ON A FAMILY OF HEREDITARILY JUST INFINITE PROFINITE GROUPS WHICH ARE NOT VIRTUALLY PRO-P

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Declaration of Authorship

I hereby certify that this thesis has been composed by me and is based on my own work, unless stated otherwise. No other person's work has been used without due acknowledgement in this thesis. All references and verbatim extracts have been quoted and all sources of information have been specifically acknowledged.

Signed:

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Abstract

Just infinite profinite groups have proved to be a very fascinating family of groups and a rich source of interesting examples in Group Theory. For a long time after the classification of finite simple groups, just infinite groups where considered the next classifiable family of groups. This proved to be much harder than expected and one of the many obstacles to this classification was the existence of hereditarily just infinite groups.

This thesis is concerned with the study of a generalisation of a new family of hereditarily just infinite profinite groups which are not virtually pro-p introduced by John Wilson in 2010, the *generalised Wilson groups*, GW groups for short. Even though these examples are important, only few properties are known. We start this work with a short overview of the known properties of generalised Wilson groups. Then, generalising a result of Bondarenko, we show that GW groups are finitely generated and we manage to produce explicit generators for GW groups. We then consider other generation-related profinite properties such as lower rank and finite presentability. We show that some generalised Wilson groups are new examples of profinite groups with finite lower rank. Moreover, we show that the direct product of certain hereditarily just infinite groups of finite lower rank still has finite lower rank. On the other hand we show that "most" GW groups are not finitely presentable. In later chapters we look more closely at the subgroup structure of generalised Wilson groups. In particular we prove an embedding theorem for finitely generated profinite groups with specified composition factors in GW groups with the same set of composition factors. We study subgroup growth functions for some GW groups. Then, we prove that these groups are new examples of profinite groups with complete Hausdorff dimension spectrum. Finally, we analyze which GW groups are self-similar.

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List of Symbols

$Y \subseteq X$**:** Y is a subset of the set X.

$H \leq G$**:** H is a subgroup of the group G.

$H \lhd G$**:** H is a normal subgroup of the group G.

$H \leq_c G$**:** H is a closed subgroup of the topological group G.

$H \leq_o G$**:** H is a open subgroup of the topological group G.

$\langle X \rangle$**:** the subgroup generated by the subset X.

$\overline{X}$**:** the topological closure of the subspace X.

$G \cong H$**:** the (topological) group G is isomorphic to the (topological) group H.

$\mathrm{d}(G)$**:** the number of (topological) generators of the (topological) group G.

H^G**:** the normal closure in the group G of the subgroup H.

$(a_i)_{i \in I}$**:** the sequence of objects $a_1, a_2, \ldots$ indexed by the set I.

$\mathrm{Sym}(n)$**:** the symmetric group acting in the right on n objects.

$G \leq \mathrm{Sym}(n)$**:** the permutation group G acting on n objects.

$A \mathrm{wr} B$**:** the abstract wreath product of A by $B \leq \mathrm{Sym}(n)$.

$A \wr B \leq \mathrm{Sym}(mn)$**:** the permutational wreath product of $A \leq \mathrm{Sym}(m)$ by $B \leq \mathrm{Sym}(n)$.

$A \circledwr B \leq \mathrm{Sym}(m^n)$**:** the exponentiation of $A \leq \mathrm{Sym}(m)$ by $B \leq \mathrm{Sym}(n)$.

Chapter 1

Preliminaries and Notation

We begin with some basic definitions and properties of abstract and profinite groups. Most of the content of this chapter is well-known and in many cases we will include references to the location of the proofs. On the other hand, we included the proof of a few elementary lemmas to help the reader become familiar with some of the notions.

Throughout this thesis we will write $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of positive integers and $\boldsymbol{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. The identity element of a group will be denoted by e.

The symmetric group *acting on the right* on a set X will be denoted by $\mathrm{Sym}(X)$, i.e. for every function $\sigma \in \mathrm{Sym}(X)$ and every point $x \in X$ we set $\sigma \cdot x = \sigma^{-1}(x)$. This is because, even though we will write functions on the left, all the actions considered in this thesis will be *right actions*. We will use the convention $\mathrm{Sym}(n) = \mathrm{Sym}(\boldsymbol{n})$ for $n \in \mathbb{N}$.

