpha_1)}) \tag{1.2}$$

$$+ \sum_{\substack{\nu(0)\le\alpha_1<1/2 \\ \nu(\alpha_1)\le\alpha_2<\min(\alpha_1,(1-\alpha_1)/2)}} S(\mathcal{B}_{p_1p_2}, p_2) + O\left(\frac{\delta_{\mathcal{B}} y}{(\log y)^2}\right).$$

The objective is to obtain asymptotic formulae of the form

$$\sum_m a_m S(\mathcal{A}_m, z) = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} \sum_m a_m S(\mathcal{B}_m, z) + \frac{\delta_{\mathcal{A}} y}{\log y}(A(x,y) + o(1)),$$

where

$$a_m = \sum_{\substack{p_i \sim P_i \\ p_1 \dots p_k = m}} 1$$

for certain P_i. We choose $\nu(\alpha)$ such that we find asymptotic formulae of this type for the first and second terms on the right hand side of (1.1). We will also find asymptotic formulae for part of the third term often decomposing part of this term two or a higher even number of iterations further of Buchstab's identity. Finally we will discard any parts of the decomposition that cannot be dealt with via further asymptotic formulae.

To conclude this section we state the fundamental theorem of the sieve of Harman (see [14] p51 for the proof) as a lemma since this is the key result on which the sieve method is based.

Lemma 1 (The Fundamental Theorem)**.** *Let $\mathcal{B} = \mathbb{Z} \cap [x/2, x)$ and let $\mathcal{A} \subseteq \mathcal{B}$. Suppose that for any sequences of complex numbers a_m,b_m that satisfy $|a_m| \le 1$, $|b_m| \le 1$ we have, for some $\lambda > 0, \alpha > 0, \beta \le 1/2, M \ge 1$, that*

$$\sum_{\substack{mn\in\mathcal{A} \\ m\le M}} a_m = \lambda \sum_{\substack{mn\in\mathcal{B} \\ m\le M}} a_m + O(Y)$$

and

$$\sum_{\substack{mn\in\mathcal{A} \\ x^\alpha \le m \le x^{\alpha+\beta}}} a_m b_n = \lambda \sum_{\substack{mn\in\mathcal{B} \\ x^\alpha \le m \le x^{\alpha+\beta}}} a_m b_n + O(Y).$$

Let c_r *be a sequence of complex numbers such that* $|c_r| \leq 1$, *and if* c_r *non-zero then* $p|r \implies p > x^{\epsilon}$, *for some* $\epsilon > 0$. *Then if* $x^{\alpha} < M, 2R < \min(x^{1-\alpha}, M)$ *and* $M > x^{1-\alpha}$ *if* $2R > x^{\alpha+\beta}$, *we have that*

$$\sum_{r \sim R} c_r S(\mathcal{A}_r, x^{\beta}) = \lambda \sum_{r \sim R} c_r S(\mathcal{B}_r, x^{\beta}) + O(Y \log^3 x).$$

THE VINOGRADOV NOTATION AND IMPLIED CONSTANTS

We make a remark on implied constants when using the standard Vinogradov notation. The notation $x \ll y$ means there is a positive constant C such that $|x| < Cy$, where C may depend on any of the fixed parameters in a theorem or lemma (ϵ, η etc). All implied constants could in theory be calculated. However, as is the usual practice in analytic number theory, we do not explicitly calculate these. It is worth pointing out that certain constants in analytic number theory cannot be calculated with current knowledge, these are referred to as the *ineffective* constants that arise when using results on primes in arithmetic progressions. No such constants appear in this thesis.

DIRICHLET POLYNOMIALS

A *Dirichlet polynomia*l is a finite Dirichlet series

$$N(s) = \sum_{1 \leq n \leq N} a_n n^{-s}, \text{ where } a_n \text{ are complex coefficients.}$$

Throughout the remainder of this thesis the term *polynomial* will be understood to be a *Dirichlet polynomial*.

The product of polynomials is a polynomial.

A *Dirichlet polynomial* of *length* N is a finite Dirichlet series of the form

$$N(s) = \sum_{n \sim N} a_n n^{-s}, \text{ where } a_n \text{ are complex coefficients.}$$

As a *convention* we denote the length of a polynomial by the same letter as the polynomial itself.

We note that the product of k Dirichlet polynomials of lengths $N_1, \dots, N_k$ is the sum of k Dirichlet polynomials of lengths $M, 2M, \dots, 2^{k-1}M$, where $M = N_1 \cdots N_k$. Such a product can therefore be treated essentially as a Dirichlet polynomial of length M.

All polynomials we use will be *divisor-bounded* which means the coefficients satisfy $|c_m| \leq \tau(m)^C$ for some C , where τ is the divisor function. However this also implies that $c_m \ll m^{\epsilon}$ for any $\epsilon > 0$ and

$$\sum_{m \sim M} |c_m| \ll M(\log M)^{2^C - 1}.$$

Many of the polynomials used will satisfy the following condition: A polynomial $R(s) = \sum_{r \sim R} c_r r^{-s}$ is said to be *prime-factored* if there exists a constant $c > 0$ such that

$$\left| \sum_{r \sim R} c_r r^{-1/2+it} \right| \ll R^{\frac{1}{2}} \exp(-c(\log x)^{\frac{13}{60}})$$

for all $t \geq T_0 = \exp(\frac{1}{8}(\log x)^{1/2})$.

A Dirichlet polynomial is called a *zeta-factor* if it is of type

$$K(s) = \sum_{k \sim K} k^{-s} \text{ or } K(s) = \sum_{k \sim K} (\log k) k^{-s}.$$

Zeta-factors are prime-factored when $K > \exp((\log x)^{9/10})$.

We often work with polynomial of the form (for p and p_i prime)

$$P_i(s) = \sum_{p \sim P_i} p^{-s},$$

and products of these types:

$$M(s) = \sum_{\substack{p_1 < ... < p_n \\ p_i \sim P_i}} (p_1 ... p_n)^{-s}.$$

In this form of polynomial we can remove the dependence between some of the p_i at the cost of a factor $(\log x)^C$ in the error term (see [12], Lemma 1).

We often use the generalized Vaughan identity [16] (see also [14] Chapter 2) to decompose polynomials of type $M(s)$. Applying it to $P_i(s)$ with $P_i > x^{1/8}$, by partial summation we obtain

$$|P_i(s)| \leq g_1(s) + ... + g_r(s) \text{ where } r \leq (\log x)^C,$$

and each G_i is of form

$$(\log x)^B \prod_{i=1}^{h} |N_i(s)|, \text{ and } h \ll 1, N_1 ... N_h \leq x,$$

$N_i(s)$ is a zeta factor when $N_i > x^{1/8}$ and all $N_i(s)$ with $N_i > x^{\eta}$ are prime factored.

Chapter 2

Products of Three Pairwise Coprime Integers in Short Intervals

2.1 Introduction

We investigate the existence of a product of three pairwise coprime integers in the interval $(x, x+y]$ where $y = x^{\frac{1}{2}}$. The approach to the problem is to suppose that one of the integers is a prime p where $p \sim P$ and that the remaining two integers m and n are coprime and we let $n \sim N$. Here N, P satisfy $NP \leq x^{\frac{3}{4}}$ where N is a positive integer such that $2N$ is less than P. We then count products of integers mnp in the interval with $p \nmid m$ and show that for sufficiently large x an asymptotic formula exists for this sum. This is achieved by considering the cases where there are no divisibilty conditions between p and m and the case $p \mid m$, where $n < p$. The former case generates a main term and all the error terms arising from the sums in the remaining cases are shown to be smaller than this main term. As a corollary to this result we prove the existence of three such integers where the order of each integer is $x^{\frac{1}{3}}$ and show that there are pairwise coprime integers of this form in the interval for sufficiently large x.

Certain problems relating to the existence of such pairwise coprime integers have originated in the study of elliptic curve cryptography. It was in a discussion at Royal Holloway with Professor Glyn Harman that Professor Steven Galbraith pointed out that no formal proof of the existence of three pairwise coprime integers existed for short intervals despite forming the basis of certain protocols. In particular it had been noted that Bentahar [6] required the existence of three coprime integers of roughly equal size $x^{\frac{1}{3}}$ in relation to such an elliptic curve cryptographic protocol. The arguments used for their existence in his paper are heuristic with a probabilitic reasoning but without formal proof. Similarly Muzereau et al [29] consider products of three primes in short intervals. Both papers assume the existence of these numbers in applications to public key cryptography and the motivation for the present chapter is to produce a formal proof of the result assumed by these papers. However a more general result is proved in the form of Theorem 1 and the particular case of equal order terms is provided as a corollary.

Let $I = (x, x+y]$ be an interval with $y = x^{\frac{1}{2}}$. We count products $mnp \in I$ such that $(m, n) = 1$. Since $2N < P$ we must have $n < p$. Hence consider the sum:

$$\sum_{\substack{mnp \in I \\ (m,n)=1, p\nmid m}} 1 = \sum_{\substack{mnp \in I \\ (m,n)=1}} 1 - \sum_{\substack{mnp \in I \\ (m,n)=1, p|m}} 1. \tag{2.1}$$

We begin by considering the first sum on the right of (2.1) with no divisibilty conditions between p and m. This sum may be re-expressed as a double sum involving the Mobius function in the following way:

$$\sum_{\substack{mnp\in I\\(m,n)=1}} 1 = \sum_{mnp\in I}\sum_{r|(m,n)} \mu(r)$$
$$= \sum_{r}\mu(r) \sum_{\substack{mnp\in I\\ r|m, r|n}} 1$$
$$= \sum_{r}\mu(r) \sum_{m'n'pr^2\in I} 1. \tag{2.2}$$

Hence we require the counting of integers of the form $m'n'pr^2 \in I$ where now in the inner sum of (2.2) r is a common divisor of m and n and $m = m'r, n = n'r$.

It will be shown that for the case that r is a large common factor, greater than a certain power of $\log x$ the bound is quickly obtained by elementary methods. The case for r being a smaller common factor is more involved and Fourier methods will be required to obtain a suitable nontrivial bound to a Type I sum to achieve the result.

The main term (see (2.11)) will be obtained for small r and will be of order $\gg y/\log x$, whilst the error term which we obtain will be

$$O\left(\frac{y}{(\log x)^2} + x^{\frac{1}{3}+3\epsilon} + \frac{y}{x^\eta} + x^{\frac{2}{5}} + yx^{\epsilon-\frac{1}{8}}\right). \tag{2.3}$$

We prove the following theorem and corollary:

Theorem 1. *Given $\epsilon > 0$, there exists $x_0(\epsilon) > 0$ such that for all $x \geq x_0(\epsilon)$ and all positive integers N and P with $x^\epsilon < 2N < P < x^{\frac{2}{5}-\epsilon}$ and*

$$NP \leq x^{\frac{3}{4}}$$

there exist numbers $mnp \in (x, x + x^{\frac{1}{2}}]$ with $n \sim N$, $p \sim P$ and m, n, p are pairwise coprime.

Corollary 1. *For all sufficiently large x there exist integers $mnp \in (x, x + x^{\frac{1}{2}}]$, with $n < p$ where*

$$\frac{x^{\frac{1}{3}}}{2} \leq m, n, p \leq 2x^{\frac{1}{3}}$$

and m, n, p are pairwise coprime.

In order to count the number of times p divides m we observe that if $x^\epsilon < P < x^{\frac{1}{8}}$ we can give a completely elementary proof to the whole of Theorem 1 quickly since we have, $NP^2 \leq x^{\frac{3}{8}} < x^{\frac{1}{2}}$ (i.e. we can always count the number of integers in intervals like $(x/np^2, (x+y)/np^2]$ accurately). The elementary proof when $NP^2 < x^{\frac{3}{8}}$ is as follows:

$$\begin{aligned}\sum_{n,p} \sum_{\substack{mnp \in I \\ (m,np)=1}} 1 &= \sum_{p \sim P} \sum_{d \leq 2N} \mu(d) \sum_{n \sim N/d} \sum_{\substack{mnpd^2 \in I \\ p \nmid m}} 1 \\ &= \sum_{p \sim P} \sum_{d \leq 2N} \mu(d) \sum_{n \sim N/d} \left(\frac{y}{nd^2 p} + O(1) - \frac{y}{nd^2 p^2} \right) \\ &= \frac{6y \log 2}{\pi^2} \sum_{p \sim P} \frac{1}{p} + O(X).\end{aligned}$$

Where X consists of several error terms but all of smaller order than the main term. We may henceforth assume that $P \geq x^{1/8}$.

2.2 Case: Large Common Factors $r > (\log x)^A$

Consider that part of the inner sum in (2.2) for which r is larger than a power of $\log x$. We essentially count the number of integers of the form $m'n'pr^2 \in I$ or equivalently we count integers of the form $m'n'p \in \left(\frac{x}{r^2}, \frac{x+y}{r^2}\right]$. Since the number of such products $m'n'p$ is bounded by the three-fold divisor function $\tau_3(k) = \sum_{a_1a_2a_3=k} 1$, we have

$$\sum_{m'n'pr^2 \in I} 1 \leq \sum_{x/r^2 \leq k \leq x/r^2 + y/r^2} \tau_3(k).$$

We appeal to the following lemma by P.Shiu [33].

Lemma 2. *Given any $\epsilon > 0$ and $Z > W^\epsilon$*

$$\sum_{W \leq k \leq W+Z} \tau_3(k) \ll Z(\log W)^2$$

where $\tau_3(k)$ is the three-fold divisor function.

Since $n < p$ and r divides both m and n, if $n > x^{\frac{1}{3}}$ then $p > x^{\frac{1}{3}}$ then $mnp \in (x, x + x^{\frac{1}{2}}]$ only if $m < 2x^{\frac{1}{3}}$. Hence we have the restriction $r < 2x^{\frac{1}{3}}$ on the size of r (which is smaller than the bound $r < x^{\frac{2}{5}-\epsilon}$ implied by the hypotheses). However, by this lemma with $W = x/r^2$ and $Z = y/r^2$, the condition $Z > W^\epsilon$ is satisfied only for $r < x^{\frac{1}{4}-\epsilon}$ and in this range we quickly obtain the bound

$$\sum_{m'n'pr^2 \in I} 1 \ll \frac{y}{r^2}\left(\log \frac{x}{r^2}\right)^2 \ll \frac{y}{r^2}(\log x)^2.$$

Letting $L = (\log x)^A$ we obtain the result that for this part of the required sum (2.2) we have the bound

$$\sum_{r>L} \mu(r) \sum_{m'n'pr^2 \in I} 1 \ll \sum_{r>L} \frac{y}{r^2}(\log x)^2 = y(\log x)^2 \sum_{r>L} \frac{1}{r^2}.$$

By comparison with an integral the final sum provides the bound

$$\sum_{r>L}\mu(r)\sum_{m'n'pr^2\in I}1\ll\frac{y(\log x)^2}{L}.$$

Thus for large x and suitable choice of A in $L=(\log x)^A$, this bound will be smaller than the main term, as discussed in the introduction to this chapter (see (2.3) for the explicit error term which is to be obtained).

Next consider the range $x^{\frac{1}{4}-\epsilon}<r<2x^{\frac{1}{3}}$. Using $\tau_3(n)\ll n^\epsilon$ we have

$$\sum_{m'n'pr^2\in I}1=\sum_{m'n'p\in\left(\frac{x}{r^2},\frac{x+y}{r^2}\right]}1\leq\sum_{n\in\left(\frac{x}{r^2},\frac{x+y}{r^2}\right]}\tau_3(n)\leq\sum_{n\in\left(\frac{x}{r^2},\frac{x+y}{r^2}\right]}n^\epsilon$$

$$\ll x^\epsilon(1+y/r^2)\leq 2x^{3\epsilon},$$

since in the range of r under consideration $y/r^2<x^{2\epsilon}$. Therefore the sum (2.2) for this range gives the bound

$$\sum_{x^{\frac{1}{4}-\epsilon}<r<2x^{\frac{1}{3}}}\mu(r)\sum_{m'n'pr^2\in I}1\ll\sum_{x^{\frac{1}{4}-\epsilon}<r<2x^{\frac{1}{3}}}\sum_{m'n'pr^2\in I}1\ll x^{\frac{1}{3}+3\epsilon},$$

which will be smaller than the main term (see section 1 and (2.3)).

Hence for the complete range of possible values of $r>L$ we obtain

$$\sum_{r>L}\mu(r)\sum_{m'n'pr^2\in I}1\ll\frac{y(\log x)^2}{L}+x^{\frac{1}{3}+3\epsilon}. \tag{2.4}$$

2.3 Case: p divides m

Before proceeding to deal with that part of sum (2.2) in the case of smaller common factors r (see the next section) we deal with the second sum on the right of (2.1) (the case $p|m$).

Letting $m = m'p$, so that $mnp = m'np^2$, the total number of solutions with $p|m$ is

$$\leq \sum_{p \sim P} \sum_{n \sim N} \sum_{m'np^2 \in I} 1 \ll \sum_{p \sim P} \left(1 + \frac{y}{p^2}\right) x^{\epsilon} = x^{\epsilon} \sum_{p \sim P} 1 + y x^{\epsilon} \sum_{p \sim P} \frac{1}{p^2}$$
$$\ll x^{\frac{2}{5}} + y x^{\epsilon - \frac{1}{8}},$$

where we use the bounds $x^{\frac{1}{8}} < P < x^{\frac{2}{5} - \epsilon}$ in the first sum on the right of the above equality. For the second sum on the right observe that there are no more than P terms each of which is less than $1/P^2$ so that the sum is bounded by $1/P$ and hence by $x^{-\frac{1}{8}}$. The bound obtained here is of smaller order than the main term (see (2.3)).

2.4 Case: Small Common Factors $r < (\log x)^A$

We next consider the inner sum (2.2) in the case that r is smaller than a power of $\log x$. Suppose $r < L$ then since $m'n'pr^2 \in (x, x + y]$ we have

$$x \leq m'n'pr^2 \leq x + y \quad \text{so that} \quad \frac{x}{pr^2} \leq m'n' \leq \frac{x + y}{pr^2}.$$

Letting $J = \left(\frac{x}{pr^2}, \frac{x+y}{pr^2}\right]$ we may write

$$\sum_{m'n'pr^2 \in I} 1 \;=\; \sum_{\substack{m'n' \in J \\ n = n'r \sim N, p \sim P}} 1 \;=\; \sum_{\substack{m'n' \in J \\ n' \sim N/r, p \sim P}} 1.$$

Next, define $\chi(m)$ as the number of integers in J divisible by m (using square brackets to denote the integral part) as follows,

$$\chi(m) = \sum_{\substack{k \in J \\ m|k}} 1 = \left[\frac{x + y}{pmr^2}\right] - \left[\frac{x}{pmr^2}\right].$$

Letting $\psi = \{x\} - \frac{1}{2}$, where the brace denotes the fractional part, we may write $\chi(m)$ as:

$$\chi(m) = \frac{y}{pmr^2} + \psi\left(\frac{x}{pmr^2}\right) - \psi\left(\frac{x+y}{pmr^2}\right).$$

This may be used to re-express the sum under consideration as a main term with fractional parts:

$$\begin{aligned}\sum_{\substack{m'n' \in J \\ n' \sim N/r, p \sim P}} 1 &= \sum_{\substack{n' \sim N/r \\ p \sim P}} \chi(n') \\ &= \sum_{\substack{n' \sim N/r \\ p \sim P}} \frac{y}{n'pr^2} + \sum_{\substack{n' \sim N/r \\ p \sim P}} \left(\psi\left(\frac{x}{n'pr^2}\right) - \psi\left(\frac{x+y}{n'pr^2}\right)\right) \\ &= S_1 + S_2, \text{ say.}\end{aligned}$$

The sum S_2 will be expressed as an exponential sum with an error term. We aim to show that sufficient saving may be achieved in the subsequent exponential sum such that the error terms will be smaller than the term S_1 and the main term (refer to (2.11) and (2.3)) which it will generate.

Before proceeding we consider the sum S_1 in more detail by first writing

$$S_1 = \sum_{\substack{n' \sim N/r \\ p \sim P}} \frac{y}{n'pr^2} = \frac{y}{r^2} \sum_{n' \sim N/r} \frac{1}{n'} \sum_{p \sim P} \frac{1}{p}.$$

The first sum on the right being over consecutive integers may be approximated by using the standard asymptotic formula

$$\sum_{n \leq A} \frac{1}{n} = \log A + C + O\left(\frac{1}{A}\right),$$

from which we obtain

$$\sum_{n'\sim N/r} \frac{1}{n'} = \log 2 + O\left(\frac{r}{N}\right).$$

To deal with the second sum on the right-hand side of the above expression for S_1 we observe that in order to obtain a final expression for the main term of a suitable order (refer to (2.3)) that application of Mertens Prime Number Theorem (see Theorem 22.8 [9, p.466]) introduces an error term of order $O(1/\log P)$ which is of the same order as the main term, $\log 2/\log P$, obtained from Merten's Theorem for the sum over the range $p \sim P$. Explicity, from Merten's Theorem

$$\sum_{p\sim P} \frac{1}{p} = \log\left(\frac{\log 2P}{\log P}\right) + O\left(\frac{1}{\log P}\right).$$

Then by applying the Taylor's series for $\log(1+A)$ to $\log(\log 2P/\log P)$ with $A = \log 2P/\log P - 1$ and noting that $|A| < 1$ and that A simplifies to $A = \log 2/\log P$ we obtain the following expression for the main term of the above

$$\log\left(\frac{\log 2P}{\log P}\right) = \frac{\log 2}{\log P} + O\left(\frac{1}{(\log P)^2}\right),$$

which is of the same order as the error term in Merten's Theorem.

Fortunately, however, it is possible to obtain this same main term $\log 2/\log P$, for the sum with an error term of order $O(1/(\log P)^2)$ using partial summation as detailed in the following discussion. To proceed we observe that

$$\int_2^N \frac{1}{\log x}\,dx = \sum_{n=2}^{N} \frac{1}{\log n} + O(1),$$

and we note that any error from the Prime Number Theorem with the logarithmic integral as the main term will be the same as that which is obtained by using the sum on the right hand side of the above expression as the main term. Hence, using the Prime Number Theorem in the form

$$\sum_{p \leq N} 1 = \sum_{n=2}^{N} \frac{1}{\log n} + O\left(\frac{N}{(\log N)^2}\right),$$

partial summation gives

$$\sum_{p \sim P} \frac{1}{p} = \frac{\log 2}{\log P}\left(1 + O\left(\frac{1}{\log P}\right)\right).$$

The argument for this partial summation (see [14, p.13]) is:

$$\begin{aligned}
\sum_{p \sim P} \frac{1}{p} &= \sum_{n \sim P}\left(\frac{1}{n} - \frac{1}{n+1}\right) \sum_{P \leq p \leq n} 1 + \frac{1}{2P+1} \sum_{p \sim P} 1 \\
&= \sum_{n \sim P}\left(\frac{1}{n} - \frac{1}{n+1}\right) \sum_{P \leq m \leq n} \frac{1}{\log m} + \frac{1}{2P+1} \sum_{m \sim P} \frac{1}{\log m} + O\left(\frac{1}{(\log P)^2}\right) \\
&= \sum_{n \sim P} \frac{1}{n \log n} + O\left(\frac{1}{(\log P)^2}\right) \\
&= \frac{\log 2}{\log P}\left(1 + O\left(\frac{1}{\log P}\right)\right),
\end{aligned}$$

since the third from final line in the above is essentially what is obtained by applying partial summation to the second from final line.

It will be noted that whilst the error term in the Prime Number Theorem can be as small as $O(N \exp(-(\log N)^{\alpha}))$ for $\alpha < \frac{3}{5}$ the larger error $O(N(\log N)^{-2})$ is sufficient since a similar size error is introduced in the last line of the partial summation argument above.