1.1 Group actions

Definition 1.1.1. *Let G be a group and let X be a set. A (right)* ***action*** *of G on X is a function $f : X \times G \to X$ such that*

1. *$f(f(x, g), h) = f(x, gh)$ for every $x \in X$ and $g, h \in G$;*

2. *$f(x, e) = x$ for every $x \in X$.*

As it is customary to do, when there is no room for confusion we will drop the symbol f and we will write the image of the element (x, g) via f as x^g. In our new notation the previous two properties become

1. $(x^g)^h = x^{gh}$ for every $x \in X$ and $g, h \in G$;

2. $x^e = x$ for every $x \in X$.

An action f of the group G on the set X induces a homomorphism φ_f from G to $\mathrm{Sym}(X)$ defined by $\varphi_f(g)(x) = x^g$. It is standard to verify that this is indeed a homomorphism and, vice versa, any homomorphism $\psi : G \to \mathrm{Sym}(X)$ from a group G to the symmetric group on a set X induces an action f_ψ of G on X by setting $f_\psi(x, g) = \psi(g) \cdot x$ for $x \in X$ and $g \in G$.

Definition 1.1.2. *An action f of the group G on the set X is said to be **faithful** if the kernel of the associated homomorphism φ_f is trivial.*

Equivalently, an action of the group G on the non-empty set X is faithful if and only if for every non-identity element $g \in G$ there exists $x \in X$ such that $x^g \neq x$. In general, we can define more than one action of isomorphic abstract groups. For instance, consider the actions f_1 and f_2 of the group $\mathrm{Alt}(5)$ on the sets $\mathbf{5}$ and $\mathrm{Alt}(5)$ respectively defined by

$$\begin{array}{rccc} f_1 : & \mathbf{5} \times \mathrm{Alt}(5) & \to & \mathbf{5} \\ & (i, \sigma) & \mapsto & \sigma^{-1}(i) \end{array} \quad \text{and} \quad \begin{array}{rccc} f_2 : & \mathrm{Alt}(5) \times \mathrm{Alt}(5) & \to & \mathrm{Alt}(5) \\ & (x, g) & \mapsto & x \cdot g \end{array},$$

the action f_1 is called the *natural* action of $\mathrm{Alt}(5)$ and f_2 is called the *(right) regular* action of $\mathrm{Alt}(5)$. The action f_1 moves 5 points, while the action f_2 moves 60 points. The next definition is a way of telling when different actions "are the same".

Definition 1.1.3. *Let $f_1 : X \times G \to X$ and $f_2 : Y \times H \to Y$ be two actions of the groups G and H on the sets X and Y respectively. We say that f_1 and f_2 are **equivalent** if there exist an isomorphism $\varphi : G \to H$ and a bijection*

$\lambda : X \to Y$ *such that*

$$\lambda(f_1(x,g)) = f_2(\lambda(x), \varphi(g))$$

for every $x \in X$ *and every* $g \in G$.

In particular, if $f_1 : X \times G \to X$ and $f_2 : Y \times H \to Y$ are equivalent we have $|X| = |Y|$. Therefore the natural action and the regular action of $\mathrm{Alt}(5)$ are not equivalent.

Up to equivalence of group actions we can always suppose, without loss of generality, that an action of the group G on the set X of size n is an action of G on the set $\boldsymbol{n}$. This identification will be carried out in all this thesis without special mention.

Definition 1.1.4. *A* ***permutation group*** G *is a subgroup of the symmetric group* $\mathrm{Sym}(n)$ *acting on the right on* n *elements.*

In this thesis we will work only with permutation groups that are not the trivial subgroup of the symmetric group, so when we will write "permutation group" we will intend "non-trivial permutation group".

Specifying a permutation group is equivalent to give a couple (G, f) where f is a faithful action of the finite group G. We will say that two permutation groups are equivalent if the associated actions are.

Definition 1.1.5. *Let* $G \leq \mathrm{Sym}(n)$ *be a permutation group. The* ***stabiliser*** *of* $x \in \boldsymbol{n}$ *is the subgroup of* G *defined by*

$$\mathrm{St}_G(x) = \{g \in G | x^g = x\}.$$

The ***orbit*** *of* $x \in \boldsymbol{n}$ *is the subset of* $\boldsymbol{n}$ *defined by*

$$x^G = \{x^g \mid g \in G\}.$$

The permutation group $G \leq \mathrm{Sym}(n)$ *is said to be:*

- ***transitive*** *if for every* $x, y \in \boldsymbol{n}$ *there exists* $g \in G$ *such that* $x^g = y$*;*
- ***imprimitive*** *if there exists a non-trivial partition* $\mathcal{B}$ *of* $\boldsymbol{n}$ *that is preserved by the action of* G*. In this case, the elements of* $\mathcal{B}$ *are called* ***blocks*** *for the action of* G*;*
- ***primitive*** *if* G *is transitive and it is not imprimitive;*
- ***regular*** *if* G *is transitive and the stabiliser of every point in* $\boldsymbol{n}$ *is trivial.*

Since a permutation group has both a group structure and an action built in, we can consider permutation groups with both group-theoretical and action-theoretical properties. When this is the case, we write first all the group-theoretical properties and then all the action-theoretical ones. For example, when we will write *d-generated perfect transitive permutation group* we intend a transitive permutation group that is perfect and d-generated as an abstract group.

1.2 Wreath products and their actions

The main current of research in Group Theory of the last century was the study of the "building blocks" of finite groups, namely the *finite simple groups*. After the completion of the Classification of Finite Simple Groups, the main open problem in this current is the classification of the possible ways of building a finite group out of these fundamental components, in other words the study of *group extensions*.

Definition 1.2.1. *Let* A*,* B *and* G *be finite groups. We say that* G *is an* ***extension*** *of* A *by* B *if there exists a normal subgroup* N *of* G *such that* $N \cong A$ *and* $G/N \cong B$*.*

No classification of all the extensions of an arbitrary group is known. Some particular extensions can be classified via *homological methods*, but these will not be mentioned in this work. Here we are going to define an easy construction which gives an "universal extension" in a sense that we are going to specify below.

Let A be a finite group and let $B \leq \mathrm{Sym}(n)$ be a permutation group. We can define an action of B on A^n by: for $(a_1, \ldots, a_n) \in A^n$ and $b \in B$

$$(a_1, \ldots, a_n)^{b^{-1}} = (a_{1^b}, \ldots, a_{n^b}). \tag{1.2.1}$$

Thus, the previous action defines an homomorphism $B \to \mathrm{Aut}(A^n)$.

Definition 1.2.2. *Let A be a group and let $B \leq \mathrm{Sym}(n)$ be a permutation group. The **abstract wreath product** of A and B is the finite group $A\mathrm{wr}B = A^n \rtimes B$ with the semidirect product obtained from the homomorphism induced by the action* (1.2.1)*. The subgroup $A^n \leq A\mathrm{wr}B$ is called the **base group** of $A\mathrm{wr}B$. The projection of the semidirect product $\pi : A\mathrm{wr}B \to B$ is called the **standard projection** of $A\mathrm{wr}B$.*

We report the following classical theorem about wreath products.

Theorem 1.2.3. ([7, Theorem 2.6A], Universal Embedding Theorem) *Let G be a finite group and $N \lhd G$. Set $K = G/N$ and consider K as a permutation group $K \leq \mathrm{Sym}(|K|)$ with the action on itself by right multiplication. Then there is an embedding $\phi : G \to N\mathrm{wr}K$ such that ϕ maps N onto $\mathrm{Im}\phi \cap N^{|K|}$ where $N^{|K|}$ is the base group of $N\mathrm{wr}K$.*

By the previous theorem, $N\mathrm{wr}K$ contains an isomorphic copy of every extension of N by K. In this sense the wreath product of N by K is an "universal extension" of N by K.

Lemma 1.2.4. *Let A be a finite non-abelian simple group and let n be a natural number. Suppose that N is a normal subgroup of A^n. For $i \in \boldsymbol{n}$, let π_i be the projection from A^n to the i-th coordinate. Then*

$$N = \pi_1(N) \times \ldots \times \pi_n(N)$$

where either $\pi_i(N)$ is either trivial or the whole of A. In particular, N is isomorphic to A^j for some $j = 0, 1, \ldots, n$.