Hence we obtain the estimate for S_1

$$S_1 = \frac{y(\log 2)^2}{r^2 \log P} + O\left(\frac{y}{r^2(\log P)^2}\right) + O\left(\frac{y}{rN \log P}\right). \tag{2.5}$$

Next consider S_2. We use the truncated Fourier series for ψ (see, for example, [14, p.108])

$$\psi(x) = -\frac{1}{2\pi i}\sum_{0<|h|<H}\frac{e(hx)}{h} + O\left(\min\left(1, \frac{1}{H||x||}\right)\right).$$

We next use this expression for ψ and let t be the value of the argument of ψ in S_2 and write $c_h = -\frac{1}{2\pi i h}$. As a result of the application of this truncated Fourier series we note that two error terms will be generated for $\psi(t)$ at each value of its argument $t = x/n'pr^2$ and $t = (x+y)/n'pr^2$. Explicitly these will be

$$O\left(\sum_{\substack{n'\sim N/r\\ p\sim P}}\min\left(1, \frac{1}{H||x/n'pr^2||}\right)\right) + O\left(\sum_{\substack{n'\sim N/r\\ p\sim P}}\min\left(1, \frac{1}{H||(x+y)/n'pr^2||}\right)\right).$$

We must choose the largest of these two errors and for brevity we write this as

$$O\left(\sum_{\substack{n'\sim N/r\\ p\sim P}}\max_{n'pr^2t=x \text{ or } (x+y)}\min\left(1, \frac{1}{H||t||}\right)\right),$$

with the understanding that the maximum is being taken over t and can occur only at either of the two values of the argument t of $\psi(t)$ in S_2. Hence we now write

$$S_2 = \sum_{\substack{n' \sim N/r \\ p \sim P}} \left(\sum_{0<|h|<H} c_h e\left(\frac{hx}{n'pr^2}\right) - \sum_{0<|h|<H} c_h e\left(\frac{h(x+y)}{n'pr^2}\right) \right)$$

$$+ O\left(\sum_{\substack{n' \sim N/r \\ p \sim P}} \max_{n'pr^2 t = x \text{ or } (x+y)} \min\left(1, \frac{1}{H||t||}\right) \right)$$

$$= S_3 + S_4, \text{ say.}$$

The sum S_3 may, after changing the order of summation be written as

$$S_3 = -\sum_{0<|h|<H} \frac{1}{2\pi i h} \sum_{\substack{n' \sim N/r \\ p \sim P}} \left(e\left(\frac{hx}{n'pr^2}\right) - e\left(\frac{h(x+y)}{n'pr^2}\right) \right).$$

Next by observing that

$$-\left(e\left(\frac{hx}{n'pr^2}\right) - e\left(\frac{h(x+y)}{n'pr^2}\right) \right) = \frac{2\pi i h}{n'pr^2} \int_x^{x+y} e\left(\frac{Yh}{n'pr^2}\right) dY,$$

we may write

$$S_3 = \int_x^{x+y} \frac{1}{r^2} \sum_{0<|h|<H} \sum_{\substack{n' \sim N/r \\ p \sim P}} \frac{1}{n'p} e\left(\frac{Yh}{n'pr^2}\right) dY.$$

The integrand is the product of $1/r^2$ and the sum

$$\sum_{0<|h|<H} \sum_{\substack{n' \sim N/r \\ p \sim P}} \frac{1}{n'p} e\left(\frac{Yh}{n'pr^2}\right).$$

By applying partial summation to the variable n' (this being over consecutive integers) the coefficient $1/n'$ may now be removed. On performing partial summation (see for example [14, p13]) we may now re-express the sum as

$$\sum_{\substack{0<|h|<H \\ p\sim P}} \frac{1}{p} \left(\sum_{n'\sim N/r} \left(\frac{1}{n'} - \frac{1}{n'+1} \right) \sum_{N/r \le s \le n'} e\left(\frac{Yh}{spr^2} \right) \right)$$

$$+ \sum_{\substack{0<|h|<H \\ p\sim P}} \frac{1}{p} \left(\frac{1}{[2N/r]+1} \sum_{s\sim N/r} e\left(\frac{Yh}{spr^2} \right) \right).$$

In the above expression we now have two exponential sums, one of which is a truncated form of the other. Hence we require a bound for the sum (where we replace the dummy variable s as in the original sum by n' for clarity)

$$\sum_{0<|h|<H} \sum_{\substack{N/r<n'<S \\ p\sim P}} \frac{1}{p} e\left(\frac{Yh}{n'pr^2} \right) \text{ for } S \le 2N/r.$$

We will deal with the sum over $n' \sim N/r$. The same argument gives the identical bound for the truncated sum. By this process we have reduced the problem of bounding S_3 essentially to demonstrating a nontrivial bound for the sum

$$\sum_{0<|h|<H} \sum_{\substack{n'\sim N/r \\ p\sim P}} \frac{1}{p} e\left(\frac{Yh}{n'pr^2} \right).$$

The variable n' runs over consecutive integers whilst the variable p runs over primes. We therefore define c_ℓ as the function

$$c_\ell = \begin{cases} \frac{p}{\ell} & \text{if } \ell \text{ is prime,} \\ 0 & \text{otherwise} \end{cases}$$

and write the sum as

$$\frac{1}{p}\sum_{0<|h|<H}\sum_{\substack{n'\sim N/r\\ \ell\sim P}} c_\ell e\left(\frac{\xi h}{n'\ell}\right) \text{ where } \xi=\frac{Y}{r^2} \tag{2.6}$$

and we note that in particular $|c_\ell|\le 1$ (as will be required for the subsequent lemma). This is a Type I sum (using the nomenclature of Vaughan) in which the variable n' runs over consecutive integers and we next appeal to the following Lemma (see section 2 of [25] for a proof and where the result we require follows immediately from Corollary 2).

Lemma 3. *Let $X>1$ and suppose $X\le\xi<2X$. Suppose $v\in(X^{\frac{1}{2}},X^{\frac{4}{5}}]$ and $K=(v,ev]$. Suppose $m\sim M$ where $X^{\frac{1}{8}}\ll M\ll X^{\frac{2}{5}-\epsilon/2}$, and $|a_m|\le 1$. Then*

$$\sum_h\sum_{m,n} a_m e\left(\frac{\xi h}{mn}\right)\ll vX^{-2\eta}$$

where $mn\in K$, $h\le vX^{-\frac{1}{2}+3\eta}$ for some $\eta=\eta(\epsilon)>0$.

To apply this lemma to the Type I sum (2.6) we note that we have already seen that we can assume that $P>x^{\frac{1}{8}}$. We can then apply Lemma 3 with $X=x/r^2, v=NP/r, K=(NP/r,eNP/r], \xi=Y/r^2, M=P$. We then have

$$P\le x^{\frac{2}{5}-\epsilon}\ll(x/r^2)^{\frac{2}{5}-\epsilon/2}$$

All the other conditions are easily checked to be valid. We note that for the sum (2.6) to satisfy Lemma 3 we must have $H\le NP(x/r^2)^{-\frac{1}{2}+3\eta}=(vx^{3\eta}/y)r^{1-6\eta}$ for some $\eta=\eta(\epsilon)>0$. We now have by this Lemma that

$$\sum_{0<|h|<H}\sum_{\substack{n'\sim N/r\\ \ell\sim P}} c_\ell e\left(\frac{Yh}{n'pr^2}\right)\ll vx^{-2\eta}. \tag{2.7}$$

We emphasise the importance of the range $H<(vx^{3\eta}/y)r^{1-6\eta}$ in the above discussion as this will be required in the bound for S_4 in what follows.

The bound on sum S_3 is now readily obtained.

$$S_3 \ll \int_x^{x+y} \frac{1}{r^2}\frac{r}{N}\frac{vx^{-2\eta}}{p}dY \ll yx^{-2\eta}.\frac{v}{rNP} \ll yx^{-2\eta}$$

since $v/rNP \ll 1/r^2$.

In fact we also find that $S_4 \ll yx^{-\eta}$. This is achieved by choosing

$$H = vx^{3\eta}/y$$

(which is within the allowable range $H < (vx^{3\eta}/y)r^{1-6\eta}$ for Lemma 3 as detailed in the discussion above) for $x^{\frac{2}{5}} < v < x^{\frac{3}{4}}$ and any $\eta > 0$. To show this we first appeal to the following Lemma [2, p.18-21]. Note that for notational convenience the letter ℓ has been employed in the subsequent lemma and discussion regarding sum S_4 but it is understood that this ℓ is different from that used in the previous discussion regarding S_3 and will take a different range of values.

Lemma 4. *Let*

$$\chi(z) = \begin{cases} 1 & \textit{if } ||z|| < \delta \\ 0 & \textit{otherwise} \end{cases}$$

and let L be an integer at least of size δ^{-1}. Then there are coefficients a_ℓ^+ and a_ℓ^- with $|a_\ell^+| \ll \delta$ and $|a_\ell^-| \ll \delta$ such that

$$2\delta - \frac{1}{L+1} + \sum_{0<|\ell|\leq L} a_\ell^- e(\ell z) \leq \chi(z) \leq 2\delta + \frac{1}{L+1} + \sum_{0<|\ell|\leq L} a_\ell^+ e(\ell z)$$

with

$$|a_\ell^{+/-}| \leq \min\left(2\delta + \frac{1}{L+1}, \frac{3}{2\ell}\right).$$

By this lemma, choosing the upper bound and letting $L = \delta^{-1}$ we see that given $\chi(z)$ as defined in the Lemma we have for $|a_\ell| \ll \delta$ (where the plus superscipt is omitted on the understanding that we are dealing with the upper bound)

$$\chi(z) \ll \delta + \sum_{\ell=1}^{\delta^{-1}} a_\ell e(\ell z).$$

Using this bound and $|a_\ell| \ll \delta$ we observe that if m is a positive integer and $m \sim M$ then given a sequence of real numbers z_m we also have the following bound

$$\sum_{\substack{||z_m||<\delta \\ m\sim M}} 1 \ll M\delta + \delta \sum_{\ell=1}^{\delta^{-1}} \left| \sum_{m\sim M} e(\ell z_m) \right|. \tag{2.8}$$

We note the similarity in the form of this bound to the Erdős-Turán Theorem (see Theorem 2.1 [2, p.19]) but here we use a constant coefficient δ rather than the harmonic coefficient in that theorem. We may employ this to estimate sum S_4 after suitable re-expression. The approach we take is to majorize the sum over terms $\min(1, 1/H||t||)$ in S_4 by comparing the term $1/H||t||$ with dyadic blocks of size 2^{-j} for integers j. To achieve this we therefore introduce a new variable j which takes positive integer values and define $Q := H2^{-j}$. Let ξ denote the value either x or $x + y$ at which $t = \xi/n'pr^2$ achieves a maximum for the sum S_4 (refer to previous discussion after (2.5) regarding t). Then for some integer j we have $1/H||t|| \geq Q/H = 2^{-j}$ whenever $||t|| = ||\xi/n'pr^2|| < 1/Q$. The condition $||\xi/n'pr^2|| < 1/Q$ therefore enables us to majorize the sum over terms $\min(1, 1/H||t||)$ with the most saving. In the following argument the notation for a summation over $Q = H2^{-j}$ is understood to be a summation over all possible values of Q given by integer values of j. Hence we may now write

$$S_4 = \sum_{\substack{n' \sim N/r \\ p \sim P}} \max_{n'pr^2 t = x \text{ or } (x+y)} \min\left(1, \frac{1}{H||t||}\right)$$

$$\leq \sum_{\substack{n' \sim N/r \\ p \sim P}} \min\left(1, \frac{1}{H\left|\left|\frac{\xi}{n'pr^2}\right|\right|}\right)$$

$$\leq \sum_{Q = H2^{-j}} \sum_{\substack{n' \sim N/r \\ p \sim P, \left|\left|\frac{\xi}{n'pr^2}\right|\right| \leq 1/Q}} \min\left(1, \frac{Q}{H}\right).$$

We may replace the double sum conditions $n' \sim N/r$ and $p \sim P$ of the inner sum with a single sum condition upto the product of the top of each of these ranges $n \leq 4NP/r$ (where we now use the variable n in the sum over the combined range for notational convenience). We thereby majorize the previous sum so that it is

$$\leq \sum_{Q = H2^{-j}} \sum_{\substack{n \leq 4NP/r \\ \left|\left|\frac{\xi}{nr^2}\right|\right| \leq 1/Q}} \min\left(1, \frac{Q}{H}\right).$$

Lemma 3 and the remarks which followed and in particular (2.8) may now be applied to this sum with $z_n = \xi/nr^2$ and $\delta^{-1} = Q$, noting that the minimum function will select $Q/H = 2^{-j}$ by the restriction $||\xi/nr^2|| < 1/Q$, thus giving the bound

$$\ll \sum_{Q = H2^{-j}} \frac{Q}{H} \left(\frac{NP}{r} \frac{1}{Q} + \frac{1}{Q} \sum_{\ell=1}^{Q} \left| \sum_{n \leq 4NP/r} e\left(\frac{\ell \xi}{nr^2}\right) \right| \right)$$

$$= \sum_{Q = H2^{-j}} \left(\frac{NP}{Hr} + \frac{1}{H} \sum_{\ell=1}^{Q} \left| \sum_{n \leq 4NP/r} e\left(\frac{\ell \xi}{nr^2}\right) \right| \right)$$

$$\ll \frac{NP}{H}\frac{\log H}{r} + \frac{1}{H}\sum_{Q=H2^{-j}}\sum_{\ell=1}^{Q}\left|\sum_{n\leq 4NP/r} e\left(\frac{\ell\xi}{nr^2}\right)\right|.$$

The first term of the last line above is $\ll v/H$ since $NP\log H/r = v\log H/r \ll v$. The inner sum of the second term of the last line above is a simple exponential sum and by the Kusmin-Landau and van der Corput bounds [8, p.7-8 Theorem 2.1 and Theorem 2.2] is readily shown to be $\ll NP/r < v$ (recall that $v = NP/r$). We now prove this and begin by quoting these two theorems as lemmas.

Lemma 5. *(Kusmin-Landau) If f is continuously differentiable, f' is monotonic and $||f'|| \geq \lambda > 0$ on $I = (a, b]$ then*

$$\sum_{k\in I} e(f(k)) \ll \lambda^{-1}.$$

Lemma 6. *(van der Corput) Suppose that f is a real valued function with two continuous derivatives on I. Suppose also that there is some $\lambda > 0$ and some $\alpha \geq 1$ such that $\lambda \leq |f''| \leq \alpha\lambda$ on $I = (a, b]$. Then*

$$\sum_{k\in I} e(f(k)) \ll \alpha|I|\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}.$$

We take $f(k) = \ell x/kr^2$ (where we recall that this ℓ relates to discussion and treatment of S_4 and has range $1 \leq \ell \leq Q$) in the above two lemmas for which we have chosen (without loss of generality) the value x for ξ. We then obtain

$$||f'|| \geq \frac{\ell x}{n^2r^2} > 0.$$

Thus we use lemma 4 when $||f'|| < 1/2$ (when $NPr \gg \ell x$ since f' is of order

$\ell x/(NP)^2 r^2)$ so that the above remains positive. Furthermore

$$\frac{\ell x}{n^3 r^2} < |f''| < 2\alpha \frac{\ell x}{n^3 r^2} \text{ for any } \alpha \geq 1$$

for which range we use Lemma 5 (which is when $(NPQr)^2 < \ell x/2$).

Hence by the lemmas for these two ranges (where k of the lemma is now n of the sum under discussion) we obtain the bounds

$$\left| \sum_{n \leq 4NP/r} e\left(\frac{\ell\xi}{nr^2}\right) \right| \ll \frac{n^2 r^2}{\ell x} < \left(\frac{4NP}{r}\right)^2 \frac{r^2}{\ell x} \ll \frac{(NP)^2}{\ell x} = \frac{v^2}{\ell x}.$$

which after summing over ℓ as in the original sum under discussion is equal to

$$\frac{v^2 \log H}{x} \ll v.$$

We also have for the second range

$$\left| \sum_{n \leq 4NP/r} e\left(\frac{\ell\xi}{nr^2}\right) \right| \ll \alpha \frac{4NP}{r} \left(\frac{\ell x}{n^3 r^2}\right)^{\frac{1}{2}} + \left(\frac{n^3 r^2}{\ell x}\right)^{\frac{1}{2}}$$

$$\ll \frac{NP}{r} \left(\frac{r\ell x}{(NP)^3}\right)^{\frac{1}{2}} + \left(\frac{(NP)^3}{r\ell x}\right)^{\frac{1}{2}}.$$

After summing over ℓ $(1 \leq \ell \leq Q)$ the above is $\ll NPx^{-1/16} \ll v$.

Hence we now have

$$S_4 \ll \frac{v \log H}{H} \ll x^{2\eta} \frac{v}{H}.$$

We may now choose $H = vx^{3\eta}/y$ (which is in the allowable range by the discussion following Lemma 2) giving the bound $S_4 \ll yx^{-\eta}$. By virtue of

(2.7) and this bound for S_4 we have

$$S_2 = S_3 + S_4 \ll yx^{-\eta}. \tag{2.9}$$

2.5 Conclusion

We are now in a position to bring all the information regarding the sum (2.2) under investigation together and apply it to (2.1). This original sum may now be decomposed into several sums and associated error terms.

$$\begin{aligned}\sum_{\substack{mnp\in I\\(m,n)=1,p\nmid m}} 1 &= \sum_{r} \mu(r) \sum_{m'n'pr^2\in I} 1 - \sum_{\substack{mnp\in I\\(m,n)=1,p|m}} 1\\ &= \sum_{r\leq L} \mu(r) \frac{y(\log 2)^2}{r^2 \log P} + O\left(\frac{y(\log x)^2}{L} + x^{\frac{1}{3}+3\epsilon}\right) + E \qquad (2.10)\end{aligned}$$

and

$$E = E_1 + E_2 + E_3 + E_4.$$

Where the main term arises from the main term of S_1 in (2.5) and E_1 is the error resulting from this approximation. The second term of (2.10) is the bound obtained for larger common factors in (2.5). The term E_2 comes from the error in reducing to exponential sums (this is essentially S_4 and is noted to become smaller as the range of h increases).The third error term E_3 arises from the estimation of the exponential sum (this being essentially S_3). However by (2.9) we have that $E_2 + E_3 \ll yx^{-\eta}$. E_4 is the error arsing from the case p divides m in section 3, which was shown to be $\ll x^{\frac{2}{5}} + yx^{\epsilon-\frac{1}{8}}$. Hence it remains to calculate E_1 and show this is smaller than the main term.

From (2.5) we have

$$E_1 = \sum_{r \le L} \left(O\left(\frac{y}{r^2 (\log P)^2} \right) + O\left(\frac{y}{rN \log P} \right) \right)$$

$$= O\left(\frac{y}{(\log P)^2} \sum_{r \le L} \frac{1}{r^2} \right) + O\left(\frac{y}{N \log P} \sum_{r \le L} \frac{1}{r} \right).$$

Hence

$$E_1 = O\left(\frac{y}{(\log P)^2} \right) + O\left(\frac{y \log\log x}{N \log P} \right),$$

where the summation over r in the second term has introduced an extra factor $O(\log L) = O(\log\log x)$.

Since by hypothesis $P > 2N > x^{\epsilon}$, we have

$$E_1 = O\left(\frac{y}{(\log x)^2} \right) + O\left(\frac{y \log\log x}{x^{\epsilon} \log x} \right).$$

Hence

$$E_1 \ll \frac{y}{(\log x)^2}.$$

The sum over larger common factors $r > L$ in the second term of (2.10) was shown to be $\ll y(\log x)^2/L + x^{\frac{1}{3}+3\epsilon}$ (see section 2).

Also as P and $\log P$ are no larger than $x^{\frac{3}{4}}$ and $\log x$ respectively, the main term given by the first term of (2.10) is

$$\sum_{r \le L} \mu(r) \frac{y(\log 2)^2}{r^2 \log P} = \frac{y(\log 2)^2}{\log P} \sum_{r \le L} \frac{\mu(r)}{r^2} \gg \frac{y}{\log x}, \tag{2.11}$$

since the sum is finite.

From (2.10) we may then conclude that

$$\sum_{\substack{mnp\in I\\(m,n)=1,p\nmid m}} 1 = \sum_{r\le L}\mu(r)\frac{y(\log 2)^2}{r^2\log P}$$
$$+O\left(\frac{y(\log x)^2}{L}+x^{\frac{1}{3}+3\epsilon}+\frac{y}{(\log x)^2}+\frac{y}{x^\eta}+x^{\frac{2}{5}}+yx^{\epsilon-\frac{1}{8}}\right).$$

Since $L=(\log x)^A$ (see section 2) we can therefore choose $A=4$ giving $L=(\log x)^4$ thus producing the anticipated error term in (2.3). Hence, for sufficiently large x the error term E_1 is a power of $\log x$ smaller than the main term (2.11) and E_2, E_3 and E_4 are a power of x smaller than the main term. Furthermore the upper bound obtained in (2.4) suffices for larger common factors (the second term of (2.10)). We have therefore established Theorem 1.

The corollary follows immediately from Theorem 1 since for $n\sim N$ and $p\sim P$ given $n<p$ with N and P about $x^{\frac{1}{3}}$ in size we have $P<2x^{\frac{1}{3}}<x^{\frac{2}{5}-\epsilon}$ with $NP\approx x^{\frac{2}{3}}\le x^{\frac{3}{4}}$ which satisfies the conditions of Theorem 1.

The asymptotic value of the main term is given by

$$\sum_{r\le L}\mu(r)\frac{y(\log 2)^2}{r^2\log P}=\frac{y(\log 2)^2}{\log P}\sum_{r\le L}\frac{\mu(r)}{r^2}=\frac{6y(\log 2)^2}{\pi^2\log P}(1+o(1)).$$

In particular there are integers of the form required in the corollary within the interval $(x,x+x^{\frac{1}{2}}]$. The number of such integers being

$$\frac{6y(\log 2)^2}{\pi^2\log P}(1+o(1)).$$

Chapter 3

Sums of Differences Between Consecutive Primes

3.1 Introduction

We consider the sum of differences between consecutive primes p_n and p_{n+1} where p_n is the n-th prime. The result will be a generalisation of a result by Matomäki [26], since the interval $x^{1/2-\Delta}$ we shall consider will in fact be arbitrarily shorter than the interval considered in that paper which dealt with those consecutive primes with gaps greater than the fixed interval $x^{1/2}$. More specifically we show that sums of differences between consecutive primes with gaps greater than $x^{1/2-\Delta}$ where $0 \leq \Delta \leq -3 + \frac{1}{6}\sqrt{327}$ (the upper bound being a quadratic irratonal of approximate value 0.01385...) provides significant refinements of existing results. In particular we are motivated by the applications of this result. These include a significant refinement to a result on a prime-representing function [27] which is explored in chapter 4 (see Theorem 3) and a corollary providing further insight into the sum of square differences between consecutive primes [31]. There is potential for a further corollary on Diophantine approximations [10] however this will require an extension of the results on squared differences between consecutive primes. It is hoped that the result established will give further insight.

A significant finding of this investigation is that a key lemma, Lemma 13, has been shown to have the greatest dependence on T_1 (as defined in Lemma 8) arising from the $\widehat{R}^5$ term from the large and mean value results (3.38) and (3.40) and it is this which restricts further improvements that are possible via more refined large value, sieve and other approaches. As will be seen in the following sections the exponent of 5Δ in the upper bound of the main theorem, Theorem 2, arises predominently from Lemma 17 which ultimately determines the size of the exponent of this upper bound.

The result of this investigation represents in some sense a perturbation of the result in [26] however the nonlinear nature of the large and mean value results for Dirichlet polynomials results in a significant nontrivial change to many of the lemmas required to achieve the result of this paper. For instance, the final estimates for the sizes of the required polynomials are affected by the additional fixed $\Delta > 0$ in many of the exponents found in the lemmas and in the presence of additional terms as the reader will discover in the subsequent sections.

The overall approach will be to transform the problem to one involving the counting of primes in short intervals and then applying the seive method of Harman to provide asymptotic formulae used in the decomposition of the sums via iterations of Buchstab's identity. For clarity a synopsis of the method is provided in the following section.

At present it has been established that [26]:

$$\sum_{\substack{p_{n+1}-p_n > x^{1/2} \\ x \le p_n \le 2x}} p_{n+1} - p_n \ll x^{2/3} \tag{3.1}$$

where the interval between consecutive primes in the sum is greater than $x^{1/2}$.

We prove

Theorem 2. *Let p_n be the n-th prime and let $0 < \Delta < 1/48$ be a fixed positive number. Then,*

$$\sum_{\substack{p_{n+1}-p_n > x^{1/2-\Delta} \\ x \le p_n \le 2x}} p_{n+1} - p_n \ll x^{2/3+5\Delta}. \tag{3.2}$$

This theorem will be needed in the next chapter where it will be applied to produce significant results for prime-representing functions.

We point out that Theorem 2 holds true for $\Delta > 1/48$ but is no longer the most efficient approach as far as the applications of the result are concerned. Also, it will be convenient to have an upper bound on Δ for the lemmas and their proofs (for example in Case 2 of Lemma 14).

Next we obtain a corollary to the theorem which provides a result on the sum of squares differences between consecutive primes.

Corollary 2. *Let p_n be the n-th prime and let $\Delta > 0$ be a small positive number. Then*

$$\sum_{\substack{x^{1/2} \ge p_{n+1}-p_n > x^{1/2-\Delta} \\ x \le p_n \le 2x}} (p_{n+1} - p_n)^2 \ll x^{7/6+4\Delta}.$$

Proof. Let U be defined by $2^{-U} \le x^{-\Delta} < 2^{1-U}$ and V by $2^{V-1} < x^{1/40} < 2^V$. Then

$$\sum_{\substack{x^{1/2} \ge p_{n+1}-p_n > x^{1/2-\Delta} \\ p_n \in [x,2x]}} (p_{n+1} - p_n)^2 \ll \sum_{m=-U}^{V} 2^m x^{1/2} \sum_{\substack{p_{n+1}-p_n \sim 2^m x^{1/2} \\ p_n \in [x,2x]}} (p_{n+1} - p_n).$$

For $m \leq 0$, Theorem 2 gives

$$\sum_{\substack{p_{n+1}-p_n \sim 2^m x^{1/2} \\ p_n \in [x,2x]}} \ll x^{2/3} 2^{-5m}.$$

But since

$$\sum_{m=-U}^{V} 2^{-4m} \ll x^{4\Delta}$$

we have

$$\sum_{\substack{x^{1/2} \geq p_{n+1}-p_n > x^{1/2-\Delta} \\ p_n \in [x,2x]}} (p_{n+1}-p_n)^2 \ll \sum_{m=-U}^{V} x^{1/2} 2^m x^{2/3} 2^{-5m} \ll x^{7/6} \sum_{m=-U}^{V} 2^{-4m}$$

which is $\ll x^{7/6+4\Delta}$, as required.

We observe that since Peck's result (in his DPhil thesis [32]) bounds the squared differences sum by $x^{5/4}$, then if we set

$$\frac{7}{6} + 4\Delta < \frac{5}{4},$$

we now see that this corollary provides a better bound than Peck for $\Delta < \frac{1}{48}$.

We will use methods from Peck [31], Matomäki [26] and the sieve of Harman [14],[11],[12], and we begin with the proof of the following lemma which is essentially derived from the first two of these papers. The lemma will then enable the problem to be reduced to considering primes in short intervals. In the lemma we start with a formula for the number of primes in a short interval (3.3) and consider a prime p_n counted by the sum (3.5). We note that there are no primes in the interval $(y, y+\delta y)$ when $y \in (p_n, \frac{1}{2}(p_{n+1}+p_n))$ and $\delta = 1/(4x^{1/2+\Delta})$ and find that we may conclude that to estimate the sum (3.5) it suffices to obtain mean value estimates of the functions $A^{(i)}(x,y)$ of the form (3.4). We will be using 4-th and 6-th powers ($i = 4, 6$).

Lemma 7. *Let $a \geq 21/40$ and $\delta_{\mathcal{A}} = 1/(4x^{1/2+\Delta})$. If we suppose there exists a constant $c > 0$ and functions $A(x,y), A^{(4)}(x,y), A^{(6)}(x,y)$ and $E(x,y)$ such that for fixed $\Delta > 0$*

$$\pi(y + \delta_{\mathcal{A}} y) - \pi(y) \geq \frac{\delta_{\mathcal{A}} y}{\log y}(c + A(x,y) + E(x,y)), \tag{3.3}$$

where $E(x,y) = o(1)$, $A(x,y) \ll |A^{(4)}(x,y)| + |A^{(6)}(x,y)|$ and where

$$\int_x^{2x} |A^{(i)}(x,y)|^i dy \ll x^a, \text{ for } i = 4, 6. \tag{3.4}$$

Then

$$\sum_{\substack{p_{n+1}-p_n > x^{1/2-\Delta} \\ x \leq p_n \leq 2x}} p_{n+1} - p_n \ll x^a. \tag{3.5}$$

Proof. We select a prime p_n counted by the sum (3.5) and since there is at most one such prime p_n which has consecutive prime $p_{n+1} > 2x$ this can contribute to the sum in (3.5) by at most $\ll x^{21/40} \leq x^a$ by [5]. So, as this will not affect the bound in (3.5), we may assume that $p_{n+1} < 2x$. Next choose $y \in (p_n, \frac{1}{2}(p_n + p_{n+1}))$ then we see that (by writing $y/4x^{1/2+\Delta}$ in the form$(y/2x).(x^{1/2-\Delta}/2))$

$$y + \delta_{\mathcal{A}} y \leq \frac{1}{2}(p_n + p_{n+1}) + \frac{y}{2x}.\frac{x^{1/2-\Delta}}{2} < \frac{1}{2}(p_n + p_{n+1}) + \frac{y}{2x}\frac{1}{2}(p_{n+1} - p_n)$$

since we have $p_{n+1} - p_n > x^{1/2-\Delta}$. As $y \in (x, 2x)$ we note that $y/2x < 1$ hence the above inequality gives

$$y + \delta_{\mathcal{A}} y < \frac{1}{2}(p_n + p_{n+1}) + \frac{1}{2}(p_{n+1} - p_n) = p_{n+1}.$$

This implies that $\pi(y+\delta_{\mathcal{A}}y)-\pi(y)=0$ and hence $A(x,y)\gg 1$ in the interval $(p_n, \frac{1}{2}(p_n+p_{n+1}))$. For each p_n counted by (3.5) there is an interval of length $\gg p_{n+1}-p_n$ where $|A^{(4)}|+|A^{(6)}|\gg |A|\gg 1$. As these intervals are disjoint the assertion of the lemma follows by integrating over $(x,2x)$ since we have

$$\int_x^{2x} |A^{(4)}(x,y)|^4 dy + \int_x^{2x} |A^{(6)}(x,y)|^6 dy \gg \sum_{\substack{p_{n+1}-p_n>x^{1/2-\Delta} \\ p_n,p_{n+1}\in(x,2x)}} \int_{p_n}^{1/2(p_n+p_{n+1})} dy$$

$$\gg \sum_{\substack{p_{n+1}-p_n>x^{1/2-\Delta} \\ p_n,p_{n+1}\in(x,2x)}} p_{n+1}-p_n.$$

By hypothesis the integrals on the left hand side of the above inequalities are $\ll x^a$ which completes the proof.

We will proceed to prove that the assumptions of the above lemma are satisfied for the shorter interval $p_{n+1}-p_n > x^{1/2-\Delta}$ for $a = 2/3+5\Delta+\epsilon$ for any $\epsilon > 0$.

3.2 Dirichlet Polynomials and Definitions

In this section we begin by introducing some of the main parameters used in the subsequent sections and then define some of the key properties of the Dirichlet polynomials we use. Several properties of Dirichlet polynomials are further detailed for reference in the introductory chapter.

Throughout this chapter we let $\Delta > 0$ be a small *fixed* number.

The resulting theorem will generalise Theorem 1.1 of [26] and we now state the parameters which are used throughout the following sections for clarity. Let

$$\gamma = \exp{(\log x)^{9/10}}$$

$$S_0 = \exp\left(2\left(\frac{\log x}{\log \gamma}\right)^2 \log\log x\right) = \exp(2(\log x)^{1/5}\log\log x).$$

$$T_0 = \exp\left(\tfrac{1}{8}(\log x)^{1/2}\right).$$

Let η and η_i for $i = 1, 2$ be small *arbitrary* positive numbers.

We use variables x and x_1 (defined later) which will be related by the inequality

$$x \ll x_1^{1+\eta}.$$

We let,

$$T_1 = x^{1/2+\Delta+\eta}.$$

By the above relations we have

$$T_1 \ll x_1^{1/2+\Delta+2\eta}.$$

We now introduce assumptions regarding the shape of polynomials we consider. We wish to consider those polynominals which consist of one of two main forms. Firstly to enable the optimum use of mean value and large value results for Dirichlet polynomials we wish to work with polynominals and their products which are of short length. The second form of polynomial which we would like to work with are those which may consist of products of zeta factors since results for these can be achieved by relatively straightforward methods.

Heath-Brown generalised Vaughan Identity fortunately often provides Dirichlet polynomials with either longer zeta factors or many prime factored short Dirichlet polynomials. We raise these Dirichlet polynomials to powers to obtain another Dirichlet polynomial whose length is as near T_1 (as defined in Lemma 8) in value as possible and this can be achieved if the polynomials are short.

For these reasons we will consider only polynomials which are composed of a finite product of several Dirichlet polynomials and with properties which restrict the overall shape as detailed in the following assumption:

ASSUMPTION: We restrict the choice of polynomials $F(s)$ to the following type. Let

$$F(s) = \left(\prod_{i=1}^{k} G_i(s) \right) H(s) = G(s)H(s), \tag{3.6}$$

where $F \sim x$, $k \geq 2$ is bounded, all the polynomials are divisor-bounded, $H \ll x^{o(1)}$, $G_i > \eta = \exp((\log x)^{9/10})$ and all $G_i(s)$ are prime-factored. Further, we assume that all the polynomials $G_i(s)$ with length greater than $G^{1/8}$ are zeta factors.

We now develop the machinery required to obtain asymptotic formulae of the type

$$\sum_{m \in \mathcal{A}} c_m = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} \sum_{m \in \mathcal{B}} c_m + \frac{\delta_{\mathcal{A}} y}{\log y} (A(x,y) + O((\log x)^{-B})). \tag{3.7}$$

We may consider c_m as coefficients of the Dirichlet polynomial

$$F(s) = \sum_{m \sim x} c_m m^{-s}$$

and we next use the Perron formula to convert the problem of obtaining an asymptotic formulae of type (3.7) to that of bounding Dirichlet polynomials using this polynomial $F(s)$ in the following lemma (see also [14] Lemma 7.2 p122):

Lemma 8. *Let*

$$F(s) = \sum_{m \sim x} c_m m^{-s}$$

be a divisor-bounded Dirichlet polynomial. Then

$$\sum_{m \in \mathcal{A}} c_m = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} \sum_{m \in \mathcal{B}} c_m \tag{3.8}$$

$$+\frac{\delta_{\mathcal{A}} y}{\log y} \left(\frac{\log y}{2\pi i} \left(\int_{T_0}^{T_1} + \int_{-T_1}^{-T_0} \right) f(t) y^{-\frac{1}{2}+it} dt + O((\log x)^{-B}) \right)$$

where $f(t) \ll |F(\frac{1}{2}+it)|$.

Proof. The proof is based on [16] and can also be found in [26]. By the Perron formula (using the same form of the Perron formula as in [16] p1371)

$$\sum_{y<n\leq y+\delta y} c_n = \frac{1}{2\pi i} \int_{1/2-iT_1}^{1/2+iT_1} F(s) y^s \frac{(1+\delta)^s - 1}{s} ds \tag{3.9}$$

$$+O\left(x^{\eta/2} \left(1 + \frac{x}{T_1} \right) \right)$$

for $\delta = \delta_{\mathcal{A}}$ and also for $\delta = \delta_{\mathcal{B}}$. Let $s = 1/2 + it$ and let

$$C(\delta, s) = \frac{(1+\delta)^s - 1}{s}.$$

We separate the integral into regions $|t| \leq T_0$ and $T_0 \leq |t| \leq T_1$. Since $\delta T_0 < 1$ we now have

$$C(\delta, s) = \begin{cases} \delta + O(T_0 \delta^2) & \text{if } |t| \leq T_0, \\ O(\delta) & \text{if } T_0 \leq |t| \leq T_1. \end{cases}$$

We may therefore write the integral as

$$\frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s)y^s\frac{(1+\delta)^s-1}{s}ds = \delta\frac{1}{2\pi i}\int_{1/2-iT_0}^{1/2+iT_0} F(s)y^s ds$$

$$+O(T_0^2\delta^2 x(\log x)^C).$$

Hence in (3.9), letting

$$f(t) = F\left(\frac{1}{2}+it\right)\left(\frac{C(\delta_{\mathcal{A}},s)}{\delta_{\mathcal{A}}} - \frac{C(\delta_{\mathcal{B}},s)}{\delta_{\mathcal{B}}}\right), \tag{3.10}$$

the proof is complete.

Next we identify the integral between positive limits T_0 to T_1 in (3.8) and call it $B_1(x,y)$:

$$B_1(x,y) = \int_{T_0}^{T_1} f(t)y^{-\frac{1}{2}+it}dt.$$

We are interested in using mean and large value estimates for Dirichlet polynomials to bound $B_1(x,y)$ when the Dirichlet polynomial $F(s)$ (as in (3.10)) is of a certain type.

We note firstly that $B_1(x,y)$ is bounded by the absolute value of the integral between T and $2T$ for some $T_0 \le T \le T_1$ at which the absolute value within these limits is maximum. Explicitly

$$B_1(x,y) \ll (\log T_1)\max_{T_0\le T\le T_1}\left|\int_T^{2T} f(t)y^{-\frac{1}{2}+it}dt\right|.$$

In view of this bound we exploit large value estimates for polynomials $F(s)$ and we introduce integrals $J_{n,T}$ (for positive integers n where $T-1<n<2T$), which are essentially a collection of the integrals of the form $B_1(x,y)$ but in integer steps of limits of integration between $\max(n,T)$ and $\min(n+1,2T)$. These $J_{n,T}$ will be of key importance in the remainder of the sections and will form the basic objects whose bounds and bounds of power moments will be used to achieve the proof of Lemma 9 and identify polynomials for which there exist asymptotic formulae for the sieve. The *definition* is

$$J_{n,T} = \int_{\max(n,T)}^{\min(n+1,2T)} f(t) y^{-\frac{1}{2}+it} dt. \tag{3.11}$$

We follow the argument of Peck in [31] and write

$$\left| \int_T^{2T} f(t) y^{-\frac{1}{2}+it} dt \right| = \left| \sum_{\substack{T-1<n<2T \\ n\equiv 0 \mod 2}} J_{n,T} + \sum_{\substack{T-1<n<2T \\ n\equiv 1 \mod 2}} J_{n,T} \right|$$

$$\ll \left| \sum_{\substack{T-1<n<2T \\ n\equiv \kappa \mod 2}} J_{n,T} \right| \text{ for either of } \kappa = 0 \text{ or } \kappa = 1 \ .$$

Next select the value of t for which $F(1/2+it)$ is maximum in the range $\max(n,T) \leq t \leq \min(n+1,2T)$, where $n \equiv \kappa \mod 2$. We *define* this value of t to be t_n.

Observe that the points $t_n \in [T, 2T]$ are *well-spaced*: $|t_m - t_n| \geq 1$ for all $m \neq n$.

We next divide x into dyadic blocks and write $u_r = x/2^r$ for some $0 \leq r \leq R$ and where $x/2^R < x^{-1} \leq x/2^{R-1}$. Using this definition of u_r for convenience we *define* the k-tuple

$$\mathbf{u} = (u_0, ..., u_k) \text{ and } u_i = \frac{x}{2^r} \text{ for some } 0 \leq r \leq R \tag{3.12}$$

as a means of referring to the collection of all u_i.

We now introduce the set $I(\mathbf{u})$ which is of *central importance* in this investigation and is *defined* as

$$I(\mathbf{u}) = \text{ The set of } n \text{ for which } t_n \text{ satisfies both} \tag{3.13}$$

$$u_i < \left|G_i\left(\frac{1}{2}+it_n\right)\right| \leq 2u_i \text{ and } u_0 < \left|H\left(\frac{1}{2}+it_n\right)\right| \leq 2u_0,$$

where G_i and H are as defined in (3.6).

Crucially it is the number of elements of this set (or cardinality) *defined* as

$$R = |I(\mathbf{u})| \tag{3.14}$$

which we use to find the key bounds for the sums of the $J_{n,T}$ and their power moments.

From the definitions we note that there are $O((\log x)^{k+1})$ sets $I(\mathbf{u})$. If there is a factor of size $\ll x^{-1}$ (where the size of a polynomial $K_i(s)$ is defined as w if $|K_i(s)| \sim w$) for some t_n then

$$|f(t_n)y^{-1/2+it_n}| \ll x^{-1}x^{1/2}x^{-1/2}(\log x)^C,$$

so these factors contribute in total $\ll x^{-1}T_1(\log x)^C$. We therefore have

$$B_1(x,y) \ll (\log x)^C \max_{T,u}\left|\sum_{n\in I(\mathbf{u})} J_{n,T}\right| + O((\log x)^{-B}).$$

In this expression we can now estimate the $J_{n,T}$ by using, in (3.11) the bound

$$|f(t)| \ll \prod_i u_i. \tag{3.15}$$

Fortunately the cardinality R of the set $I(\mathbf{u})$ is restricted by the fact that there exist exactly $R = |I(\mathbf{u})|$ well-spaced points t_n satisfying $|G_i(s)| \geq u_i$ and $|H(s)| \geq u_0$ by definition. Hence we are able to use established mean and large value estimates for Dirichlet polynomials (see Lemma 12 and in particular (3.31)) to bound $|I(\mathbf{u})|$ and therefore obtain bounds for $B_1(x, y)$ thereby obtaining the requisite asymptotic formulae of the form (3.7).

We now introduce the definition of a *good Dirichlet polynomial* $F(s)$. This will be a polynomial which via our definition (3.11) for $J_{n,T}$, which is an integral with $f(t)$ in its integrand, and also by (3.10) (relating $f(t)$ by its definition to the polynomial $F(t)$) will by definition immediately provide asymptotic formulae of the type (3.7).

Definition: A Dirichlet Polynomial $F(s)$ for this problem is a *good* Dirichlet Polynomial if there exists a partition $K_1 \cup K_2 \cup K_3$ of possible values of $(T, \mathbf{u})$ then :

$$\left| \sum_{n \in I(\mathbf{u})} J_{n,T} \right| \ll (\log x)^{-B}, \text{ if } (T, \mathbf{u}) \in K_1 , \tag{3.16}$$

$$\int_x^{2x} \left| \sum_{n \in I(\mathbf{u})} J_{n,T} \right|^4 dy \ll x^{2/3+5\Delta+\epsilon}, \text{ if } (T, \mathbf{u}) \in K_2 , \tag{3.17}$$

$$\int_x^{2x} \left| \sum_{n \in I(\mathbf{u})} J_{n,T} \right|^6 dy \ll x^{2/3+5\Delta+\epsilon}, \text{ if } (T, \mathbf{u}) \in K_3 \tag{3.18}$$

for any $\epsilon > 0$ and fixed $\Delta > 0$.

We will prove the following lemma which will form the basis for identifying good polynomials in the following sections. The proof will occupy several of the subsequent sections and the approach to the proof will be structured to follow of the stages of the arguments of Matomäki [26] and Peck [31]. However, we will in this chapter be required to make some major adaptations to the arguments of those works. In Matomäki [26] the results from Peck [31] could be in certain instances (for example the case for longer zeta factors) quoted verbatim, whereas the shorter intervals required in Theorem 2 cause technical difficulties in certain ranges. Examples of some of the more significant adaptations can be seen, for instance, in the section on longer zeta factors and in the final estimates Lemma 17.

Lemma 9. *Let $F(s) = G(s)H(s)$ be a Dirichlet polynomial of the assumed form (3.6). Let $N_i(s)$ be a Dirichlet polynomial of length $N_i = x^{\beta_i}$. Let $K(s), K_1(s)$ and $K_2(s)$ be zeta factors. Then $F(s)$ is good if one of the following hold:*

(i) $G(s) = N_1(s)N_2(s)N_3(s)K(s)$,

$$\beta_1 \leq \frac{1}{2}, \beta_3 \leq \beta_2, 2\beta_2 + \beta_3 \leq \frac{1}{2} \text{ and } \frac{3}{4}\beta_2 + \beta_3 \leq \frac{1}{4}$$

(ii) $G(s) = N_1(s)N_2(s)N_3(s)K(s)$,

$$\beta_1 \leq \frac{1}{2}, \beta_3 \leq \beta_2, 2\beta_2 + \beta_3 \leq \frac{1}{2} \text{ and } \beta_3 \leq \frac{1}{8}$$

(iii) $G(s) = N_1(s)N_2(s)N_3(s)K(s)$,

$$\beta_1 \leq \frac{1}{2}, \beta_3 \leq \frac{1}{16}, \text{ and either } \beta_2 \leq \frac{1}{4} \text{ or } N_2 \text{ is a zeta factor.}$$

(iv) $G(s) = N_1(s)N_2(s)K(s)$,

$$\beta_1 \leq \frac{1}{2} \text{ and } \beta_2 \leq \frac{9}{32}$$

(v) $G(s) = N_1(s)N_2(s)$,

$$\text{where } \frac{13}{27} \leq \beta_i \leq \frac{14}{27}$$

(vi) $G(s) = N_1(s)N_2(s)$,

$$\text{where } \frac{7}{15} \leq \beta_i \leq \frac{8}{15}$$

and $G(s)$ *can be grouped into products* $H(s)$ *such that* $G(s) = \prod_{i=1}^{j} H_i(s)$, *where each* $H_i = x^{\gamma_i}$ *is either a zeta-factor of length* $\geq x^{\frac{1}{4}-\eta_2}$ *or* $\gamma_i \in \mathcal{G}$ *where* $\mathcal{G}$ *is the union of the following intervals*

$$\mathcal{G} = \left(0, \frac{41}{180}\right] \cup \left[\frac{13}{54}, \frac{1}{4} - \eta_2\right] \cup \left[\frac{1}{3}, c\right]$$

where $c = \frac{751+\sqrt{11041}}{1920} = 0.445873...$ *(see* (3.49) *for full derivation of* c*).*

(vii) $G(s) = K_1(s)K_2(s)N_1(s)N_2(s)N_3(s)$,

$$\text{where } \beta_3 \leq \beta_2 \leq \frac{1}{8} \text{ and } \beta_1 + \beta_2 \leq \frac{1}{2}$$

(viii) $G(s) = K_1(s)K_2(s)N_1(s)N_2(s)$

$$\text{where } \beta_2 \leq \frac{1}{8} \text{ and } \beta_1 \leq \frac{1}{2}.$$

The lemma will provide several shapes of Dirichlet polynomial which will be used to provide asymptotic formulae to be applied to the sieve in the final sections. We begin in the next section by setting up the main definitions and terminology required for the remainder of this chapter and introduce the fundamentally important functions $R^{(k)}$ which enable the required bounds (3.16), (3.17) and (3.18) to be established for polynomials.

3.3 Functions R and $R^{(k)}$: Establishing Good Polynomials

The main objective of this section is to introduce and define the functions R and $R^{(k)}$ which will prove to be of fundamental importance in providing bounds for the left hand sides of (3.16), (3.17) and (3.18). The functions will then enable, by our definition, the idenitfication of *good* polynomials.

Before defining R and $R^{(k)}$ we start by introducing some further terminology and definitions which we use for Dirichlet polynomials.

DEFINITIONS

For clarity we initially list here the definitions for $a \sim A$ and for the size and length of polynomials from the introductory chapter.

We use the expression $a \sim A$ when $A < a \leq 2A$.

The **length** of the general Dirichlet polynomial $R(s) = \sum_{r \sim A} f(r) r^{-s}$ is defined as A.

We define the **size** of a polynomial $K_i(s)$ as w if $|K_i(s)| \sim w$.

From our general assumption about the shape of the required polynomials we recall that

$$F(s) = G(s)H(s) = \prod_{i=1}^{k} G_i(s)H(s),$$

where we now define x_1 to be the length of $G(s)$ and hence by definition the product of the lengths of the $G_i(s)$

$$x_1 = \prod_{i=1}^{k} G_i = G, \tag{3.19}$$

and we defiine x_2 as the length of $H(s)$ whose value H we use to estimate trivially. Let

$$x_2 = H.$$

We see that these definitions allow us to write

$$x_1 x_2 = F \text{ where } F \sim x.$$

Next we define the size of the $G_i(s)$ at $s = 1/2 + it_n$ as u_i so that

$$\left| G_i\left(\frac{1}{2} + it_n\right) \right| \sim u_i \tag{3.20}$$

where we now define $0 \leq \sigma_{G_i} \leq 1$ by

$$u_i = G_i^{\sigma_{G_i} - \frac{1}{2}} \text{ for } i = 1, ..., k.$$

This enables the size of $G_i(s)$ to be expressed as a power of its length with the trivial value corresponding to $\sigma_{G_i} = 1$.

In particular we define $0 \leq \sigma \leq 1$ by

$$\prod_{i=1}^{k} u_i = x_1^{\sigma - \frac{1}{2}} = G^{\sigma - \frac{1}{2}} \text{ where } \sigma = \sigma_G.$$

This definition of σ will be central to the proof of the main theorem as we will establish the existence of non-trivial bounds for all values of sigma $0 \leq \sigma \leq 1$.

In a similar manner we define the size of $H(s)$ at $s = 1/2 + it_n$ as u_0 so that

$$\left|H\left(\frac{1}{2}+it_n\right)\right| \sim u_0 \tag{3.21}$$

but this time we note that

$$u_0 = x_2^{1/2}$$

since we will take the trivial estimate for $H(s)$.

The definition of σ_{G_i} above will be extended in a natural way for products of the $G_i(s)$ so that if $G = G_1 ... G_k = x_1^{\kappa_1} ... x_1^{\kappa_k}$ we have

$$\sigma = \sigma_G = \sum_{i=1}^{k} \frac{\log G_i}{\log x_1} \sigma_{G_i} = \sum_{i=1}^{k} \kappa_i \sigma_{G_i}$$

$$\sigma_N = \max_i(\sigma_{G_i})$$

We will use the following notation throughout the following sections:

$$N(s) = G_i(s) \text{ with the largest } \sigma_N \text{ amongst the } G_i(s)$$

In the case of section (vi) of the lemma at the end of the last section we use this $N(s)$ to denote the same but for $H_i(s)$ instead.

So in particular we may now write

$$\left|\prod_{i=1}^{k} G_i\left(\frac{1}{2}+it_n\right)\right| \sim \prod_{i=1}^{k} u_i = x_1^{\sigma-\frac{1}{2}},$$

which along with the definition of u_0 gives an important expression for the size of the assumed shape of the polynomial :

$$\left|F\left(\frac{1}{2}+it_n\right)\right| = \left|\prod_{i=1}^{k} G_i\left(\frac{1}{2}+it_n\right) H\left(\frac{1}{2}+it_n\right)\right| \sim \left(\prod_{i=1}^{k} u_i\right) u_0$$

$$= x_1^{\sigma-\frac{1}{2}} x_2^{\frac{1}{2}}.$$

We now use the above definitions and the expression defining $f(t)$ in terms of the Dirichlet polynomial $F(\frac{1}{2}+it)$ in the previous section together with the assumption regarding the shape of $F(s)$ as $F(s) = \prod_{i=1}^{k} G_i(s)H(s)$ to write an upper bound for $f(t)$ in terms of powers of the sizes of these polynomials. From the above list of definitions we may now write

$$|f(t)| \ll \left|F\left(\frac{1}{2}+it\right)\right| \ll x_1^{\sigma-\frac{1}{2}} x_2^{\frac{1}{2}}. \tag{3.22}$$

We recall the following important definition introduced in (3.13) which will be key to the proofs

$$I(\mathbf{u}) = \text{The set of } n \text{ for which } t_n \text{ satisfy (3.20) and (3.21)}$$

where (each u_i as defined in (3.20) and (3.21)) the $\mathbf{u}$ was defined in (3.12).

We point out that in definition (3.12) we restrict the smallest values of u_i by $x/2^R < x^{-1} \leq x/2^{R-1}$ so that there are $O((\log x)^{k+1})$ sets $I(\mathbf{u})$.

Using the important definition for R in (3.14) together with (3.15) and (3.22) we may bound the left and side of (3.16):

$$\left|\sum_{n\in I(\mathbf{u})} J_{n,T}\right| \ll x_1^{\sigma-\frac{1}{2}} x_2^{\frac{1}{2}} \left|\sum_{n\in I(\mathbf{u})} 1\right| \ll x_1^{\sigma-\frac{1}{2}} x_2^{\frac{1}{2}} x^{-\frac{1}{2}} R \ll x_1^{\sigma-1} R.$$

We now introduce functions $R^{(2)}$ and $R^{(3)}$ to bound the left hand sides of (3.17) and (3.18) respectively. The beauty of these functions is the ability to estimate them using methods introduced by Heath-Brown [18],[19]. We proceed by expanding the left hand side of (3.17):

$$\int_x^{2x} \left| \sum_{n\in I(\mathbf{u})} J_{n,T} \right|^4 dy = \int_x^{2x} \left| \sum_{n\in I(\mathbf{u})} \int_{(n)} f(t) y^{-1/2+it} dt \right|^4 dy$$

$$= \int_x^{2x} \sum_{n\in I(\mathbf{u})} \int_{(n_1)} ... \int_{(n_4)} f(t_1)f(t_2)\overline{f(t_3)f(t_4)} y^{-2+i(t_1+t_2-t_3-t_4)} dt_1..dt_4 dy$$

Since $f(t)$ is independent of y, we can integrate the above firstly by y to find that it is

$$\ll x_1^{4\sigma-2} x_2^2 \sum_{n_1,...,n_4} \int_{(n_1)} \int_{(n_2)} ...$$

$$... \int_{(n_4)} \left| \frac{(2x)^{-1+i(t_1+t_2-t_3-t_4)} - x^{-1+i(t_1+t_2-t_3-t_4)}}{-1+i(t_1+t_2+t_3+t_4)} \right| dt_1...dt_4.$$

But we have

$$|(2x)^{-1+i(t_1+t_2-t_3-t_4)} - x^{-1+i(t_1+t_2-t_3-t_4)}| \ll x^{-1}$$

and

$$|-1+i(t_1+t_2+t_3+t_4)| \gg 1+|t_1+t_2+t_3+t_4| \gg 1+|n_1+n_2-n_3-n_4|,$$

so that we now have the left hand side of (3.17) bounded as follows:

$$\int_x^{2x} \left| \sum_{n\in I(\mathbf{u})} J_{n,T} \right|^4 dy \ll x_1^{4\sigma-3} x_2 \sum_{n_1,...,n_4} \frac{1}{1+|n_1+n_2-n_3-n_4|}.$$

We also have

$$\sum_{n_1,\dots,n_4} \frac{1}{1+|n_1+n_2-n_3-n_4|} = \sum_{-2T\le m\le 2T} \frac{g(m)}{1+|m|}, \tag{3.23}$$

where $g(m)$ is the number of solutions of $n_1+n_2-n_3-n_4=m$ for $n_i \in I(\mathbf{u})$.

We claim that

$$g(m) \le g(0) \text{ for integers } m \ge 0 \ . \tag{3.24}$$

This is seen by first writing

$$N_2(a) = |\{(n_1,n_2) : n_i \in I(\mathbf{u}), n_1+n_2=a\}|,$$

whence

$$g(m) = \sum_{\substack{a,b\in\mathbb{Z}\\ a-b=m}} N_2(a)N_2(b) = \sum_{b\in\mathbb{Z}} N_2(b)N_2(b+m).$$

By Cauchy-Schwarz inequality we see this is

$$\le \left(\sum_{b\in\mathbb{Z}} N_2(b)^2\right)^{1/2} \left(\sum_{b\in\mathbb{Z}} N_2(b+m)^2\right)^{1/2} = \sum_{b\in\mathbb{Z}} N_2(b)^2 = g(0).$$

Hence the claim (3.24) is proved.

From (3.24) we see that the sum on the right hand side of (3.23) is $\ll (\log T) R^{(2)}$ where we have defined $R^{(2)}$ as the function

$$R^{(2)} = |\{(n_1,n_2,n_3,n_4) : n_i \in I(\mathbf{u}), n_1+n_2=n_3+n_4\}|.$$

Therefore we may now bound the left hand side of (3.17) as follows:

$$\int_x^{2x} \left| \sum_{n\in I(\mathbf{u})} J_{n,T} \right|^4 dy \ll (\log T) x_2 x_1^{4\sigma-3} R^{(2)},$$

We define $R^{(3)}$ in a similar way and bound (3.18) with this function (see (3.27)) and write in general for positive integer k:

$$R^{(k)} = |\{(n_1, ..., n_{2k}) : n_1 + ... + n_k = n_{k+1} + ... + n_{2k} \text{ for } n_i \in I(\mathbf{u})\}|.$$

Note that $R = R^{(1)}$.

THE BOUNDS FOR SUMS OF $J_{n,T}$ IN TERMS OF $R^{(k)}$

Employing the above terminology along with the fact that we have R well spaced points $s = 1/2 + it_n$ satisfying $|G_i(s)| \geq u_i$ we now list the expression for the bounds for sum of the $J_{n,T}$ and its power moments as in the left hand sides of (3.16), (3.17) and (3.18) in terms of R and $R^{(k)}$:

$$\left| \sum_{n \in I(\mathbf{u})} J_{n,T} \right| \ll x_1^{\sigma-1} R, \tag{3.25}$$

$$\int_x^{2x} \left| \sum_{n \in I(\mathbf{u})} J_{n,T} \right|^4 dy \ll (\log T) x_2 x_1^{4\sigma-3} R^{(2)}, \tag{3.26}$$

$$\int_x^{2x} \left| \sum_{n \in I(\mathbf{u})} J_{n,T} \right|^6 dy \ll (\log T) x_2 x_1^{6\sigma-5} R^{(3)}. \tag{3.27}$$

These bounds, now expressed in terms of $R^{(k)}$ will enable the development of a series of lemmas and the use of existing large and mean value results for Dirichlet polynomials to enable the identification of good polynomials and prove the assertions of the Lemma 9. In the next section we develop the main estimates required to achieve this objective.

3.4 Polynomial Products $M(s)$ and Estimates for R and $R^{(k)}$

We begin by estimating R. Recall that R is the number of well-spaced points $s = 1/2 + it_n$ satisfying $|G_i(s)| \geq u_i$.

Firstly we consider a general product of g of the Dirichlet polynomials $G_i(s)$ and we allow for the possibility of repetitions.

$$G(s) = \prod_{j=1}^{g} G_{i_j}(s) \text{ for any } 1 \leq i_j \leq k \text{ not necessarily distinct.}$$

We now introduce a change of symbol to emphasize this is a finite product and for clarity (as we will be using the letter g for other objects in the following sections) and we *define* this product to be $M(s)$. We may also write the product as the sum

$$M(s) = \sum_{M<k\leq M_0} m(k)k^{-s} \text{ with } M \leq x.$$

We may write this since in general the product of any two Dirichlet polynomials is a Dirichlet polynomial as detailed in the introductory chapter (**??**).

As we have a product of g polynomials in $M(s)$ we have $M_0 = 2^g M$. So our assumption that each $G_i(s)$ is bounded below by u_i at the R well-spaced points $s = 1/2 + it_n$ (so that $n \in I(u)$) provides a lower bound for the product $M(s)$. We call this lower bound w and write

$$|M(s)| \geq w$$

Using the notation of the previous section we write

$$w = M^{\sigma_M - \frac{1}{2}}.$$

We now introduce two Lemmas which will be used to obtain bounds in particular situations in the following sections. These are estimates based upon Deshouillers-Iwaniec mean value theorem [7] and Watt's mean value theorem [34] respectively.

Lemma 10. *Let $M(s) = N_1(s)N_2(s)K(s)$ with $N_i(s)$ some product of factors $G_i(s)$ with boundedly many repetitions, $N_i(s) = x^{\beta_i}$ and $K(s) = G_i(s)$ a zeta factor with $\beta_2 \leq \beta_1$, $2\beta_1 + \beta_2 \leq \frac{1}{2}$ and $\frac{3}{4}\beta_1 + \beta_2 \leq \frac{1}{4}$. Then*

$$R \ll T_1^{1+\epsilon} w^{-2}.$$

Proof.

From the proof of the Deshouillers-Iwaniec mean value theorem [7] and from Matomäki [26] we see that we have for $M_2 \leq M_1$

$$\int_T^{2T} \left| L\left(\frac{1}{2} + it\right) M_1\left(\frac{1}{2} + it\right) M_2\left(\frac{1}{2} + it\right)\right|^2 dt$$

$$\ll T^{\epsilon}(T + T^{1/2}M_1^{3/4}M_2 + T^{1/2}M_1M_2^{1/2} + M_1^{7/4}M_2^{3/2}). \tag{3.28}$$

In this expression $L(s) = \sum_{L \leq l \leq L_1} l^{-s}$ and $L < L_1 \leq 2L \leq T$. We use the following identity (see for instance Iwaniec and Kowalski [23] p 233) to convert the continuous mean value theorem to a discrete mean value theorem: for a smooth function $f(t)$ on $[0, 1]$ by partial integration we have (where we have put the actual value 1/2 in the argument instead of any arbitrary value in $[0,1]$)

$$\left|f\left(\frac{1}{2}\right)\right| = \left|\int_0^1 f(t)dt + \int_0^{1/2} tf'(t)dt + \int_{1/2}^1 (t-1)f'(t)dt\right|$$

$$\leq \int_0^1 \left(|f(t)| + \frac{1}{2}|f'(t)|\right) dt.$$

We let $f(t) = M(\frac{1}{2} + i(t - \frac{1}{2} + t_0))^2$ for each $t_0 \in I(\mathbf{u})$ and obtain

$$\sum_{t_0 \in I(\mathbf{u})} \left|M\left(\frac{1}{2} + it_0\right)\right|^2$$

$$\leq \int_T^{2T} \left(\left|M\left(\frac{1}{2} + it\right)\right|^2 + \left|M'\left(\frac{1}{2} + it\right) M\left(\frac{1}{2} + it\right)\right|\right) dt.$$

Then by applying (3.28) to the first of the integrands on the right and side we can now obtain

$$\int_T^{2T} |M(s)|^2 dt = \int_2^{2T} |K(s)N_1(s)N_2(s)|^2 dt \ll T_1^{1+\epsilon}.$$

To deal with the second integrand we note that

$$|M(s)M'(s)| \ll |N_1'(s)N_1(s)N_2(s)^2K(s)^2| + |N_2'(s)N_2(s)N_1(s)^2K(s)^2|$$

$$+|N_1(s)^2N_2(s)^2K(s)K'(s)|. \qquad (3.29)$$

We group the products on the right hand side of (3.29) in the first sum in as $(N_1(s)N_2(s)K(s)) = A$ and $(N_1'(s)N_2(s)K(s)) = B$, say. Then applying the Cauchy-Schwarz inequality (with limits suppressed) in the form $\int AB \leq (\int A^2)^{1/2}(\int B^2)^{1/2}$ we may write the integral of the first sum of the right hand side of (3.29) as

$$\int_T^{2T} |N_1'(s)N_1(s)N_2(s)^2K(s)^2|dt$$

$$\leq \left(\int_T^{2T} |N_1'(s)N_2(s)K(s)|^2dt\right)^{1/2} \left(\int_T^{2T} |N_1(s)N_2(s)K(s)|^2dt\right)^{1/2}.$$

Repeating this process of applying Cauchy-Schwarz inequality to the other two sums on the right and side of (3.29) and then factorizing we obtain

$$\int_T^{2T} |M(s)M'(s)|dt \ll \left(\int_T^{2T} |N_1(s)N_2(s)K(s)|^2dt\right)^{1/2} (I_1 + I_2 + I_3)$$

where

$$I_1 = \left(\int_T^{2T} |N_1'(s)N_2(s)K(s)|^2dt\right)^{1/2}$$

$$I_2 = \left(\int_T^{2T} |N_1(s)N_2'(s)K(s)|^2dt\right)^{1/2}$$

$$I_3 = \left(\int_T^{2T} |N_1(s)N_2(s)K'(s)|^2dt\right)^{1/2}.$$

We may now apply (3.28) as before since $F'(s)$ and $F(s)$ have the same length and the logarithm in the coefficients may be included in the error term. The last term in I_3 above involves $K'(s)$ rather than $K(s)$ however the logarithm can be removed by partial summation. Hence we have

$$\sum_{t\in I(\mathbf{u})} \left|M\left(\frac{1}{2}+it\right)\right|^2 \ll T_1^{1+\epsilon}.$$

However $|M(1/2+it)|^2 \geq w^2$ hence

$$w^2R = w^2 \sum_{t\in I(\mathbf{u})} 1 \leq \sum_{t\in I(\mathbf{u})} \left|M\left(\frac{1}{2}+it\right)\right|^2 \ll T_1^{1+\epsilon}$$

and we obtain

$$R \ll T_1^{1+\epsilon} w^{-2},$$

as required.

Lemma 11. *Let $M(s) = N_1(s)^2 K(s)^4$ with $N_1(s)$ some product of factors $G_i(s)$ with boundedly many repetitions $N_1(s) = x^{\beta_1}$ with $\beta_1 \leq \frac{1}{8}$ and $K(s) = G_i(s)$ a zeta factor. Then*

$$R \ll T_1^{1+\epsilon} w^{-1}.$$

Proof.

In the case that $N_1 \geq T$ we have $K < T \leq x^{1/8}$ so that by the Dirichlet Mean Value Theorem

$$R \ll (T + N_1 K^2) x_1^{\epsilon} w^{-1} \ll T_1^{1+\epsilon} w^{-1}.$$

Hence we may now assume $N_1 < T$. Then by [5] Lemma 2, which in turn is based on Watt's mean value theorem [34] and following the reasoning in Matomäki [26] we have

$$\int_T^{2T} \left| K_1 \left(\frac{1}{2} + it \right)^4 M_1 \left(\frac{1}{2} + it \right)^2 \right| dt \ll T^{1+\epsilon} (1 + M_1^2 T^{-1/2}),$$

with $K_1(s)$ a zeta factor and $M_1 \leq T$. The remainder of the proof follows the same approach as the previous lemma but using Hölder's inequality with weights $1/4, 3/4$ instead of the Cauchy-Schwarz inequality to separate $K(s)$ and $K'(s)$ in the term $4K(s)^3 K'(s) N_1(s)^2$. This completes the proof.

We will now denote by R_1 the best of the estimates achieved by the lemmas above (based on the Deshouillers-Iwaniec mean value theorem [7] and Watt's mean value therorem [34]). We compare these estimates to those achieved via Montgomery's mean value theorem for Dirichlet polynomials (see for instance [14] p 346), Huxley's large value theorem (see for instance

[14] Lemma 7.1) and estimates from the fourth and twelfth power moments of the Riemann zeta function. The combined set of results are amalgamated in the following lemma which will form the main point of reference for the estimates of Dirichlet polynomials throughout the paper.

Lemma 12. *(Estimates for Dirichlet Polynomials) Let T_1 be the limit of the integral specified in* (3.8) *with $T_0 \leq T \leq T_1$ where T_0 is as defined in the beginning of section 3.2 and let $J_{n,T}$ be as defined in* (3.11)*. Let S_0 be as defined in section 3.2 and x_1 as definition* (3.19) *(with $x \ll x_1^{\eta}$ for arbitrary $\eta > 0$) and let $C > 0$ be arbitrary. Then*

$$\left| \sum_{n \in I(\boldsymbol{u})} J_{n,T} \right| \ll S_0^4 (\log x)^C x_1^{\sigma-1} \widehat{R} \tag{3.30}$$

where

$$\widehat{R} = \min\big(\min_{M(s)=K_i(s)} (T_1 w^{-4}, T_1^2 w^{-12}), \tag{3.31}$$

$$\min_{M(s)} (M w^{-2} + T_1 w^{-2}, M w^{-2} + T_1 M w^{-6}), R_1 \big),$$

where the second minimum runs over all the zeta-factors $K_i(s) = G(s)$ and the the third minimum over all the products $M(s) = \sum_{k \sim M} m(k) k^{-s}$, possibly with repeats, of g of the Dirichlet polynomials $G_i(s)$ satisfying $M \leq x$ and $|M(s)| \sim w = M^{\sigma - \frac{1}{2}}$ specified at the beginning of this section.

We now conclude this section by stating the key lemma which will enable bounds to be establised for the left hand sides of (3.17) and (3.18). The full proof can be found in [31] section 10. For the purpose of the derivation it should be noted that the trivial bound $R^{(k)} \leq R^2 R^{(k-1)}$ together with (3.26) and (3.27) are used along with the bounds $(\log T) x_2 \ll x^{\eta_1}$ and $R \ll x^{\eta_1} \widehat{R}$, for arbitrary $\eta_1 > 0$.

Lemma 13.

$$\int_x^{2x} \left| \sum_{n\in I(\boldsymbol{u})} J_{n,T} \right|^4 dy \ll x^{14\eta_1} x_1^{4\sigma-3} \widehat{R}^{(2)},$$

and

$$\int_x^{2x} \left| \sum_{n\in I(\boldsymbol{u})} J_{n,T} \right|^6 dy \ll x^{24\eta_1} x_1^{6\sigma-5} \widehat{R}^{(3)},$$

where

$$\widehat{R}^{(2)} = \min(\widehat{R}^3, \min_{M(s)}(\widehat{R}^3 w^{-2} + T_1^{1/4}\widehat{R}^{21/8} w^{-2} + \widehat{R}^{5/2} M^{1/2} w^{-2} \tag{3.32}$$

$$+T_1^{4/5}\widehat{R}^{9/5} w^{-16/5} + T_1^{2/5}\widehat{R}^{8/5} M^{4/5} w^{-16/5} + \widehat{R} M^2 w^{-4})),$$

and

$$\widehat{R}^{(3)} = \min(\widehat{R}^5, \min_{M(s)}(\widehat{R}^5 w^{-2} + T_1^{1/4}\widehat{R}^{(2)3/8}\widehat{R}^{7/2} w^{-2} \tag{3.33}$$

$$+\widehat{R}^{(2)1/2}\widehat{R}^3 M^{1/2} w^{-2} + T_1^{4/5}\widehat{R}^{(2)3/5}\widehat{R}^2 w^{-16/5}$$

$$+T_1^{2/5}\widehat{R}^{(2)4/5}\widehat{R}^{6/5} M^{4/5} w^{-16/5} + \widehat{R}^{(2)} M^2 w^{-4})).$$

Here $\eta_1 > 0$ is arbitrary and the latter minima run over all the products $M(s)$ with possible repetitions, of g of the Dirichlet polynomials $G_i(s)$ satisfying $M \leq x$.

3.5 Longer Zeta Factors and the case $\sigma_N > \frac{5}{6} + \Delta + 3\eta_2$ or $\sigma_N < \frac{1}{2}$

We now show that at least one of the required bounds (3.16),(3.17) and (3.18) will be satisfied when the largest σ_N corresponds to a zeta factor of length greater than $x^{1/4-\delta}$. We will also show that σ_N larger than $5/6+\eta_2$ or smaller than $1/2$ will also result in the bounds being satisfied.

First, we consider the case $\sigma_N \leq 1/2$. Using the trivial bound $\widehat{R}^{(2)} \ll T_1^3$, we obtain

$$x_1^{4\sigma_N-3}\widehat{R}^{(2)} \ll x_1^{4\sigma_N-3}x_1^{3(\frac{1}{2}+\Delta+2\eta)} \ll x_1^{\frac{1}{2}+3\Delta+6\eta} \ll x_1^{\frac{2}{3}+5\Delta+\epsilon}.$$

In the following we may therefore assume without further comment that $\sigma_N > 1/2$.

We remark that its is in the next lemma that we are required to make a major adaptation of the work of Peck [31] and Matomäki [26]. In Matomäki [26] the results from Peck [31] could be quoted verbatim as Peck had proved much more than he needed, however the shorter intervals required in the proof of the result of this chapter instantly cause a problem in this range and requires significant adaptation.

Lemma 14. *Suppose δ is given with $0 < \delta \leq \eta$. Let $N(s)$ be the factor with largest σ_N. If this is a zeta factor of length $N > x^{\frac{1}{4}-\delta}$ then one of the bounds* (3.16),(3.17) *and* (3.18) *is satisfied.*

To see this we consider five cases. The first four of which correspond to those of section 11 of Peck [31].

First as defined in section 3.3 of this chapter we let $N(s)$ be a zeta factor with largest σ_N. Therefore $N(s) = G_i(s)$ with the largest σ_N among the $G_i(s)$ in all cases of lemma 9 except case (vi) where $N(s) = H_i(s)$ with the largest σ_N among the $H_i(s)$.

In the following we will let $w = N^{\sigma_N-1/2}$ and recall that $T_1 \ll x_1^{1/2+\Delta+2\eta}$ where $\Delta > 0$ is fixed and $\eta > 0$ is a small arbitrary postive number.

For simplicity of notation we suppress the N and write σ instead of σ_N in the following cases.

Case 1: $N \geq x^{1/4-\delta}$ and $\sigma \leq \frac{2}{3} + 2\Delta$. Using the trivial bound for $\hat{R}^{(2)}$ from (3.32) we have

$$\hat{R}^{(2)} \leq \hat{R}^3 \leq (T_1 w^{-4})^3 = T_1^3 w^{-12} = T_1^3 N^{6-12\sigma}\,.$$

Since $\sigma > 1/2$ this decreases with N so we can substitute the value $N = x_1^{1/4-\eta}$ to obtain

$$x_1^{4\sigma-3}\hat{R}^{(2)} \ll x_1^{4\sigma-3/2+3\Delta+6\eta} x_1^{(1/4-\eta)(6-12\sigma)} \ll x_1^{\sigma+3\Delta+12\eta}\,.$$

This will be $\ll x_1^{\frac{2}{3}+5\Delta+\epsilon}$ for $\sigma \leq \frac{2}{3} + 2\Delta$ which satisfies the required bound (3.17).

Case 2: $N > x^{1/3}$ and $\sigma \geq 2/3$. We include this case simply to be able to assume $N < x^{1/3}$ in case 4. We could have considered a case $N \geq x^{(18\Delta+1)/(48\Delta+4)+a\eta}$ for some fixed real $a > 0$ and $\sigma \geq \frac{2}{3} + 2\Delta$ if need be here. We use the following estimate from (3.31)

$$\hat{R} \leq T_1 w^{-4} \leq T_1 N^{2-4\sigma} \ll x_1^{1/2+\Delta+2\eta} x_1^{(2-4\sigma)/3}.$$

Hence

$$x_1^{\sigma-1}\widehat{R} \ll x_1^{\Delta+1/6-\sigma/3+2\eta}.$$

As the right hand side of the last bound decreases with increasing σ we substitute $\sigma = \frac{2}{3}$ and we obtain

$$x_1^{\sigma-1}\hat{R} \ll x_1^{2\eta+\Delta-1/18} \ll x_1^{-\eta}\,,$$

using $\Delta < 1/48$. We have thus established the required bound (3.16) for this case.

Case 3: $N \geq x^{1/4+2\eta}$ and $\sigma > \frac{3}{4} + \Delta$.

We use the bound from (3.31).

$$\hat{R} \leq T_1^2 w^{-12} \ll (x_1^{1/2+\Delta+2\eta})^2 N^{6-12\sigma}$$

$$\ll (x_1^{1/2+\Delta+2\eta})^2 (x^{1/4+2\eta})^{6-12\sigma} \ll x_1^{1-3\sigma+3/2+2\Delta+12\eta-24\sigma\eta+4\eta}$$

Hence

$$x_1^{\sigma-1}\hat{R} \ll x_1^{3/2-2\sigma+2\Delta++16\eta-24\eta\sigma}.$$

As the bound is decreasing with increasing σ we may substitute $\sigma = \frac{3}{4}+\Delta$ so we now obtain

$$x_1^{\sigma-1}\hat{R} \ll x_1^{4\eta-6\eta(1+4\Delta)} \ll x_1^{-\eta}.$$

Hence we see this satisfies (3.16).

Case 4: $N \geq x^{1/4-\delta}$ and $2/3 + \Delta \leq \sigma \leq \frac{8}{7}\left(\frac{2}{3} + \frac{17}{8}\Delta\right)$.

We take the bound for $\widehat{R}^{(2)}$ in (3.32) (using $\hat{R} \leq T_1 w^{-4} \leq T_1 N^{2-4\sigma}$) and then observe that the dominant term in the bound for $x_1^{4\sigma-3}\hat{R}^{(2)}$ in the range we are considering is the second term $(T_1^{1/4}\widehat{R}^{21/8}w^{-2})$. We justify this statement below. Assuming the truth of our assertion we obtain

$$x_1^{4\sigma-3}\hat{R}^{(2)} \ll x_1^{4\sigma-3}(x_1^{1/2+\Delta+2\eta})^{1/4}(x_1^{1-\sigma+\Delta+2\eta})^{21/8} x_1^{(1/4)(1-2\sigma)}$$

$$\ll x_1^{\frac{7}{8}\sigma+\frac{23}{8}\Delta+\frac{23}{4}\eta} \ll x_1^{\frac{2}{3}+5\Delta+\epsilon},$$

for $\sigma \leq \frac{8}{7}\left(\frac{2}{3} + \frac{17}{8}\Delta\right)$ as required.

We now check the contribution from the other terms in (3.32).

(a) $\hat{R}^3 w^{-2}$. This leads to a bound

$$x_1^{4\sigma-3} T_1^3 N^{7-14\sigma} \ll x_1^{4\sigma-3+3/2+3\Delta+6\eta+(1/4-\eta)(7-14\sigma)} .$$

This is no more than

$$x_1^{\sigma/2+1/4+3\Delta+13\eta} \leq x^{2/3+5\Delta+\epsilon} ,$$

for $\sigma \leq 5/6$. Hence this term gives a suitable contribution.

(b) $\hat{R}^{5/2} N^{1/2} w^{-2}$. This leads to a bound

$$x_1^{4\sigma-3} T_1^{5/2} N^{1/2-6(2\sigma-1)} .$$

For $\sigma > 2/3$ this is decreasing in N so the bound is

$$\leq x_1^{4\sigma-3+5\Delta/2+5/4+2\eta+(1/4-\delta)(1/2-6(2\sigma-1))} \ll x_1^{2/3+5\Delta+\epsilon}$$

for $\sigma < 19/24 + 5\Delta/2$. This is another suitable bound in the range.

(c) $T_1^{4/5} \hat{R}^{6/5} w^{-16/5}$. Working as above this give a suitable bound for $\sigma < 5/6 + 3\Delta/2$.

(d) $T_1^{2/5} \hat{R}^{8/5} N^{4/5} w^{-16/5}$. Working as in the previous cases this gives the correct bound for $\sigma < 19/24 + 15\delta/8$.

(e) $\hat{R} N^2 w^{-4}$. This leads to a bound

$$x_1^{4\sigma-3} x^{1/2+\Delta+2\eta} N^{6-8\sigma} .$$

By case 2 we can suppose $N < x^{\frac{1}{2}}$ and so the above is an increasing function of σ and for $\sigma > 3/4$ it is a decreasing function of N. We can therefore obtain a suitable bound when $\sigma < 5/6 + 2\Delta$.

Our claim that all the other terms in (3.32) give valid bounds in the range covered by this case is therefore vindicated.

Case 5: $x_1^{1/4-\delta} < N < x_1^{1/4+2\eta}$ and $\sigma > 16/21 + 17\Delta/7$. We argue as in Case 3. Our upper bound is now

$$x_1^{\sigma-1}x_1^{1+2\Delta+4\eta+(1/4-\delta)(6-12\sigma)} < x_1^{9\eta-1/42-20\Delta/7} < x_1^{-\eta}$$

for $\sigma > 16/21 + 17\Delta/7$ and assuming, as we may, that $\eta < 1/420$. This concludes the proof of this case and so the lemma is established.

We may now assume that in all cases (except case (vi) of Lemma 9) that the zeta factor $N(s)$ with the largest σ (which we defined as σ_N) has length $N < x^{1/4-\delta}$.

Next we note that for arbitrary $\delta > 0$ since

$$2.[1/6 - \delta/2, 1/4 - \delta] \subseteq [1/3 - \delta, 1/2 - \delta]$$

since the length of the left hand side is $1/6-\delta$ and the right hand side is $1/6$ and the end points of the left hand interval are contained in the right hand interval.

We also note that

$$1/6 - \delta/2 \leq (1/2 - \delta) - (1/3 - \delta)$$

so that we are able to assume that for some $k > 1$, $M_1 = N^k$ satisfies

$$x^{1/3-\delta} \leq M_1 \leq x^{1/2-\delta}. \tag{3.34}$$

We can now consider the case when σ_N is large. More specifically when $\sigma_N > 5/6 + \Delta + 3\eta$. We have by (3.31) the bound

$$\hat{R} \leq \min_{M_1(s)} (M_1 w^{-2} + T_1 w^{-2}, M_1 w^{-2} + T_1 M_1 w^{-6}),$$

where the minimum runs over all the products $M_1(s)$, possibly with repititions, of g of the Dirichlet polynomials $G_i(s)$ satisfying $M \leq x$.

We have $w = M_1^{\sigma-1/2}$ hence we write (again supressing the N in σ_N for ease of notation)

$$\hat{R} \leq M_1^{2-2\sigma} + \min(T_1 M_1^{1-2\sigma}, T_1 M_1^{4-6\sigma}).$$

Case: $M_1^{2-2\sigma}$ dominates. Here, since $x^{1/3-\delta} \leq M_1 \leq x^{1/2-\delta}$ we have

$$\hat{R} \leq (x_1^{1/2-\delta})^{2-2\sigma} \leq x_1^{1-\sigma-2\delta+2\sigma\delta}.$$

Hence

$$x_1^{\sigma-1}\hat{R} \leq x_1^{\sigma_N-1}\hat{R} \ll x_1^{2\delta(\sigma-1)}.$$

However, we also have $x_1 \geq N$ (so, since $\sigma - 1 < 0$ we can replace x_1 by N in the above) and the fact that $N(s)$ is prime factored which by definition means

$$|N(-1/2+it)| \ll N^{\frac{1}{2}} \exp(-c(\log x)^{13/60}),$$

so that

$$x_1^{\sigma-1}\hat{R} \ll x_1^{2\delta(\sigma_N-1)} \leq N^{2\delta(\sigma_N-1)} \ll \exp(-c(\log x)^{13/60})(\log x)^2.$$

Hence we achieve bound (3.16) via (3.30). As this bound is not dependent on the value of σ_N we may now assume that

$$\hat{R} \ll \min(T_1 M_1^{1-2\sigma_N}, T_1 M_1^{4-6\sigma_N}).$$

We now consider this case when $\sigma_N > 5/6 + \Delta + 3\eta_2$ for arbitrary $\eta_2 > 0$. Since $T_1 \ll x^{1/2+\Delta+2\eta}$ we have

$$\begin{aligned}\widehat{R} &\ll T_1 M_1^{4-6\sigma_N} \ll T_1 x_1^{(4-6\sigma_N)(\frac{1}{3}-\delta)} \\ &\ll x_1^{\frac{1}{2}+\Delta+2\eta} x_1^{\frac{4}{3}-2\sigma_N-\delta(4-6\sigma_N)}.\end{aligned}$$

Therefore

$$\begin{aligned}x_1^{\sigma_N-1}\widehat{R} &\ll x_1^{(\sigma_N-1)+(\frac{1}{2}+\frac{4}{3}-2\sigma_N+\Delta+2\eta-\delta(4-6\sigma_N))} \\ &\ll x_1^{\frac{5}{6}-\sigma_N+\Delta+2\eta-\delta(4-6\sigma_N)}.\end{aligned}$$

The expression on the right hand side decreases when σ_N increases, for small η so that we may take $\sigma_N = 5/6 + \Delta + 3\eta$ in the range $\sigma_N > 5/6 + \Delta + 3\eta$ (where we drop the subscript 2 in η, since it is arbitrary, we let it equal the arbitrary $\eta > 0$ in the bound for T_1). This gives

$$x_1^{\sigma_N-1}\widehat{R} \ll x_1^{-\eta+\delta(1+6\Delta+18\eta)}$$

So we once again achieve bound (3.16) via (3.30).

We now make a crucial observation which will be required in section 3.10.

Remark. We note that the lower limit for σ in the above is determined by the lower limit on M_1. We observe that if $M_1 \geq x_1^{\frac{3}{8}}$ then σ does not need to be much more than $\frac{4}{5}$. So, a result we shall require later is that

$$(3.16) \text{ holds if } M_1 \geq x_1^{\frac{3}{8}} \text{ and } \sigma \geq \frac{5}{6}. \qquad (3.35)$$

We now conclude this section having ascertained that we may assume from now on that $\frac{1}{2} \leq \sigma \leq \frac{5}{6} + \Delta + 3\eta_2$ for arbitrary $\eta_2 > 0$.

3.6 Estimates for $\widehat{R}$ in the range $\frac{1}{2} \leq \sigma \leq \frac{5}{6} + \Delta + 3\eta_2$

We now consider bounds for $\widehat{R}$ in the range $1/2 \leq \sigma \leq 5/6 + \Delta + 3\eta_2$ in relation to the special conditions on the Dirichlet polynomials in Lemma 9.

The following convexity lemma ([31], p59) will be used throughout this and subsequent sections of the paper.

Lemma 15 (The Convexity Lemma)**.** *Let $A_1, , ..., A_n$ be arbitrary positive numbers. Then*

$$\min(A_1, ..., A_n) \leq A_1^{p_1} ... A_n^{p_n}$$

where the indices p_i are positive numbers satisfying $\sum_{i=1}^{n} p_i = 1$. We say this convexity relation has indices $(p_1, ..., p_n)$.

In the following lemma we prove the key bound on $\widehat{R}$ which will be required throughout the subsequent sections and in particular the region $1/2 \leq \sigma \leq 5/6 + \Delta + 3\eta_2$. The lemma will establish the bound in this range of σ for each of the cases of Lemma 9.

Lemma 16. *Let $\eta > 0$ and $\delta > 0$ be arbitrary numbers and let $0 < \Delta < 1/48$ a fixed number.*

Let θ be a positive number such that $\theta \in \{0, 1/27, 1/15\}$ then in Lemma 9 write in cases(i)-(iv) and (vii)-(viii) that $\theta = 0$, in case (v) $\theta = 1/27$ and in case (vi) $\theta = 1/15$. Then assuming one of the conditions of that lemma hold we have

$$\hat{R} \ll x_1^{1-\sigma+\theta/4+\Delta+2\eta} \text{ if } \tfrac{1}{2} \leq \sigma \leq \tfrac{3}{4},$$

$$\hat{R} \ll x_1^{11/8+\theta/4-3\sigma/2+\Delta+2\eta} + x_1^{1-\sigma+\Delta+2\eta} \ if \ \tfrac{3}{4} \le \sigma \le \tfrac{5}{6} + \Delta + 3\eta_2.$$

Proof. Let, in Lemma 9 $N_i(s)$ be of size w_i, $K(s)$ of size v, $K_i(s)$ of size v_i and $G(s)$ of size w. Also we recall that $w = G^{\sigma-1/2} = x_1^{\sigma-1/2}$ and that $x \ll x_1^{1+\eta}$ for arbitrarily small $\eta > 0$ and $T_1 = x^{1/2+\Delta+\eta}$ so that $T_1 \ll x_1^{1/2+\Delta+2\eta}$.

Case (i) By Lemma 10 for polynomial $M(s) = N_2(s)N_3(s)K(s)$ and for $M(s) = N_1(s)$ and convexity lemma with indices $(1/2, 1/2)$

$$\hat{R} \ll \min\{T_1^{1+\eta}(w_2w_3v)^{-2}, T_1w_1^{-2}\} \ll (T_1^{1+\eta}(w_2w_3v)^{-2})^{1/2}(T_1w_1^{-2})^{1/2}$$

$$\ll T_1^{1+\eta/2}(w_1w_2w_3v)^{-1} \ll x_1^{1/2+\Delta+2\eta}w^{-1} \ll x_1^{1-\sigma+\Delta+2\eta}.$$

Case (ii): we use Lemma 11 for $M(s) = N_3(s)^2K(s)^4$ and (3.31) for $M(s) = N_1(s)$ and $M(s) = N_2(s)^2N_3(s)$ so that by the convexity lemma with indices $(1/4, 1/2, 1/4)$

$$\widehat{R} \ll \min(T_1^{1+\eta_1}w_3^{-2}v^{-4}, T_1w_1^{-2}, T_1w_2^{-4}w_3^{-2}) \ll x_1^{1-\sigma+\Delta+2\eta_1}$$

Case (iii): Using Lemma 11 for $M(s) = N_3(s)^4K(s)^4$ and (3.31) for $M(s) = N_1(s)$ and $M(s) = N_2(s)$ or $M(s) = N_2(s)^2$ we have

$$\widehat{R} \ll \min(T_1^{1+\eta_1}w_3^{-4}v^{-4}, T_1w_1^{-2}, T_1w_2^{-4}) \ll x_1^{1-\sigma+\Delta+2\eta_1}$$

Case (iv):The proof of this is essentially the same as the corresponding case in Lemma 6.1 of [26].

If $\beta_2 \le \frac{1}{4}$, we use part (ii) with $N_3(s) = 1$. So we assume that $\beta_2 > \frac{1}{4}$.

Letting $N_2(s) = L_1...L_k(s)$, where $L_i = x^{\delta_i}$, $\delta_i \leq \delta_j$ for $i < j$ and each $L_i(s)$ with $\delta_i > 1/8$ is a zeta factor.

First, suppose that no subproduct of the $L_1...L_k$ lies in $[x^{1/16}, x^{\beta_2 - 1/16}]$ then the product of all L_i that are less than $x^{1/16}$ is itself by this restriction still going to be less than $x^{1/16}$. There will therefore be only one remaining factor since $2(\beta_2 - 1/16) > \beta_2$. Hence $L_k > x^{\beta_2 - 1/16}$ and L_k is a zeta factor. The claim now follows from (iii). Next suppose that there is a subproduct

x^δ of the L_i with $\delta \in [1/16, \beta_2 - 1/16]$. Then let $\gamma_1 = \min(\delta, \beta_2 - \delta)$ and $\gamma_2 = \beta_2 - \gamma_1$. Then $\gamma_1 \leq \gamma_2$, therefore

$$2\gamma_2 + \gamma_1 = 2\beta_2 - \gamma_1 \leq 9/16 - 1/16 = 1/2$$

and

$$3\gamma_2/4 + \gamma_1 = 3\beta_2/4 + \gamma_1/4 \leq 7\beta_2/8 \leq 1/4.$$

Then the claim follows from (i).

Cases (v) and (vi) : We have $\frac{1-\theta}{2} \leq \beta_i \leq \frac{1+\theta}{2}$
where $w_i = x^{\beta_i}$ is the size of $N_i(s)$ where $i = 1, 2$. For these cases we have $G(s) = N_1(s)N_2(s)$. These restrictions provide the contraints $N_i \leq x^{(1+\theta)/2}$ and we will use that by definition $w = w_1 w_2 = x_1^{\sigma - 1/2}$ and $G = N_1 N_2 = x_1$.

By the standard bounds in (3.31) we have for $\sigma \leq 3/4$ that

$$\hat{R} \leq \min\{(N_1 + N_2)w_1^{-2}, (N_2 + T_1)w_2^{-2}\}$$

which by convexity with indices $(1/2, 1/2)$ gives

$$\hat{R} \ll ((N_1 N_2)^{1/2} + T_1^{1/2}(N_1^{1/2} + N_2^{1/2}) + T_1)(w_1 w_2)^{-1}$$

$$\ll (x_1^{1/2} + (x_1^{1/2+\Delta+2\eta})^{1/2}(x_1^{\frac{1+\theta}{2}})^{1/2} + x_1^{1/2+\Delta+2\eta})(x_1^{\sigma-1/2})^{-1}$$

$$\ll x_1^{1-\sigma+\Delta/2+\theta/4+2\eta}$$

which for $\sigma \le 3/4$ is $\ll x_1^{1-\sigma+\theta/4+\Delta+2\eta}$ as required.

Similarly for the range $\sigma \ge 3/4$ we use the estimates from (3.31)

$$\hat{R} \le \min\{N_1 w_1^{-2} + T_1 N_1 w_1^{-6}, N_2 w_2^{-2} + T_1 N_2 w_2^{-6}\}$$

$$\ll (N_1 N_2)^{1/2}(w_1 w_2)^{-1} + T_1^{1/4} N_1^{3/4} N_2^{1/4}(w_1 w_2)^{-3/2}$$
$$+T_1^{1/4} N_1^{1/4} N_2^{3/4}(w_1 w_2)^{-3/2} + T_1 (N_1 N_2)^{1/2}(w_1 w_2)^{-3}$$

$$\ll x_1^{1/2}(x_1^{\sigma-1/2})^{-1} + (x_1^{1/2+\Delta+2\eta})^{1/4} x_1^{1/4}(x_1^{\frac{1+\theta}{2}})^{1/2}$$
$$+x_1^{1/2+\Delta+2\eta} x_1^{1/2}(x_1^{\sigma-1/2})^{-3}$$

$$\ll x_1^{1-\sigma} + x_1^{11/8+\theta/4-3\sigma/2+\Delta/4+\eta/2} + x_1^{\frac{5-6\sigma}{2}+\Delta+2\eta}.$$

Now $\frac{5-6\sigma}{2} \le 1-\sigma$ is clearly satisfied when $\sigma \ge 3/4$ hence we can now conclude that

$$\hat{R} \ll x_1^{1-\sigma+\Delta+2\eta} + x_1^{11/8+\theta/4-3\sigma/2+\Delta+2\eta}$$

as required.

Case (vii):

Using Lemma 11 with $M(s) = N_2(s)^2 K_(s)^4$ and $M(s) = N_3(s)^2 K_2(s)^4$ and by (3.30) with$M(s) = N_1(s)N_2(s)$ and $M(s) = N_1(s)N_3(s)$. Then

$$\widehat{R} \ll \min(T_1^{1+\Delta+\eta} v_1^{-4} w_2^{-2}, T_1^{1+\Delta+\eta} v_2^{-4} w_3^{-2}, T_1 w_1^{-2} w_3^{-2}, T_1 w_1^{-2} w_2^{-2})$$

$$\ll T_1^{1+\Delta+\eta} w^{-1} \ll x_1^{1-\sigma+\Delta+2\eta}.$$

Finally,

Case (viii):

Using Lemma 11 with $M(s) = N_2(s)^2 K_1(s)^4$ and $M(s) = N_2(s)^2 K_2(s)^4$ and (3.30) with $M(s) = N_1(s)$, then

$$\widehat{R} \ll \min(T_1^{1+\Delta+\eta} v_1^{-4} w_2^{-2}, T_1^{1+\delta+\eta} v_2^{-4} w_3^{-2}, T_1 w_1^{-2})$$

$$\ll T_1^{1+\delta+\eta} w^{-1} \ll x_1^{1-\sigma+\Delta+2\eta}.$$

This completes the proof of the lemma.

We have established the lemma and in the next section we proceed to establish the final estimates for R, $R^{(2)}$ and $R^{(3)}$.

3.7 Final Estimates for R, $R^{(2)}$ and $R^{(3)}$

We proceed by using the bounds from the previous sections obtained from $M_1(s)$ as previously defined and which for small $\delta > 0$ was found to satisfy

$$x^{1/3-\delta} \leq M_1 \leq x^{1/2-\delta}.$$

Our choice of bound is determined by the values of M_1 and σ and we will ensure all possible combinations of the values of these are accounted for.

By factorising the $x_1^{2\eta}$ from Lemma 16 from the previous section and by using

$$\widehat{R} \ll \min\{T_1 M_1^{1-2\sigma}, T_1 M_1^{4-6\sigma}\}$$

from the section on long zeta factors (where the above bound is then determined by the value of σ), we define

$$\widehat{R} \ll x_1^{2\eta} \widetilde{R}, \tag{3.36}$$

where from Lemma 16

$$\widetilde{R} = \begin{cases} \min\left(x_1^{1-\sigma+\theta/4+\Delta}, T_1 M_1^{1-2\sigma}\right) & \text{if } \sigma \le \frac{3}{4} \\ \min\left(x_1^{11/8+\theta/4-3\sigma/2+\Delta} + x_1^{1-\sigma+\Delta}, T_1 M_1^{4-6\sigma}\right) & \text{if } \frac{3}{4} \le \sigma \le \frac{5}{6} + \Delta + 3\eta_2. \end{cases}$$

Or since $T_1 \ll x_1^{1+\Delta+2\eta}$ we can rewrite the above expressions in the following lemma. The second half of the lemma deals with the case when M_1 is large.

Lemma 17. *We have*

$$\widehat{R} \ll x_1^{2\eta} \widetilde{R},$$

where

$$\widetilde{R} = \begin{cases} \min\left(x_1^{1-\sigma+\theta/4+\Delta}, x_1^{1/2+\Delta} M_1^{1-2\sigma}\right) & \text{if } \sigma \le \frac{3}{4} \\ \min\left(x_1^{11/8+\theta/4-3\sigma/2+\Delta} + x_1^{1-\sigma+\Delta}, x_1^{1/2+\Delta} M_1^{4-6\sigma}\right) & \text{if } \frac{3}{4} \le \sigma \le \frac{5}{6} + \Delta + 3\eta_2 \end{cases}$$

for η and $\delta > 0$, $\theta \in \{0, 1/27, 1/15\}$ and fixed $\Delta > 0$. Furthermore, we have $\widehat{R}^{(2)} \ll x_1^{6\eta_1+2\eta_2} \bar{R}^{(2)}$ and $\widehat{R}^{(3)} \ll x_1^{16\eta_1+4\eta_2} \bar{R}^{(3)}$ for small arbitrary η_1 and $\eta_2 > 0$, where

$$\bar{R}^{(2)} = \widetilde{R}^3 M_1^{1-2\sigma} + \widetilde{R}^{5/2} M_1^{(3-4\sigma)/2} x_1^{\Delta/2} + \widetilde{R} M_1^{4-4\sigma} x_1^{2\Delta} \tag{3.37}$$

and

$$\bar{R}^{(3)} = \widetilde{R}^5 M_1^{1-2\sigma} + \bar{R}^{(2)1/2} \widetilde{R}^3 M_1^{(3-4\sigma)/2} x_1^{\Delta/2} + \bar{R}^{(2)} M_1^{4-4\sigma} x_1^{2\Delta}. \tag{3.38}$$

If the last term dominates in $\bar{R}^{(2)}$ then the second and third terms dominate in $\bar{R}^{(3)}$.

When $M_1 = N^g$ for $g > 1$,we define $M_3(s) = N(s)^{[(g+1)/2]}$ such that M_3 satisfies

$$M_1^{1/2} \le M_3 \le M_1^{2/3}.$$

In this case we have

$$\widehat{R}^{(2)} \ll x_1^{6\eta_1+2\eta_2}\widetilde{R}^{(2)}$$

and

$$\widehat{R}^{(3)} \ll x_1^{16\eta_1+4\eta_2}\widetilde{R}^{(3)}$$

where

$$\widetilde{R}^{(2)} = \min(\bar{R}^{(2)}, \widetilde{R}^3 M_3^{1-2\sigma} + x_1^{1/8}\widetilde{R}^{21/8}M_3^{1-2\sigma}x_1^{\Delta/4} + \widetilde{R}^{5/2}M_3^{(3-4\sigma)/2} \quad (3.39)$$

$$+x_1^{2/5}\widetilde{R}^{9/5}M_3^{(8-16\sigma)/5}x_1^{4\Delta/5} + x_1^{1/5}\widetilde{R}^{8/5}M_3^{(12-16\sigma)/5}x_1^{2\Delta/5} + \widetilde{R}M_3^{(4-4\sigma)})$$

and

$$\widetilde{R}^{(3)} = \widetilde{R}^5 M_1^{1-2\sigma} + \widetilde{R}^{(2)1/2}\widetilde{R}^3 M_1^{(3-4\sigma)/2}x_1^{\Delta/2} + \widetilde{R}^{(2)}M_1^{4-4\sigma}x_1^{2\Delta}. \qquad (3.40)$$

Proof. By (3.32) and (3.33) we may estimate $\widehat{R}^{(2)}$ and $\widehat{R}^{(3)}$ following a similar line of argument to Peck [31] section 14, we let $M(s) = M_1(s)$.

We show that

$$\widehat{R}^3 w_1^{-2} + T_1^{1/4}\widehat{R}^{21/8}w_1^{-2} + \widehat{R}^{5/2}M_1^{1/2}w_1^{-2} + T_1^{4/5}\widehat{R}^{9/5}w_1^{-16/5}$$

$$+T_1^{2/5}\widehat{R}^{8/5}M_1^{4/5}w_1^{-16/5} + \widehat{R}M_1^2 w_1^{-4}$$

$$\ll x_1^{2\eta_2}(\widehat{R}^3 w_1^{-2} + \widehat{R}^{5/2}M_1^{1/2}x_1^{\Delta/2}w_1^{-2} + \widehat{R}M_1^2 x_1^{2\Delta}w_1^{-4}).$$

Then using

$$M_1 \geq x^{1/3-\eta_2} \text{ and } w_1 \geq M_1^{\sigma-1/2}$$

for arbitrary $\eta_2 > 0$, and then by either / or and convexity arguments with indices $(1/4, 3/4), (2/5, 3/5), (2, 5, 3/5)$ respectively the above inequality will now be shown to be true.

To see this observe that

$$x_1^{2\eta_2}(\widehat{R}^3 w_1^{-2} + x_1^{\Delta/2}\widehat{R}^{5/2}M_1^{1/2}w_1^{-2}) \gg x_1^{2\eta_2}\widehat{R}^{3/4+15/8}M_1^{3/8}w_1^{-2} \gg T_1^{1/4}\widehat{R}^{21/8}w_1^{-2}$$

since $M_1 \geq T_1^{2/3-\eta_2}$. Also we have

$$x_1^{2\eta_2}(\widehat{R}^3 w_1^{-2}+x_1^{2\Delta}\widehat{R}M_1^2 w_1^{-4}) \gg x_1^{2\eta_2}\widehat{R}^{6/5+3/5}M_1^{6/5}w_1^{-4/5-12/5} \gg T_1^{4/5}\widehat{R}^{9/5}w_1^{-16/5}.$$

Finally,

$$\begin{aligned} &x_1^{2\eta_2}(x_1^{\Delta/2}\widehat{R}^{5/2}M_1^{1/2}w_1^{-2} + x_1^{2\Delta}\widehat{R}M_1^2 w_1^{-4}) \\ &\gg x_1^{2\eta_2}(\widehat{R}^{1+3/5}M_1^{1/5+6/5}w_1^{-4/5-12/5}) \gg T_1^{2/5}\widehat{R}^{8/5}M_1^{4/5}w_1^{-16/5} \end{aligned}$$

and hence the claim follows.

So by (3.36), (3.32) and $w_1 \geq M_1^{\sigma-1/2}$ as $|M_1(s)| \sim M_1^{\sigma-1/2}$ at $s = 1/2 + it_n$, we obtain the estimate by factorisation:

$$\widehat{R}^{(2)} \ll x_1^{6\eta_1+2\eta_2}\bar{R}^{(2)}$$

where

$$\bar{R}^{(2)} = (\widetilde{R}^3 M_1^{1-2\sigma} + \widetilde{R}^{5/2}M_1^{(3-2\sigma)/2}x_1^{\Delta/2} + \widetilde{R}M_1^{4-4\sigma}x_1^{2\Delta}). \tag{3.41}$$

In an analogous manner we may prove that

$$\widehat{R}^{(3)} \ll x_1^{10\eta_1+2\eta_2}(\widetilde{R}^5 M_1^{1-2\sigma} + \widehat{R}^{(2)1/2}\widetilde{R}^3 M_1^{(3-4\sigma)/2}x_1^{\Delta/2} + \widehat{R}^{(2)}M_1^{4-4\sigma}x_1^{2\Delta}).$$

Hence by factorization we obtain the estimate

$$\widehat{R}^{(3)} \ll x_1^{16\eta_1+4\eta_2}\bar{R}^{(3)}.$$

where

$$\bar{R}^{(3)} = \widetilde{R}^5 M_1^{1-2\sigma} + \bar{R}^{(2)1/2}\widetilde{R}^3 M_1^{(3-4\sigma)/2} x_1^{\Delta/2} + \bar{R}^{(2)} M_1^{4-4\sigma} x_1^{2\Delta}. \qquad (3.42)$$

In the case that $\bar{R}^{(2)}$ is dominated by the last term in (3.41) then

$$\bar{R}^{(3)} \ll \bar{R}^{(2)1/2}\widetilde{R}^3 M_1^{(3-4\sigma)/2} x_1^{\Delta/2} + \bar{R}^{(2)} M_1^{4-4\sigma} x_1^{2\Delta}. \qquad (3.43)$$

To see this note that in the case that $\bar{R}^{(2)}$ is dominated by the last term in (3.41) then

$$\widetilde{R}^{5/2} M_1^{(4-4\sigma)/2} \ll \min(\widetilde{R} M_1^{4-4\sigma} x_1^{\Delta/2}, \bar{R}^{(2)}) \ll \bar{R}^{(2)1/2}\widetilde{R}^{1/2} M_1^{2-2\sigma} x_1^{\Delta/2}$$

hence multiplying by $\widetilde{R}^{5/2} M_1^{-1/2}$ gives

$$\widetilde{R}^5 M_1^{1-2\sigma} \ll \bar{R}^{(2)1/2}\widetilde{R}^3 M_1^{(3-4\sigma)/2} x_1^{\Delta/2}$$

as required by the claim.

Next consider the case when M_1 is large and define $M_3(s) = N(s)^{[(g+1)/2]}$, so that M_3 satisfies

$$M_1^{1/2} \le M_3 \le M_1^{2/3}.$$

Using $M(s) = M_3(s)$ in Lemma 13 together with $\widehat{R} \ll x_1^{2\eta_1}\widetilde{R}$ and $w_3 \ge M_3^{\sigma-1/2}$ we obtain

$$\widehat{R}^{(2)} \le \widehat{R}^3 w_3^{-2} + T_1^{1/4}\widehat{R}^{21/8} w_3^{-2} + \widehat{R}^{5/2} M_3^{1/2} w_3^{-2} + T_1^{4/5}\widehat{R}^{9/5} w_3^{-16/5}$$

$$+T_1^{2/5}\widehat{R}^{8/5} M_3^{4/5} w_3^{-16/5} + \widehat{R} M_3^2 w_3^{-4}$$

$$\ll x_1^{6\eta_1}(\widetilde{R}^3 M_3^{1-2\sigma} + x_1^{1/8}\widetilde{R}^{21/8} M_3^{1-2\sigma} x_1^{\Delta/4} + \widetilde{R}^{5/2} M_3^{(3-4\sigma)/2}$$

$$+x_1^{2/5}\widetilde{R}^{9/5} M_3^{(8-16\sigma)/5} x_1^{4\Delta/5} + x_1^{1/5}\widetilde{R}^{8/5} M_3^{(12-16\sigma)/5} x_1^{2\Delta/5}$$

$$+\widetilde{R}M_3^{4-4\sigma}).$$

We now *define*

$$\widetilde{R}^{(2)} = \min(\bar{R}^{(2)}, \widetilde{R}^3 M_3^{1-2\sigma} + x_1^{1/8}\widetilde{R}^{21/8}M_3^{1-2\sigma}x_1^{\Delta/4} + \widetilde{R}^{5/2}M_3^{(3-4\sigma)/2} \tag{3.44}$$

$$+x_1^{2/5}\widetilde{R}^{9/5}M_3^{(8-16\sigma)/5}x_1^{4\Delta/5} + x_1^{1/5}\widetilde{R}^{8/5}M_3^{(12-16\sigma)/5}x_1^{2\Delta/5} + \widetilde{R}M_3^{4-4\sigma}),$$

and

$$\widetilde{R}^{(3)} = \widetilde{R}^5 M_1^{1-2\sigma} + \widetilde{R}^{(2)1/2}\widetilde{R}^3 M_1^{(3-4\sigma)/2}x_1^{\Delta/2} + \widetilde{R}^{(2)}M_1^{4-4\sigma}x_1^{2\Delta}, \tag{3.45}$$

so that

$$\widehat{R}^{(2)} \ll x_1^{6\eta_1+2\eta_2}\widetilde{R}^{(2)}$$

and

$$\widehat{R}^{(3)} \ll x_1^{16\eta_1+4\eta_2}\widetilde{R}^{(3)}$$

which concludes the proof of the lemma.

Note that we will frequently require the use of the trivial bounds $\widehat{R}^{(2)} \ll \bar{R}^{(2)}$ and $\widetilde{R}^{(3)} \ll \bar{R}^{(3)}$.

We observe that the highest power of $\widetilde{R}$ in the above Lemma 17 is $\widetilde{R}^5$ (in the expressions for $\bar{R}^{(3)}$ and $\widetilde{R}^{(3)}$). It is for this reason that the additional fixed exponent $\Delta > 0$ for x_1 in the expression for $\widetilde{R}$ therefore produces a maximum additional fixed exponent of 5Δ for x_1 in the upper bound for the original sum of the main theorem of this chapter, Theorem 2.

In Lemma 17 above we note that if $M_1(s)$ itself is a factor of $G(s)$ rather than a power of a factor (in other words $g = 1$) we may not use $M_3(s)$. However from Lemma 9 in cases (i)-(v) and (vii),(viii) we may assume that $N \leq x^{1/4}$ so that $g \geq 2$. Since in these cases, all possible longer factors $N(s)$

are zeta factors and by Lemma 14 these are dealt with. We prove Lemma 9 using the previous lemma with $\theta = 1/27$ for all

$$M_1 = x_1^{\kappa} \tag{3.46}$$

where $1/3 - \eta_1 \leq \kappa \leq 1/2$ and $g > 1$. This in turn implies the result for $\theta = 0$.

In case (vi) of Lemma 9 $N(s)$ is in fact one of the $H_i(s)$ in that lemma we may not assume $N \leq x_1^{1/2-\eta_2}$. Consequently we may use $M_3(s)$ in this case only when $N \leq x_1^{1/4-\eta_2}$, in which case $g > 1$.

The next lemma will be used to restrict the size x^{κ} of M_1.

Lemma 18. *Let $\beta \in \mathcal{G}$ where $\mathcal{G}$ is defined in Lemma 9 part (vi) as $\mathcal{G} = \left(0, \frac{41}{180}\right] \cup \left[\frac{13}{54}, \frac{1}{4} - \eta_2\right] \cup \left[\frac{1}{3}, c\right]$ and let $\kappa = g\beta$ be some multiple of of β where $M_1 = N^g$ and $N = x^{\beta}$ so that $M_1 = N^g = x^{g\beta} = x^{\kappa}$. Then the following the constraints apply to κ and g:*

$$(i)\kappa \in \left[\frac{1}{3}, c\right] \text{ and } g \geq 1,$$

$$(ii)\kappa \in \left[c, \frac{41}{90}\right] \cup \left[\frac{13}{27}, \frac{1}{2}\right] \text{ and } g > 1,$$

$$(iii)\kappa \in \left[\frac{41}{90}, \frac{13}{27}\right] \text{ and } g = 3.$$

Proof.

If $\beta \in [1/3, c]$ take $g = 1$ and (i) is satisfied so $\kappa = g\beta \in [1/3, c] \subset \mathcal{G}$.

Further if $\beta \in [1/6, 41/180] \cup [13/54, 1/4 - \eta_2]$ take $g = 2$ and either (i) or (ii) is satisfied so $\kappa = g\beta \in [13/27, 1/2 - 2\eta_2] \subset \mathcal{G}$.

If $\beta \in [1/9, 1/6]$ take $g = 3$ and either one of (i),(ii) or (iii) is satisfied.

We therefore obtain κ in the appropriate intervals.

If $\beta \leq 1/9$ there is a g such that $\kappa = g\beta \in [1/3, c]$ since $1/9 < c - 1/3$ which concludes the proof of the lemma.

In the specific case (vi) of Lemma 9 we need only prove the result for $M_1 = N^g = x^\kappa$ satisfying one of (i) - (iii) of the previous lemma.

We may now proceed by proving that for the range $1/2 \leq \sigma \leq 5/6 + \Delta + 3\eta_2$ one of the the following is true:

$$\widetilde{R} \ll x_1^{1-\sigma-3\eta_2},$$

$$\widehat{R}^{(2)} x_1^{4\sigma-3} \ll x_1^{2/3+5\Delta+6\eta_1+2\eta_2+\epsilon},$$

$$\widehat{R}^{(3)} x_1^{6\sigma-5} \ll x_1^{2/3+5\Delta+16\eta_1+4\eta_2+\epsilon}.$$

3.8 Case: $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$

For $\sigma \leq 3/4$ the bounds for $x_1^{4\sigma-3}\bar{R}^{(2)}$, $x_1^{6\sigma-5}\bar{R}^{(3)}$, $x_1^{4\sigma-3}\widetilde{R}^{(2)}$ and $x_1^{6\sigma-5}\widetilde{R}^{(3)}$ increase with increasing σ. We may therefore consider just the case $\sigma = 3/4$. Then by Lemma 17

$$\widetilde{R} = \min(x_1^{1+\theta/4-3/4+\Delta}, x_1^{1/2+\Delta} M_1^{1-2\sigma})$$

and

$$\bar{R}^{(2)} = \widetilde{R}^3 M_1^{1-2\sigma} + \widetilde{R}^{5/2} x_1^{\Delta/2} M_1^{(3-4\sigma)/2} + \widetilde{R} x_1^{2\Delta} M_1^{4-4\sigma}$$

$$= \widetilde{R}^3 M_1^{-1/2} + \widetilde{R}^{5/2} x_1^{\Delta/2} + \widetilde{R} x_1^{2\Delta} M_1, \text{ as } \sigma = 3/4$$

$$\ll x_1^{\frac{3}{4}(1+\theta)+3\Delta-\frac{1}{2}(1/3-\eta_2)} + x_1^{\frac{5}{8}(1+\theta)+3\Delta} + \min(x_1^{\frac{1}{4}(1+\theta)+3\Delta}M_1, x_1^{\frac{1}{2}+3\Delta}M_1^{\frac{1}{2}})$$

for $\theta \leq 1/15$. Hence we may now assume that

$$\bar{R}^{(2)} \ll \min(x_1^{\frac{(1+\theta)}{4}+3\Delta}M_1, x_1^{\frac{1}{2}+3\Delta}M_1^{\frac{1}{2}}).$$

So

$$\bar{R}^{(3)} \ll \bar{R}^{(2)1/2}\widetilde{R}^3 x_1^{\Delta/2} + \bar{R}^{(2)}M_1 x_1^{2\Delta}$$

by (3.43) as $\bar{R}^{(2)}$ is dominated by its last term. So from the bound above for $\bar{R}^{(2)}$ and the expression for $\widetilde{R}$

$$\bar{R}^{(3)} \ll \min(x_1^{\frac{7}{8}(1+\theta)+5\Delta}M_1^{\frac{1}{2}}, x_1^{\frac{7}{4}+5\Delta}M_1^{-\frac{5}{4}}) + \min(x_1^{\frac{(1+\theta)}{4}+5\theta}M_1^2, x_1^{\frac{1}{2}+5\Delta}M_1^{\frac{3}{2}}),$$

which by convexity in the first minimum with indices $(5/7, 2/7)$ is

$$\ll x_1^{\frac{5}{7}(\frac{7}{8}(1+\theta)+5\Delta)+\frac{2}{7}(\frac{7}{4}+5\Delta)} + \min(x_1^{\frac{(1+\theta)}{4}+5\Delta}M_1^2, x_1^{\frac{1}{2}+5\Delta}M_1^{\frac{3}{2}})$$

$$\ll x_1^{\frac{2}{3}+\frac{1}{2}+5\Delta} + \min(x_1^{\frac{(1+\theta)}{4}+5\Delta}M_1^2, x_1^{\frac{1}{2}+5\Delta}M_1^{\frac{3}{2}})$$

for $\theta \leq 1/15$. Here we have $x_1^{\frac{(1+\theta)}{4}+5\Delta}M_1^2 \leq x_1^{\frac{2}{3}+\frac{1}{2}+5\Delta}$ when

$$2\kappa + \frac{(1+\theta)}{4} + 5\Delta \leq \frac{2}{3} + \frac{1}{2} + 5\Delta$$

so that

$$\kappa \leq \frac{(11-3\theta)}{24}.$$

This gives

$$\kappa = \frac{49}{108} \text{ when } \theta = \frac{1}{27}$$

$$\kappa = \frac{9}{20} \text{ when } \theta = \frac{1}{15}.$$

We may therfore now assume that $\kappa \geq \frac{(11-3\theta)}{24}$. We have also established the required bound for the case $g = 1$ and may assume $g > 1$.

We next use the estimate $\widetilde{R} \ll x_1^{1/2+\Delta} M_1^{1-2\sigma}$ with $\sigma = 3/4$ giving $\widetilde{R} \ll x_1^{1/2+\Delta} M_1^{-1/2}$. Recalling that by hypothesis we also have $M_1^{1/2} \leq M_3 \leq M_1^{2/3}$ which by the expression for $\widetilde{R}^{(2)}$ in Lemma 17 gives by dominance of the second term (see Peck [31] p66 after (15.3))

$$\widetilde{R}^{(2)} \ll x_1^{\frac{23}{16}+\frac{23}{8}\Delta} M_1^{-\frac{25}{16}}.$$

Similarly by the expression for $\widetilde{R}^{(3)}$ in Lemma 17 gives by dominance of the third term

$$\widetilde{R}^{(3)} \ll x_1^{\frac{23}{16}+\frac{39}{8}\Delta} M_1^{-\frac{9}{16}} \ll x_1^{\frac{2}{3}+\frac{1}{2}+5\Delta}$$

for $M_1 \geq x_1^{13/27}$.

We next use the estimate $\widetilde{R} \leq x_1^{(1+\theta)/4+\Delta}$ (from the beginning of this section) and $M_1^{1/2} \leq M_3 \leq M_1^{2/3}$ which by Lemma 17 gives

$$\widetilde{R}^{(2)} \ll x_1^{\frac{3}{4}(1+\theta)+3\Delta} M_1^{-\frac{1}{4}} + x_1^{\frac{25}{3}+\frac{21}{32}\theta+\frac{21}{8}\Delta} M_1^{-\frac{1}{4}} + x_1^{\frac{5}{8}(1+\theta)+\frac{5}{2}\Delta}$$

$$+x_1^{\frac{17}{20}+\frac{9}{20}\theta+\frac{13}{5}\Delta} M_1^{-\frac{4}{5}} + x_1^{\frac{3}{5}+\frac{2}{5}\theta+2\Delta} + x_1^{\frac{(1+\theta)}{4}+\Delta} M_1^{\frac{2}{3}}.$$

We observe that the third, fifth and sixth terms are $\ll x_1^{2/3+5\Delta}$ for $\theta \leq 1/15$ and the second term dominates the first and the fourth terms when $1/27 \leq \theta \leq 1/15$ and $\kappa \geq 9/20$. Hence we may now assume that

$$\widetilde{R}^{(2)} \ll x_1^{\frac{25}{3}+\frac{21}{32}\theta+\frac{21}{8}\Delta} M_1^{-\frac{1}{4}}.$$

Then by Lemma 17

$$\widetilde{R}^{(3)} \ll x_1^{\frac{5}{4}(1+\theta)+5\Delta} M_1^{-\frac{1}{2}} + x_1^{\frac{73}{64}+\frac{69}{64}\theta+\frac{77}{16}\Delta} M_1^{-\frac{1}{8}} + x_1^{\frac{25}{32}+\frac{21}{32}\theta+\frac{37}{8}\Delta} M_1^{\frac{3}{4}}$$

which is $\ll x_1^{2/3+1/2+5\Delta}$ for $\theta = 1/27$ in the remaining range $\kappa \in [49/108, 13/27]$ and for $\theta = 1/15$ in the range $\kappa \in [9/20, 41/90]$.

Finally we consider the remaining case $\theta = 1/15$ and when $g = 3$ so that $x_1^{41/90} \leq N^3 \leq x_1^{13/27}$. In this case, by Lemma 17, $M_1 = N^3$ and $M_3 = N^2 = M_1^{2/3}$. Since $\widetilde{R} \ll x_1^{(1+\theta)/4+\Delta}$ for $\theta = 1/15$ we have $\widetilde{R} \ll x_1^{4/15+\Delta}$ from which we obtain the estimate

$$\widetilde{R}^{(2)} \ll x_1^{\frac{4}{5}+3\Delta} M_1^{-\frac{1}{3}} + x_1^{\frac{33}{40}+\frac{23}{8}\Delta} M_1^{-\frac{1}{3}} + x_1^{\frac{2}{3}+\frac{5}{2}\Delta}$$

$$+ x_1^{\frac{22}{25}+\frac{13}{5}\Delta} M_1^{-\frac{8}{15}} + x_1^{\frac{47}{75}+2\Delta} + x_1^{\frac{4}{15}+\Delta} M_1^{\frac{2}{3}}.$$

If the second term does not dominate we have $\widetilde{R}^{(2)} \ll x_1^{2/3+5\Delta}$ for $\kappa \in [41/90, 13/27]$. Thus we may assume that the second term dominates. Then by Lemma 17

$$\widetilde{R}^{(3)} \ll x_1^{\frac{4}{3}+\Delta} M_1^{-\frac{1}{2}} + x_1^{\frac{97}{80}+\frac{71}{16}\Delta} M_1^{-\frac{1}{6}} + x_1^{\frac{33}{40}+\frac{39}{8}\Delta} M_1^{\frac{2}{3}} \ll x_1^{\frac{2}{3}+\frac{1}{2}+5\Delta}$$

for $x_1^{41/90} \leq M_1 \leq x_1^{13/27}$ which completes the proof required in this section.

3.9 Case: $\frac{3}{4} \leq \sigma \leq \frac{3}{4} + \frac{\theta}{2}$

We now use, for $\sigma > 3/4$, the bound (from Lemma 17)

$$\widetilde{R} \ll \min\left(x_1^{11/8+\theta/4-3\sigma/2+\Delta} + x_1^{1-\sigma+\Delta}, x_1^{1/2+\Delta} M_1^{4-6\sigma} \right).$$

We observe that in the range under consideration we have

$$x_1^{11/8+\theta/4-3\sigma/2+\Delta} \geq x_1^{1-\sigma+\Delta}.$$

Hence by Lemma 17 we obtain

$$\bar{R}^{(2)} \ll x_1^{\frac{33}{8}+\frac{3}{4}\theta-\frac{9}{2}\sigma+3\Delta} M_1^{1-2\sigma} + x_1^{\frac{55}{16}+\frac{5}{8}\theta-\frac{15}{4}\sigma+\frac{1}{2}\Delta} M_1^{\frac{3-4\sigma}{2}}$$

$$+\min(x_1^{\frac{11}{8}+\frac{\theta}{4}-\frac{3}{2}\sigma+3\Delta} M_1^{4-4\sigma}, x_1^{\frac{1}{2}+2\Delta} M_1^{8-10\sigma}).$$

If the first or second term dominates then the bound for $x_1^{4\sigma-3}\bar{R}^{(2)}$ decreases as σ increases and we may therefore take $\sigma = 3/4$. In this case, since $M_1 \geq x_1^{1/3-\eta}$ we have the bound $\bar{R}^{(2)} \ll x_1^{2/3+5\Delta}$ for $\theta \leq 1/15$. We may therefore now assume from now on that

$$\bar{R}^{(2)} \ll \min(x_1^{\frac{11}{8}+\frac{\theta}{4}-\frac{3}{2}\sigma+3\Delta} M_1^{4-4\sigma}, x_1^{\frac{1}{2}+2\Delta} M_1^{8-10\sigma}).$$

Hence by Lemma 17 we use this bound to obtain

$$x_1^{6\sigma-5}\bar{R}^{(3)} \ll x_1^{\frac{3}{4}\sigma-\frac{3}{16}+\frac{7}{8}\theta+5\Delta} M_1^{\frac{7}{2}-4\sigma}$$

$$+\min(x_1^{\frac{9}{2}\sigma-\frac{29}{8}+\frac{\theta}{4}+5\Delta} M_1^{8-8\sigma}, x_1^{6\sigma-\frac{9}{2}+4\Delta} M_1^{12-14\sigma}).$$

If the first term dominates then the bound for $x_1^{6\sigma-5}\bar{R}^{(3)}$ decreases as σ increases so that we may take $\sigma = 3/4$. We observe that by using $\kappa \leq 1/2$ for $\theta = 1/27$ and $\kappa \leq c$ for $\theta = 1/15$ we obtain the bound $\ll x_1^{2/3+5\Delta}$. Hence sufficient bounds are obtained when

$$\frac{9}{2}\sigma - \frac{29}{8} + \frac{\theta}{4} + \kappa(8-8\sigma) + 5\Delta \leq \frac{2}{3} + 5\Delta$$

or

$$6\sigma - \frac{9}{2} + 4\Delta + \kappa(12-14\sigma) \leq \frac{2}{3} + 5\Delta.$$

These will be satisfied if

$$\frac{9}{2}\sigma - \frac{29}{8} + \frac{\theta}{4} + \kappa(8-8\sigma) \leq \frac{2}{3} \tag{3.47}$$

or

$$6\sigma - \frac{9}{2} + \kappa(12 - 14\sigma) \le \frac{2}{3}. \tag{3.48}$$

Next we consider which value of κ equalises the bounds (3.47) and (3.48). The first bound (3.47) requires larger values of κ as σ increases in order to satisify the inequality. Therefore if, in (3.47), we let $\sigma = 3/4 + \theta/2$, the largest value of σ in the range being considered, then we obtain the restriction $\kappa \le 45/104$. Consider next when this does not hold. Then the second bound (3.48) requires κ to become larger as σ decreases hence by letting the left hand sides of (3.47) and (3.48) be equal we find that the value of σ for which these bounds are the same value is given by

$$\sigma = \frac{4k - \frac{7}{8} - \frac{\theta}{4}}{6k - \frac{3}{2}}.$$

Then substituting this expression for σ into the equality for (3.48)

$$6\sigma - \frac{9}{2} + \kappa(12 - 14\sigma) = \frac{2}{3}$$

we obtain the following quadratic in κ:

$$960\kappa^2 - 751\kappa + 144 = 0$$

with solution

$$\kappa = \frac{751 + \sqrt{11041}}{1920}, \text{ which we call } c. \tag{3.49}$$

We now assume $k \ge c$ for the remainder of this section and and use the estimate

$$\widetilde{R} \ll \min(x_1^{\frac{11}{8} + \frac{\theta}{4} - \frac{3}{2}\sigma + \Delta}, x_1^{\frac{1}{2} + \Delta} M_1^{4 - 6\sigma})$$

and we recall

$$M_1^{\frac{1}{2}} \le M_3 \le M_1^{\frac{2}{3}}.$$

Then by Lemma 17 we obtain

$$\widetilde{R}^{(2)} \ll \min(x_1^{\frac{33}{8}+\frac{3}{4}\theta-\frac{9}{2}\sigma+3\Delta}M_1^{\frac{1}{2}-\sigma}, x_1^{\frac{3}{2}}M_1^{\frac{25}{2}-19\sigma}) \tag{3.50}$$

$$+\min(x_1^{\frac{239}{64}+\frac{21}{32}\theta-\frac{63}{16}\sigma+\frac{23}{8}\Delta}M_1^{\frac{1}{2}-\sigma}, x_1^{\frac{23}{16}+\frac{1}{4}\Delta}M_1^{11-\frac{67}{4}\sigma})$$

$$+x_1^{\frac{55}{16}+\frac{5}{8}\theta-\frac{15}{4}\sigma+\frac{5}{2}\Delta}M_1^{\frac{3}{4}-\sigma}$$

$$+\min(x_1^{\frac{23}{8}+\frac{9}{20}\theta-\frac{27}{10}\sigma+\frac{13}{5}\Delta}M_1^{\frac{4}{5}-\frac{8}{5}\sigma}, x_1^{\frac{13}{10}+\frac{4}{5}\Delta}M_1^{8-\frac{62}{5}\sigma})$$

$$+x_1^{\frac{12}{5}+\frac{2}{5}\theta-\frac{12}{5}\sigma+2\Delta}M_1^{\frac{6}{5}-\frac{8}{5}\sigma}$$

$$+x_1^{\frac{11}{8}+\frac{\theta}{4}-\frac{3}{2}\sigma+\Delta}M_1^{\frac{8}{3}-\frac{8}{3}\sigma}.$$

If the third, fifth or sixth terms dominate then we immediately estabish the bound $x_1^{4\sigma-3}\widetilde{R}^{(2)} \ll x_1^{\frac{2}{3}+5\Delta}$. For $\theta \leq 1/15$ and $3/4 \leq \sigma \leq 3/4+\theta/2$ with $\kappa \geq c$ we observe from the terms of $\widetilde{R}^{(2)}$ above that

$$x_1^{\frac{33}{8}+\frac{3}{4}\theta-\frac{9}{2}\sigma+3\Delta}M_1^{\frac{1}{2}-\sigma} \ll x_1^{\frac{239}{64}+\frac{21}{32}\theta-\frac{63}{16}\sigma+\frac{23}{8}\Delta}M_1^{\frac{1}{2}-\sigma}$$

and

$$x_1^{\frac{3}{2}}M_1^{\frac{25}{2}-19\sigma} \ll x_1^{\frac{23}{16}+\frac{1}{4}\Delta}M_1^{11-\frac{67}{4}\sigma}.$$

Hence we may assume that the second or fourth term dominate in $\widetilde{R}^{(2)}$. Assuming the second term dominates then by Lemma 17 we obtain

$$\widetilde{R}^{(3)} \ll x_1^{\frac{55}{8}+\frac{5}{4}\theta-\frac{15}{2}\sigma+5\Delta} + x_1^{\frac{767}{128}+\frac{69}{64}\theta-\frac{207}{32}\sigma+\frac{79}{16}\Delta}$$
$$+\min\left(x_1^{\frac{239}{64}+\frac{21}{32}\theta-\frac{63}{16}\sigma+\frac{23}{8}\Delta}M_1^{\frac{9}{2}-5\sigma}, x_1^{\frac{23}{16}+\frac{\Delta}{4}}M_1^{15-\frac{83}{4}\sigma}\right).$$

If in this expression the first or second term dominate we establish $x_1^{6\sigma-5}\widetilde{R}^{(3)}$ is $\ll x_1^{\frac{2}{3}+5\Delta}$ for $\kappa \geq c$ and $\theta \leq 1/15$. We may therefore assume the last term dominates in $\widetilde{R}^{(3)}$ above. Hence in the case when the second term of $\widetilde{R}^{(2)}$ in (3.50) dominates we will obtain sufficient bounds when in the exponents of the expression for $x_1^{6\sigma-5}\widetilde{R}^{(3)}$ from $\widetilde{R}^{(3)}$ above, either (with the ommision of the delta terms on each side of the inequalities as in the previous argument which led to (3.47) and (3.48))

$$\frac{33}{16}\sigma - \frac{81}{64} + \frac{21}{32}\theta + \kappa\left(\frac{9}{2} - 5\sigma\right) \leq \frac{2}{3}$$

or

$$6\sigma - \frac{57}{16} + \kappa\left(15 - \frac{83}{4}\sigma\right) \leq \frac{2}{3}$$

We observe that the first inequality gives an upper bound for κ and that for $\kappa \geq c$ it becomes more favourable as σ increases. Hence by taking $\sigma = 3/4$ we achieve a sufficient bound for

$$\kappa \leq \frac{37}{72} - \frac{7}{8}\theta.$$

This gives

$$\kappa \leq \frac{13}{27} \text{ when } \theta = \frac{1}{27}$$

and

$$\kappa \leq \frac{41}{90} \text{ when } \theta = \frac{1}{15}.$$

The second inequality gives a lower bound for κ and becomes more favourable as σ increases. Hence by taking $\sigma = 3/4$ we obtain a sufficient bound for $\kappa \geq 13/27$.

Next we assume that the fourth term of (3.50) dominates. Then

$$\widetilde{R}^{(3)} \ll x_1^{\frac{55}{8}+\frac{5}{4}\theta-\frac{15}{2}\sigma+5\Delta}M_1^{1-2\sigma} + x_1^{\frac{89}{16}+\frac{39}{40}\theta-\frac{117}{20}\sigma+\frac{76}{18}\Delta}M_1^{\frac{19}{10}-\frac{14}{5}\sigma}$$

$$+x_1^{\frac{13}{10}+\frac{4}{5}\Delta}M_1^{12-\frac{82}{5}\sigma}.$$

Hence $x_1^{6\sigma-5}\widetilde{R}^{(3)} \ll x_1^{\frac{2}{3}+5\Delta}$ for $\theta \leq 1/15$ and $\kappa \geq c$ as required.
Next we must consider the case $\theta = 1/15$ and $x_1^{\frac{41}{90}} \leq N^3 \leq x_1^{\frac{13}{27}}$. As in the preceding section we recall $M_1 = N^3$ and $M_3 = N^2 = M_1^{\frac{2}{3}}$. By the bound for $\widetilde{R}$ at the beginning of this section this gives

$$\widetilde{R} \ll x_1^{\frac{167}{120}-\frac{3}{2}\sigma+\Delta}.$$

Hence by Lemma 17 we obtain

$$\widetilde{R}^{(2)} \ll x_1^{\frac{167}{40}-\frac{9}{2}\sigma+3\Delta}M_1^{\frac{2}{3}-\frac{4}{3}\sigma} + x_1^{\frac{1209}{320}-\frac{63}{16}\sigma+\frac{23}{8}\Delta}M_1^{\frac{2}{3}-\frac{4}{3}\sigma}$$

$$+x_1^{\frac{167}{48}-\frac{15}{4}\sigma+\frac{5}{2}\Delta}M_1^{1-\frac{4}{3}\sigma} + x_1^{\frac{581}{200}-\frac{27}{10}\sigma+\frac{13}{5}\Delta}M_1^{\frac{16}{15}-\frac{32}{15}\sigma} + x_1^{\frac{182}{75}-\frac{12}{5}\sigma+2\Delta}M_1^{\frac{8}{5}-\frac{32}{15}\sigma}$$

$$+x_1^{\frac{167}{120}-\frac{3}{2}\sigma+\Delta}M_1^{\frac{8}{3}-\frac{8}{3}\sigma}.$$

In this bound the first three terms multiplied by $x_1^{4\sigma-3}$ decrease as σ increases. We let $\sigma = 3/4$ and see that if the first or third term dominates we achieve the bound$\ll x_1^{2/3+5\Delta}$. The last three terms multiplied by $x_1^{4\sigma-3}$ increase as σ increases. By setting $\sigma = 47/60$ we see that if one of these terms dominates we achieve the bound $x_1^{4\sigma-3} \ll x_1^{2/3+5\Delta}$. Hence we may assume that the second term dominates. In this case we have

$$\widetilde{R}^{(3)} \ll x_1^{\frac{167}{24}-\frac{15}{2}\sigma+5\Delta} + x_1^{\frac{3881}{640}-\frac{207}{32}\sigma+\frac{79}{16}\Delta}M_1^{\frac{11}{6}-\frac{8}{3}\sigma}$$

$$+x_1^{\frac{1209}{320}-\frac{63}{16}\sigma+\frac{39}{8}\Delta}M_1^{\frac{14}{3}-\frac{16}{3}\sigma}.$$

In this bound the terms multiplied by $x_1^{6\sigma-5}$ decrease as σ increases. Letting $\sigma = 3/4$ we now see that $x_1^{6\sigma-5}\widetilde{R}^{(3)} \ll x_1^{2/3+5\Delta}$ as required, concluding this section.

3.10 Case: $\frac{3}{4}+\frac{\theta}{2}\le\sigma\le\frac{5}{6}+\Delta+3\eta_2$

In this range where $\sigma > 3/4+\theta/2$ we now use the bound for $\widetilde{R}$ in Lemma 17 for $\sigma > 3/4$ (where the first term in the following expression for the minimum is readily shown to be the larger of the two terms in the sum in the first term of the expression shown in that lemma for the range of σ in this section; enabling the use of the more simple form of the bound used here).

$$\widetilde{R} \ll \min(x_1^{1+\Delta-\sigma}, x_1^{\frac{1}{2}+\Delta} M_1^{4-6\sigma}).$$

Observe that if

$$x_1^{\frac{1}{2}+\Delta} M_1^{4-6\sigma} \le x_1^{1-\sigma-3\eta_1}$$

then we have a sufficient bound. That is if

$$\Delta+\kappa(4-6\sigma)\le\frac{1}{2}-\sigma-3\eta_1,$$

then we have nothing to prove.

We therefore use the bound $\widetilde{R}\ll x_1^{1-\sigma+\Delta}$ to obtain

$$\bar{R}^{(2)} \ll x_1^{3-3\sigma+3\Delta} M_1^{1-2\sigma} + x_1^{\frac{5}{2}(1-\sigma)+3\Delta} M_1^{\frac{1}{2}(3-4\sigma)} + x_1^{1-\sigma+3\Delta} M_1^{4-4\sigma}. \tag{3.51}$$

If the first term dominates in the right hand side of this bound then

$$x_1^{4\sigma-3}\bar{R}^{(2)} \ll x_1^{\sigma+(\frac{1}{3}-\eta_1)(1-2\sigma)+3\Delta} \ll x_1^{\frac{2}{3}+5\Delta}.$$

We may therefore assume that the second and third terms dominate. The bounds for $x_1^{4\sigma-3}\bar{R}^{(2)}$ and $x_1^{6\sigma-5}\bar{R}^{(3)}$ increase as σ increases. Hence we assume in this case that $\sigma=5/6+\Delta+3\eta$ (where we supress the subscript 2 as it is arbitrary and we may allow it to equal the η in the bound for T_1). For this value of σ we have $\widetilde{R} \ll x_1^{\frac{1}{6}-3\eta}$ and we note that the third term in $\bar{R}^{(2)}$ dominates which gives $\bar{R}^{(2)} \ll x_1^{\frac{1}{6}+2\Delta} M_1^{\frac{2}{3}-4\Delta}$. Hence by (from Lemma 17)

$$\bar{R}^{(3)} \ll x_1^{\frac{7}{12}+\frac{3}{2}\Delta} M_1^{\frac{1}{6}-4\Delta} + x_1^{\frac{1}{6}+4\Delta} M_1^{\frac{4}{3}-8\Delta}$$

and since $M_1 = x_1^{\kappa}$ we see that whilst the first term immediately satifies the required bounds the second term of this bound will certainly be $\ll x_1^{\frac{2}{3}+5\Delta}$ whenever $\kappa \leq \frac{3}{8}$.

Now we return to considering σ in the full range under consideration in this section rather than fixing it. We may now assume $\kappa > \frac{3}{8}$.
We now appeal to the remark at the end of section 3.5, referring to (3.35). As $\kappa > \frac{3}{8}$ we have by definition (3.46) that $M_1 > x_1^{\frac{3}{8}}$ so by (3.35) we can now assume that $\sigma \leq \frac{5}{6}$.

In this case the third term of $\bar{R}^{(2)}$ in (3.51) above dominates.

Then

$$x_1^{6\sigma-5} \bar{R}^{(3)} \ll x_1^{\frac{5}{2}\sigma-\frac{3}{2}+5\Delta} M_1^{\frac{7}{2}-4\sigma} + x_1^{5\sigma-4+5\Delta} M_1^{8-8\sigma}.$$

The first term increases with increasing σ and for $\sigma = 5/6$ this term is $\ll x_1^{\frac{31}{48}+5\Delta} \ll x_1^{\frac{2}{3}+5\Delta}$. Hence a sufficient bound is achieved when

$$5\sigma - 4 + \kappa(8 - 8\sigma) + 5\Delta \leq \frac{2}{3} + 5\Delta.$$

This will be satisfied when

$$5\sigma - 4 + \kappa(8 - 8\sigma) \leq \frac{2}{3}.$$

Combining this with the inequality in kappa from the first bound considered in this section

$$\Delta + \kappa(4 - 6\sigma) \leq \frac{1}{2} - \sigma - 3\eta_1,$$

we find that in the range $\frac{3}{4}+\frac{\theta}{2} \leq \sigma \leq \frac{5}{6}+\Delta+3\eta_2$ we obtain sufficient bounds when

$$\kappa \leq \frac{\frac{14}{3}-5\sigma}{8-8\sigma}$$

or

$$\kappa \geq \frac{\sigma-\frac{1}{2}+3\eta_1+\Delta}{6\sigma-4}.$$

By elementary calculation the latter bound is smaller than the former when $\sigma \in [\frac{3}{4}+\frac{\theta}{2}, \frac{5}{6}+\Delta+3\eta_2]$ and $\theta \geq \frac{1}{27}$. Hence, since the ranges for κ overlap, in this case we obtain sufficient bounds for $\theta \in \{\frac{1}{15}, \frac{1}{27}\}$ for all values of κ as required.

3.11 Regions where $\theta = \frac{1}{15}$ may be used

We recall that we wish to consider polynomials of the form

$$F(s) = \prod_{i=1}^{k} R_i(s) \prod_{i=1}^{l} M_i(s) K(s) H(s)$$

where $M_i < x_1^{\frac{1}{15}}$, K(s) is a zeta-factor and $H(s) = x^{o(1)}$. When $R_i = x_1^{\alpha_i}$ is written then $1/27 \leq \alpha_i \leq 13/27$ and $\alpha_{i+1} \leq \min\left(\alpha_i, \frac{1}{2}(1-\alpha_1-...-\alpha_i)\right)$. Furthermore $R_i(s) = \prod_{j=1}^{j_i} R_{ij}(s)$, where all non-zeta factors have lenghth $\leq x_1^{1/8}$ and all the factors are prime-factored. We join the terms with length $\leq \eta$ arising from $P_i(s)$ to $H(s)$. Assuming our polynomials are of this particular form we obtain the following lemma (see [26] p 510-511 for the proof since the polynomials (i)-(viii) considered in lemma 2.4 of that paper are the same as in lemma 9 of this chapter and numbered in the same order).

Lemma 19. *If some product of R_i is in the range $[x_1^{7/15}, x_1^{8/15}]$ and one of the following is satisfied:*

(i) $\alpha_i \leq 1/8$ for some $1 \leq i \leq k$,

(ii) $\alpha_1 \leq 41/90$ *and* $\alpha_2 \leq 41/180$,

(iii) $k \geq 4$ *and* $\alpha_2 \leq c - 13/54$

then $F(s)$ *is good.*

3.12 Sieve Asymptotic Formulae

From this section onwards, having established Lemma 9, the working will now be identical to Matomäki [26]. The previous sections have provided the arithmetical information which will now enable the production of asymptotic formulae which we use in the sieve of Harman. We require formulae of the form

$$\sum_{m\sim M} a_m S(\mathcal{A}_m,\eta) = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}}\sum_{m\sim M} a_m S(\mathcal{B}_m,\eta) + \frac{\delta_{\mathcal{A}} y}{\log y}(A(x,y)+o(1)), \qquad (3.52)$$

where $\eta = \exp((\log x)^{9/10})$, and we will use (3.52) to obtain similar formulae with η replaced by a larger value. We also note that

$$\sum_{m\sim M} a_m S(\mathcal{A}_m,\eta) = \sum_{m\sim M} a_m \sum_{\substack{ml\in\mathcal{A}\\ d|(l,P(\eta))}} \mu(d).$$

We bound the length of the sum over d using the following two lemmas from Heath-Brown [20].

Lemma 20. *Let $\gamma > 1$, then*

$$\sum_{d|(n,P(\eta))} \mu(d) = \sum_{\substack{d|(n,P(\eta))\\ d\le\gamma}} \mu(d) + O\left(\sum_{\substack{d|(n,P(\eta))\\ \gamma\le d\le\gamma\eta}} 1\right).$$

Lemma 21. *Let a and u be positive numbers, $z_0 = x^{1/u}$ and $D = x^a$. Suppose that*

$$\frac{1}{a} < u < (\log x)^{1-\epsilon}.$$

Then

$$\sum_{\substack{d|P(z_0)\\ d>D}} \frac{1}{d} \ll \exp(\log\log z_0 + 2ua - ua\log(ua)).$$

We are now in a position to obtain some asymptotic formulae. The proof of the following lemma follows that in [4] (the proof of Lemma 12) and can also be found in relation to the present problem in [26] (proof of lemma 12.3, since the polynomials (i)-(viii) considered in lemma 2.4 of that paper are the same as in lemma 9 of this paper and numbered in the same order).

Lemma 22. *Let* $M(s) = \sum_{m \sim M} a_m m^{-s}$*, with* $M \le x_1^{1/8}$ *or with* $M(s)K(s)$ *good, where* $K = \sum_{k \sim x_1^{1-\eta_1}/M} k^{-s}$*. Then* (3.52) *holds.*

Proof. Let

$$c_k = \sum_{\substack{ml=k \\ d|(l,P(\eta))}} a_m \mu(d)$$

and write

$$c_k' = \sum_{\substack{ml=k, d \le \gamma \\ d|(l,P(\eta))}} a_m \mu(d), \; c_k'' = \sum_{\substack{ml=k, \gamma \le d < \eta\gamma \\ d|(l,P(\eta))}} |a_m|$$

with $\gamma = \eta^\tau$, $\tau = (\log\log x)^2$. Now $\gamma\eta \ll x^{\mu_1}$,for arbitrary $\mu_1 > 0$. By Lemma 20 we have

$$\sum_{k \in \mathcal{A}} c_k = \sum_{k \in \mathcal{A}} c_k' + O\left(\sum_{k \in \mathcal{A}} c_k''\right).$$

If $M\gamma\eta > x_1^{1/4}$ then we obtain asymptotic formulae for sums with c_k' and c_k'' by our assumptions since the polynomial in d can be incorporated in $H(s)$. We still need to show that the sums with c_k'' is $O((\log x)^{-B})$. By Lemma 21 we have

$$\frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} \sum_{k \in \mathcal{B}} c_k'' \ll \delta_{\mathcal{A}} y \sum_m \frac{|a_m|}{m} \exp\left(-\frac{1}{2}(\log\log x)^2 \log\log\log x\right)$$

$$\ll \frac{\delta_{\mathcal{A}y}}{\log y} (\log x)^{-B}.$$

hence the claim follows in this case. If $M\gamma\eta \le x_1^{1/4}$, the sums with $N \ll x_1^{1/4}$

are of the form

$$\sum_{\substack{nl\in\mathcal{A}\\ n\sim N}} a_n = \sum_{n\sim N} a_n\left(\frac{\delta_{\mathcal{A}}y}{n}+O(1)\right) = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}}\sum_{\substack{nl\in\mathcal{B}\\ n\sim N}} a_n + O\left(\sum_{n\sim N}|a_n|\right)$$

$$= \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}}\sum_{\substack{nl\in\mathcal{B}\\ n\sim N}} a_n + O(x^{1/4+\epsilon}).$$

We see that in this case we also have an asymptotic formula and the proof is complete.

To conclude this section we state two lemmas which extend the results for asymptotic formulae to cases where η is replaced by a larger value. We refer the reader to [26](section 12 Lemma 12.4 and 12.5) for the proofs. The first Lemma is a modification of the sieve of Harman and provides asymptotic formulae in cases (i)-(v) of Lemma 9 and 19. The second Lemma provides asymptotic formulae in certain more specific cases of Lemma 9 and Lemma 19.

Lemma 23. *Let $M(s) = N_1(s)N_2(s)N_3(s) = \prod_{i=1}^{k} R_I(s)$, and let $N_i = x_1^{\beta_i}$ be such that the polynomial $K(s)N_1(s)N_2(s)N_3(s)$ satisfies Lemma 19 or one of the conditions (i) - (v) of Lemma 9.*

Let $\theta = \frac{1}{15}$ and let

$$I_h = \left[\frac{1-\theta}{2} - h\theta, \frac{1-\theta}{2} - (h-1)\theta\right)$$

for $h \geq 1$ and

$$\nu(\alpha) = \begin{cases} \min\left(\frac{1}{8}, \frac{\frac{1+\theta}{2}-\alpha}{h}\right) & \text{if } \alpha \in I_h, \\ \theta & \text{if } \frac{7}{15} \leq \alpha \leq \frac{1}{2}. \end{cases}$$

Then

$$\sum_{m\sim M} a_m S(\mathcal{A}_m, \nu(\beta_1)) = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} \sum_{m\sim M} a_m S(\mathcal{B}_m, \nu(\beta_1)) + \frac{\delta_{\mathcal{A}} y}{\log y}(A(x,y) + o(1)).$$

Lemma 24. *Let*

$$P(s) = \sum_{p\sim P} \frac{1}{p^s}.$$

If $M(s)P(s)K(s)$ satisfies condition (v) Lemma 9 or $M(s)P(s)$ satisfies the assumptions of Lemma 19, then

$$\sum_{\substack{m\sim M\\ p\sim P}} a_m S(\mathcal{A}_{mp}, p) = \frac{\delta_{\mathcal{A}}}{\delta_{\mathcal{B}}} \sum_{\substack{m\sim M\\ p\sim P}} a_m S(\mathcal{B}_{mp}, p) + \frac{\delta_{\mathcal{A}} y}{\log y}(A(x,y) + o(1)).$$

3.13 The Final Decomposition

We now apply the sieve method. We apply Buchstab's identity twice and subsequently in certain areas we apply this identity a further two or four times (as an even number of iterations are needed in each case). The previous sections will provide a reasonable collection of regions over which asymptotic formulae can be obtained and used in the sieve. The advantage of this will be that we will not need to discard too much of the required sum.

We now introduce a piecewise-linear function $\nu(\alpha)$ as the exponent of x in our application of Buchstab's idenity. The importance of this function is that we will use the arguments (usually indexed) to plot a graphical representation of the regions representing sums derived from repeated iterations of Buchstab's identity and within certain of these regions we obtain asymptotic formulae. Let $x^{\nu(0)} \leq p_1 < 2x^{1/2}$ where $\nu(\alpha)$ is a positive piecewise linear real valued function of the real non-negative variable α. We define $x^{\alpha_1} = p_1$ where p_1 is the first prime which is indexed by 1. Similarly and in general $x^{\alpha_j} = p_j$ for the j-th indexed prime.

With this notation we decompose $S(\mathcal{A}, 2x^{1/2})$ with two iterations of Buchstab's identity and write

$$S(\mathcal{A}, 2x^{1/2}) \geq S(\mathcal{A}, x^{\nu(0)}) - \sum_{\nu(0) \leq \alpha_1 < x^{\frac{1}{2}}} S(\mathcal{A}_{p_1}, x^{\nu(\alpha_1)})$$

$$+ \sum_{\substack{\nu(0) \leq \alpha_1 < \frac{1}{2} \\ \nu(\alpha_1) \leq \alpha_2 < \min\{\alpha_1, \frac{1-\alpha_1}{2}\}}} S(\mathcal{A}_{p_1 p_2}, p_2)$$

$$= \sum\nolimits_1 - \sum\nolimits_2 + \sum\nolimits_3, \text{ say.}$$

.

We have by Lemma 23 asymptotic formulae for $\sum_1$ and $\sum_2$. Define

$$H_k = \{(\alpha_1, ..., \alpha_k) | \theta \leq \alpha_k \leq ... \leq \alpha_1 \leq \frac{13}{27}, \alpha_1 + ... + \alpha_{k-1} + 2\alpha_k \leq 1\},$$

and further define

$$G_k = \{(\alpha_1, ..., \alpha_k) \in H_k | \text{ asymptotic formulae achieveable by Lemma 24}\}.$$

Next we define sets A, B, B', C, D and E:

$A = \{(\alpha_1, \alpha_2) \in H_2 | \alpha_1 + \alpha_2 \geq \frac{14}{27}, \alpha_2 \leq \frac{1}{7}\} \setminus G_2$

$B = \{(\alpha_1, \alpha_2) \in H_2 | \alpha_1 + \alpha_2 \geq \frac{9}{32}, \alpha_1 - \alpha_2 \leq \frac{1}{27}\} \setminus G_2$

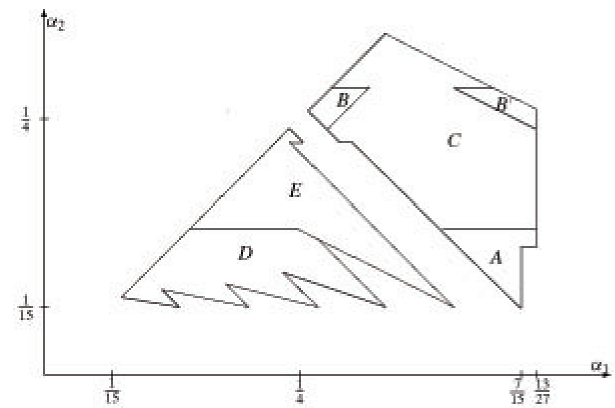
$B' = \{(\alpha_1, \alpha_2) \in H_2 | \alpha_1 + 2\alpha_2 \geq \frac{26}{27}, \alpha_2 \leq \frac{9}{32}\} \setminus G_2$

$C = \{(\alpha_1, \alpha_2) \in H_2 | \alpha_1 + \alpha_2 \geq \frac{14}{27}\} \setminus (G_2 \cup A \cup B \cup B')$

$D = \{(\alpha_1, \alpha_2) \in H_2 | \alpha_1 + 2\alpha_2 \geq \frac{13}{27}, \alpha_2 \leq \frac{1}{7}\} \setminus G_2$

$E = \{(\alpha_1, \alpha_2) \in H_2 | \alpha_1 + \alpha_2 \geq \frac{13}{27}\} \setminus (G_2 \cup D).$

The regions are illustrated in the following diagram (from [26]).



Next we split sum $\sum_3$ according to the regions defined above as follows:

$$\sum\nolimits_3 = \sum\nolimits_A + \sum\nolimits_B + \sum\nolimits_{B'} + \sum\nolimits_C + \sum\nolimits_D + \sum\nolimits_E + \sum\nolimits_{G_2}.$$

We observe that in B and B', only products of three primes are counted and that (α_1, α_2) satisfies the conditions for α_i in the definition of B' if and only if $(1 - \alpha_1 - \alpha_2, \alpha_2)$ satisfies the condition in the definition of B. Hence

$$\sum\nolimits_B = \sum\nolimits_{B'}$$

and

$$\sum\nolimits_3 = \sum\nolimits_A + 2\sum\nolimits_B + \sum\nolimits_C + \sum\nolimits_D + \sum\nolimits_E + \sum\nolimits_{G_2}.$$

First consider the sum $\sum_A$. In the region A we have $\alpha_2 \leq \frac{1}{7}$ which allows the use of Buchstab's identity twice more by appealing to case (i) of Lemma 9 and Lemma 23. We obtain

$$\sum\nolimits_A = \sum_{(\alpha_1, \alpha_2) \in A} S(\mathcal{A}_{p_1 p_2}, \nu(\alpha_1)) - \sum S(\mathcal{A}_{p_1 p_2 p_3}, \nu(\alpha_1)) + \sum S(\mathcal{A}_{p_1 p_2 p_3 p_4}, p_4)$$

The summation conditions have been suppressed for clarity. Lemma 23 ensures that we have asymptotic formulae for the first and second sums on the right hand side of the above expression. We also have asymptotic formulae for those parts of the third sum on the right hand side for which some combination of the α_i lies in $[7/15, 8/15]$ by Lemma 24 via condition (iii) of Lemma 19. More specifically we have asymptotic formulae for $\alpha_1 > \frac{7}{15}$. Hence by discarding only those parts of the third sum on the right hand side for which we do not have asymptotic formulae we can calculate the total loss from the sum $\sum_A$ for region A is:

$$\int_{\alpha_1=\frac{41}{105}}^{\frac{7}{15}} \int_{\alpha_2=\frac{8}{15}-\alpha_1}^{\frac{1}{7}} \int_{\alpha_3=\nu(\alpha_1)}^{\alpha_2} \int_{\alpha_4=\nu(\alpha_1)}^{\alpha_3} \omega\left(\frac{1-\alpha_1-\alpha_2-\alpha_3-\alpha_4}{\alpha_4}\right) \frac{d\alpha_4 d\alpha_3 d\alpha_2 d\alpha_1}{\alpha_1\alpha_2\alpha_3\alpha_4^2}.$$

where $(\alpha_1, \ldots, \alpha_4) \notin G_2$.

By numerical methods of integration this loss from region A is less than 0.018.

In region B we proceed to decompose a further two times using Lemma 9 (iv) as $\alpha_2 \leq 9/32$ and $\alpha_1 + \alpha_3 \leq \alpha_1 + (1-\alpha_1-\alpha_2)/2 \leq 14/27$:

$$\sum_B = \sum_{(\alpha_1,\alpha_2)\in B} S(\mathcal{A}_{p_1p_2}, \nu(\alpha_1)) - \sum S(\mathcal{A}_{p_1p_2p_3}, \nu(\alpha_1+\alpha_3)) + \sum S(\mathcal{A}_{p_1p_2p_3p_4}, p_4).$$

Working in the same way as in region A we see the loss from B is < 0.019.

The region C may be left without further decomposition and produces a loss of < 0.81.

In region D we may decompose via the Buchstab idenity four times as $\alpha_1 = \alpha_2 + \alpha_3 \leq 1/2$ and $\alpha_4\alpha_5 \leq 1/7$ hence by case (i) for Lemma 9. As there are six variabes many of the sums are in asymptotic formulae regions and the loss in D will be reduced to less than 0.0003.

In region E we decompose twice more and use case (iv) of Lemma 9 since here we have $\alpha_1 + \alpha_3 \leq 1/2$ and $\alpha_2 \leq 1/4$. The loss is less than 0.12.

The total loss is therefore less than $0.99 < 1$ as required.

Theorem 2 is therefore proved with exponent $a = \frac{2}{3} + 5\Delta + \epsilon$. However by continuity of the argument ϵ may be removed to obtain the final desired result.

Chapter 4

Prime-representing Functions

4.1 Application of Chapter 3

A significant application of the result of the previous chapter is an improvement to a prime representing function [27]. We therefore provide a further discussion of such functions and by doing so we give further insight into how development of results regarding differences between consecutive primes have been key in establishing the existence of prime representing functions.

We will prove the following theorem.

Theorem 3. *There exists* $\alpha > 2$ *and* $\beta = 1/(\frac{1}{2} + \Delta)$ *where* $0 < \Delta \leq -3 + \frac{1}{6}\sqrt{327}$ *such that the sequence* $[\alpha^{\beta^n}]$ *contains only prime numbers. The set of such numbers* α *has the cardinality of the continuum, is nowhere dense and has measure zero. The smallest value for* β *is* 1.946067...

Prime representing functions are typically functions of one parameter α, all of whose values are prime. The parameter will usually depend on the prime sequence which the function represents. Whilst it is not presently possible to determine the values of α which lead to prime representing functions it is possible to prove the existence of such a number. Mills [28] in 1947 showed that there exits $\alpha > 1$ such that

$$[\alpha^{3^n}] \tag{4.1}$$

is prime for all $n \in \mathbb{N}$ which was later improved by Niven [30] who reduced the exponent 3 to any real number

$$c > \frac{8}{3} = \frac{1}{1-5/8}.$$

The value 5/8 is from Ingham's [21] result that the interval $[x, x + Cx^{5/8}]$ contains primes for some $C > 0$ and sufficiently large x. The best result presently improving on Ingham's early result is by Baker, Harman and Pintz [5]:

Lemma 25. *There exists a positive constant d_0 such that*

$$\pi(x + x^{21/40}) - \pi(x) \geq d_0 \frac{x^{21/40}}{\log x}$$

for sufficiently large x.

Niven's argument then ensures that (4.1) is still true if 3 is replaced by exponent

$$c \geq \frac{1}{1-21/40}.$$

An improvement to this result was produced by Matomäki [27] who reduced the exponent to $c = 2$ by using a similar approach to Wright [35] who had established in 1954 that previous authors' results could be obtained via a more general approach. We will follow a similar approach to that paper and prove Theorem 3.

Firstly we *define* a ϕ-sequence: Let $\lambda_n(x) = x^{c_n}$, let $\phi_0(x) = x$ and $\phi_n(x)$

be the composed function

$$\phi_n(x) = \lambda_n \circ \cdots \circ \lambda_1(x) = x^{C_n}$$

where

$$C_n = c_1 \cdots c_n.$$

Here, for our purposes $c_i \in \mathbb{R}$ and $c_i > 1$ and we will be considering values of c_i less than 2.

We say that a sequence (a_n) of positive integers is a *ϕ-sequence* if for some fixed $\alpha > 1$, $a_n = [\phi_n(\alpha)]$ for every $n \in \mathbb{N}$.

We point out that in Wright's paper [35] as mentioned in Matomäki [27] the choice of function $\lambda_n(x)$ satisfy the conditions of the functions in Wright's paper so that the results of that paper may be applied. We use the following lemma from Wright [35]:

Lemma 26. *Assume that $a_0 > 2$,*

$$\lambda_{n+1}(a_n) \leq a_{n+1} \leq \lambda_{n+1}(a_n + 1) - 1$$

for all $n \in \mathbb{N}$ and

$$a_{n+1} < \lambda_{n+1}(a_n + 1) - 1$$

for infinitely many $n \in \mathbb{N}$. Then the sequence (a_n) is a ϕ-sequence.

We can now see that if it can be shown that there is a prime sequence (a_n) satisfisfying the conditions of this lemma then we will obtain a prime representing function.

In a more general setting let $\mathcal{D}$ be an infinite set of positive integers and $c \geq 2$ and $E_c(\phi, \mathcal{D})$ be the set of all $\alpha \geq c$ such that $[\phi(\alpha)] \in \mathcal{D}$. It can be shown (see [27] p 309 Lemma 6) that there are fairly straightforward conditions by which it can be determined if $E_c(\phi, \mathcal{D})$ is non-empty, has the cardinality of the continuum, is nowhere dense or is of zero measure.

In fact the set of all possible α such that $[\alpha^{C_n}]$ contains only primes in Theorem 3 is nowhere dense and has measure zero as it satisfies conditions (iii) and (iv) of lemma 6 of [27]. The non-emptiness and cardinality of the continuum will follow from Theorem 2 of this paper.

The proof of Theorem 2 (refer especially to Lemma 7) will in fact imply the following stronger result.

Lemma 27. *There exist positive constants $d' < 1$ and D' such that for every sufficiently large x the interval $[x, 2x]$ contains at most $D'x^{1/6+6\Delta}$ disjoint intervals $[n, n + n^{1/2-\Delta}]$ for which*

$$\pi(n + n^{1/2-\Delta}) - \pi(n) \leq \frac{d'n^{1/2-\Delta}}{\log n}.$$

We quote the following comparative result based upon (3.1) as a lemma:

Lemma 28. *There exist positive constants $d' < 1$ and D' such that for every sufficiently large x the interval $[x, 2x]$ contains at most $D'x^{1/6}$ disjoint intervals $[n, n + n^{1/2}]$ for which*

$$\pi(n + n^{1/2}) - \pi(n) \leq \frac{d'n^{1/2}}{\log n}.$$

We now make an interesting remark that the above two lemmas may be compared to Lemma 25 which essentially states that the number of primes in the interval $[x, x + x^{\gamma}]$ is of the expected order of magnitude by the prime number theorem when $\gamma \geq 21/40$. If the Riemann hypothesis was assumed true then this range could be extended to $\gamma \geq 1/2 + \epsilon$ for any $\epsilon > 0$. The existing result Lemma 28 then effectively show unconditionally that at $\gamma = 1/2$ there are in fact few exceptional intervals.

We state and prove the following lemma from corollary 4 of [27].

Lemma 29. *There exists $\alpha > 2$ such that the sequence $[\alpha^{2^n}]$ contains only prime numbers. The set of such α has the cardinality of the continuum, is nowhere dense and has measure zero.*

Proof. The proof is inductive. By Lemma 28 we construct a sequence (a_n) consisting of primes satisfying the conditions of Lemma 26 for a ϕ-sequence which by definition will provide a prime representing function for the sequence (a_n).

We let $\lambda_n(x) = x^2$ and note that a prime sequence (a_n) will clearly satisfy the conditions of Lemma 26 if

$$a_0 \geq 4 \text{ and } a_{n+1} \in [a_n^2, a_n^2 + a_n].$$

A sequence (a_n) may be constructed recursively. Let d' and D' be as in Lemma 28. Let a_0 be a large enough prime such that the interval $[a_0^2, a_0^2 + a_0]$ contains at least $d'a_0/(2\log a_0)$ primes. Such an a_0 exists by the prime number theorem.

Proceeding by induction let $k \geq 0$. Assume that we have chosen prime numbers $a_0, ..., a_k$ such that the interval

$$[a_j^2, a_j^2 + a_j] \text{ for } j = 0, ..., k$$

contains, by Lemma 28, at least $d'a_j/(2\log a_j)$ primes and

$$a_j \in [a_{j-1}^2, a_{j-1}^2 + a_{j-1}] \text{ for } j = 1, ..., k.$$

To complete the induction we now wish to find a prime $a_{k+1} \in [a_k^2, a_k^2 + a_k]$ such that the interval $[a_{k+1}^2, a_{k+1}^2 + a_{k+1}]$ contains at least $d'a_{k+1}/(2\log a_{k+1})$ primes.

For p prime with $p \in [a_k^2, a_k^2 + a_k] \cap \mathbb{P}$ the intervals $[p^2, p^2 + p]$ are disjoint and contained in $[a_k^{2^2}, 2a_k^{2^2}]$. By Lemma 28 at most $D'(a_k^4)^{1/6} = D'a_k^{2/3}$ of these intervals contain less than $d'p/(2\log p)$ primes.

But for large enough a_0 we have

$$D' a_k^{2/3} < d' a_k / (2 \log a_k) \le |[a_k^2, a_k^2 + a_k] \cap \mathbb{P}|.$$

We are therefore supplied with sufficient primes to choose a prime $a_{k+1} \in [a_k^2, a_k^2 + a_k] \cap \mathbb{P}$ such that the interval $[a_{k+1}^2, a_{k+1}^2 + a_{k+1}]$ contains at least $d' a_{k+1}/(2 \log a_{k+1})$ primes. This completes the induction.

Now by Lemma 26 (a_n) is a ϕ-sequence of prime numbers which by definition means there exists α such that $a_n = [\alpha^{2^n}]$. The multiple choices of a_i at each step of the recursion implies the set of all possible α has the cardinality of the continuum and the proof is complete.

We now show that Theorem 2 enables Lemma 29 above to be improved by reducing the value of the exponent β in $[\alpha^{\beta^n}]$ from $\beta = 2$ to a number $\beta < 2$ (as stated in Theorem 3). We show that we can in fact reduce the value of β to $1/(\frac{1}{2} + \Delta)$ for $0 < \Delta \le -3 + \frac{1}{6}\sqrt{327}$.

We now prove Theorem 3.

Proof. We follow a similar approach to the proof of Lemma 29. This time, however, we let $\lambda_n(x) = x^c$ for some $c > 0$ we show that there exists a $c < 2$ which satisfy the conditions of Lemma 26.

We wish to find a prime sequence (a_n) with $a_{n+1} \in [a_n^c, (a_n + \delta)^c] = [a_n^c, a_n^c + a_n^{c-1}]$ for some $0 < c < 2$. Comparing this to the proof of Lemma 29 we see that this gives the same intervals as in that proof when $c = 2$. Firstly

we note that transforming the interval $[a_n^c, a_n^c + a_n^{c-1}]$ by letting $z = a^c$ we have $a^{-1} = z^{-1/c}$ so we may write the interval as $[z, z + z^{1-1/c}]$ and by Lemma 28 this will therefore contain

$$\frac{d' a_n^{c-1}}{c \log a}$$

primes. To see this we have by Lemma 28 that $[z, z + z^{1-1/c}]$ contains at most $d' z^{(1-1/c)} / \log z$ primes and this is just $d'(a^c)^{(1-1/c)} / \log(a^c) = d' a^{c-1}/c \log a$.

Further by Theorem 2 of this paper the number of intervals containing too few primes will be $D'a_n^{c^2(1/6+6\Delta)}$. To see this note that the sum of the lengths of the intervals containing too few prime will be, by the result of this paper $\ll x^{2/3+5\Delta}$ so dividing by $x^{1/2-\Delta}$ for the interval length in Theorem 2 the exponents give $(2/3+5\Delta)-(1/2-\Delta)=1/6+6\Delta$.

We also note, as before, that a prime sequence (a_n) will clearly satisfy the conditions of Lemma 26 if for large enough $0<c<2$

$$a_0 \geq 4 \text{ and } a_{n+1} \in [a_n^c, a_n^c + a_n^{c-1}].$$

A sequence (a_n) may once again be constructed recursively.

Let d' and D' be as in Lemma 28. Let a_0 be a large enough prime such that the interval $[a_0^c, a_0^c+a_0^{c-1}]$ contains at least $d'a_0^{c-1}/(c\log a_0)$ primes. Such an a_0 exists by the prime number theorem.

Proceeding by induction let $k \geq 0$. Assume that we have chosen prime numbers $a_0, ..., a_k$ such that the interval

$$[a_j^c, a_j^c + a_j^{c-1}] \text{ for } j = 0, ..., k$$

contains, by Lemma 28, at least $d'a_j^{c-1}/(c\log a_j)$ primes and

$$a_j \in [a_{j-1}^c, a_{j-1}^c + a_{j-1}^{c-1}] \text{ for } j = 1, ..., k.$$

To complete the induction we now wish to find a prime $a_{k+1} \in [a_k^c, a_k^c + a_k^{c-1}]$ such that the interval $[a_{k+1}^c, a_{k+1}^c+a_{k+1}^{c-1}]$ contains at least $d'a_{k+1}^{c-1}/(c\log a_{k+1})$ primes.

We observe that for p prime with $p \in [a_k^c, a_k^c+a_k^{c-1}]\cap\mathbb{P}$ the intervals $[p^c, p^c+p^{c-1}]$ are disjoint and contained in $[a_k^{c^2}, 2a_k^{c^2}]$. However, as an outcome of Theorem 2 of this paper we find that by Lemma 27 at most $D'(a_k^{c^2})^{(1/6+6\Delta)} = D'a_k^{c^2(1/6+6\Delta)}$ of these intervals contain less than $d'p^{c-1}/(c\log p)$ primes.

But for large enough a_0 we have

$$D' a_k^{c^2(1/6+6\Delta)} < d' a_k^{c-1}/(c \log a_k) \le |[a_k^c, a_k^c + a_k^{c-1}] \cap \mathbb{P}|. \tag{4.2}$$

We are therefore supplied with sufficient primes to choose a prime $a_{k+1} \in [a_k^c, a_k^c + a_k^{c-1}] \cap \mathbb{P}$ such that the interval $[a_{k+1}^c, a_{k+1}^c + a_{k+1}^{c-1}]$ contains at least $d' a_{k+1}^{c-1}/(c \log a_{k+1})$ primes. This completes the induction.

Now by Lemma 26 (a_n) is a ϕ-sequence of prime numbers which by definition means there exists α such that $a_n = [\alpha^{c^n}]$ where $0 < c < 2$. The multiple choices of a_i at each step of the recursion implies the set of all possible α has the cardinality of the continuum and the proof is complete.

We next consider which value of Δ in Theorem 2 provides an improvement over the prime-representing function of Lemma 29. We consider intervals $[p^c, p^c + p^{c-1}] \subset [a_k^{c^2}, 2a_k^{c^2}]$ where p is a prime selected in the recursive approach of the preceding proofs. As a result of Theorem 2 we find that by Lemma 27 at most $D'(a_k^{c^2})^{(1/6+6\Delta)} = D' a_k^{c^2(1/6+6\Delta)}$ of these intervals contain less than $d' p^{(c-1)}/(c \log p)$ primes. We require, as we have seen in the inductive step (4.2) of the proof of the corollary above, that

$$D' a_k^{c^2(1/6+6\Delta)} < d' a_k^{c-1}/(c \log a_k).$$

By equating exponents and ignoring logs and constants we see that equality would occur when

$$c^2 \left(\frac{1}{6} + 6\Delta \right) = c - 1. \tag{4.3}$$

Transforming as before by $z = p^c$ the interval $[p^c, p^c + p^{c-1}]$ we obtain $[z, z + z^{1-1/c}]$. From Theorem 2 the interval between primes in the sum is $\frac{1}{2} - \Delta$, so we also require that

$$1 - \frac{1}{c} = \frac{1}{2} - \Delta. \tag{4.4}$$

Solving (4.3) and (4.4) simultaneously in Δ and c we obtain the quadratic equation

$$12\Delta^2 + 72\Delta - 1 = 0.$$

This has the quadratic-irrational solution

$$\Delta = -3 + \frac{1}{6}\sqrt{327} = 0.013856...$$

Therefore we now see that for

$$0 < \Delta \leq -3 + \frac{1}{6}\sqrt{327}$$

we will achieve a significant improvement in the exponent c in the prime-representing function $[\alpha^{c^n}]$ for $n \in \mathbb{N}$ reducing c from the value 2 (see Lemma 29) down to the value (using c from (4.4))

$$c = \frac{1}{\frac{1}{2} + \Delta}.$$

The smallest value that we can achieve for c being

$$c = 1.946067....$$

4.2 Primes in Beatty Sequences

In this section we apply the general prime-representing lemma of Wright, Lemma 26, and produce a result on prime-representing function only taking values which are primes in Beatty sequences.

Definition. *The sequence* $[\xi n + \eta]$, *for fixed* ξ *and* η, *is called a* Beatty sequence.

A Beatty sequence can be seen to essentially generalise the notion of an arithmetic progression and for integer values of ξ that is exactly what it is.

We write $\pi(x;\xi,\eta)$ for the number of primes of the form $[\xi n+\eta] \leq x$. We state the following lemma, which is Theorem 2 of [15]:

Lemma 30. *If $\xi > 1$ is irrational and $y = x^\theta$ with $\theta > 5/9$ then*

$$\pi(x+y;\xi,\eta) - \pi(x;\xi,\eta) > \frac{y}{10\xi \log x}(1+o(1)).$$

In the case ξ is the rational $\frac{q}{a}$ and $\delta(\xi,\eta) > 0$ this becomes

$$\pi(x+y;\xi,\eta) - \pi(x;\xi,\eta) > \frac{99y}{100 \log x}(1+o(1))\delta(\xi,\eta).$$

This lemma show that for all large x the number of primes in the Beatty equence $[\xi n+\eta]$ where ξ is an irrational is greater than 1, in the interval $[x, x+x^{\frac{5}{9}}]$, is greater than 1/10 of the expected number.

Theorem 4. *Let $\xi > 1$ be irrational and $\eta \in \mathbb{R}$. Then there exists $\alpha > 2$ such that the sequence $[\alpha^{c^m}]$ only takes values which are primes in the Beatty sequence $[\xi n+\eta]$ for*

$$c > \frac{1}{1-\frac{5}{9}} = \frac{9}{4}.$$

Proof

We wish to find sequence of primes (a_m) which is a subequence of a Beatty sequence $[\xi n+\eta]$ with $a_{m+1} \in [a_m^c, a_m^c + a_m^{c-1}]$, for some c. However we note that the sequence (a_m) will clearly satisfy the conditions of Lemma 26 if for large enough c

$$a_0 \geq 4 \text{ and } a_m \in [a_m^c, a_m^c + a_m^{c-1}].$$

We observe that Lemma 30 shows that for all large x the number of primes in the Beatty sequence $[\xi n+\eta]$ where $\xi > 1$ is an irrational, in the interval $[a_m, a_m + a_m^{\frac{5}{9}}]$, is greater than 1/10 of the expected number. Hence (a_m) is a ϕ-sequence consisting of primes in a Beatty sequence by the same inductive argument of the previous section which by definition implies there exists an α such that $a_m = [\alpha^{c^m}]$.

Transforming the interval $[a_m, a_m + a_m^{\frac{5}{9}}]$ by letting $z = a_m^c$ so that $a_m^{-1} = z^{-1/c}$ we can write the interval as $[z, z + z^{1-\frac{1}{c}}]$. Using the argument of the final part of the previous section we see that we can obtain a Beatty prime representing function with

$$1 - \frac{1}{c} < \frac{5}{9}$$

which gives

$$c > \frac{1}{1 - \frac{5}{9}} = \frac{9}{4}.$$

Chapter 5

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