Proof. It is clear that N is contained in the product of its projections. Now, fix $i \in \boldsymbol{n}$. For every $a \in A$ and for every $(k_1, \ldots, k_n) \in N$ we have that

$$[(k_1, \ldots, k_n), (e, \ldots, e, a, e, \ldots, e)] = (e, \ldots, e, [k_i, a], e, \ldots, e) \in N$$

where a and $[k_i, a]$ are in the i-th position of the above n-tuples. Therefore the subgroup $[\pi_1(N), A] \times \ldots \times [\pi_n(N), A]$ is contained in N. Now, for every $i \in \boldsymbol{n}$ the subgroup $[\pi_i(N), A]$ is normal in the non-abelian simple group A. Therefore either $[\pi_i(N), A] = A$ or $[\pi_i(N), A] = \{e\}$. In the former case we deduce that $\pi_i(N) = A$. Since A is non-abelian, in the latter case we have that $\pi_i(N) = \{e\}$. We have the other inclusion and the result follows. $\square$

The following lemma is straightforward and will be used many times in the next chapters.

Lemma 1.2.5. *Let A be a finite non-abelian simple group and let $B \leq \mathrm{Sym}(n)$ be a permutation group. Then A^n is a minimal normal subgroup of $A\mathrm{wr}B$ if and only if A is simple and B is transitive. Moreover, in such case A^n is the unique minimal normal subgroup of $A\mathrm{wr}B$.*

Proof. Suppose that A is simple and B is transitive and let N be a normal subgroup of $A\mathrm{wr}B$ contained in A^n. For $i \in \boldsymbol{n}$, let π_i be the projection from A^n to the i-th coordinate.

By Lemma 1.2.4, N is the direct product of the groups $\pi_i(N)$ for $i \in \boldsymbol{n}$. On the other hand, by the transitivity of B and the normality of N, we see that the projections $\pi_i(N)$ of N must be pair-wise isomorphic. It follows that either $N = A^n$ or N is trivial. Thus A^n is minimal normal.

For the converse, suppose that A^n is a minimal normal subgroup of $A\mathrm{wr}B$. First suppose by contradiction that A is not simple; then there exists a non-trivial proper normal subgroup N of A. But this produces a non-trivial proper normal subgroup N^n of $A\mathrm{wr}B$ contained in A^n, a contradiction. Now suppose that B is not transitive and pick one element $x \in \boldsymbol{n}$. Consider the orbit x^B of x, hence the subgroup $N = \{(a_1, \ldots, a_n) \in A^n \mid a_j = e \text{ for } j \notin x^B\}$ is normal in $A\mathrm{wr}B$ and properly contained in A^n, again a contradiction.

Suppose that A^n is minimal normal in $A\mathrm{wr}B$ and let N be another minimal normal subgroup of $A\mathrm{wr}B$. Since N and A^n are both minimal normal, we must have that $N \cap A^n = \{e\}$. Pick $(c_1, \ldots, c_n)d \in N$. Then, for all $(a_1, \ldots, a_n)$ in A^n,

$$h = ((c_1, \ldots, c_n)d)^{(a_1,\ldots,a_n)} = (c_1^{a_1}, \ldots, c_n^{a_n})(a_1, \ldots, a_n)^{-1}(a_1, \ldots, a_n)^{d^{-1}} d \in N$$

and $(c_1, \ldots, c_n)dh^{-1} \in N \cap A^n = \{e\}$. Therefore $((c_1, \ldots, c_n)d)^{(a_1,\ldots,a_n)} = (c_1, \ldots, c_n)d$ for every $(a_1, \ldots, a_n) \in A^n$ and, by the simplicity of the group A, $(c_1, \ldots, c_n) = (e, \ldots, e)$ and N is contained in B. Suppose now that $d \in N$ and pick $(a_1, \ldots, a_n) \in A^n$, then $d^{(a_1,\ldots,a_n)} = (a_1, \ldots, a_n)^{-1}(a_1, \ldots, a_n)^{d^{-1}} d \in N \leq B$, so d must fix all the n-tuples of A^n. By the faithfulness of the action of B, this implies that $N = 1$. □

If the group A is also a permutation group we can define two actions of the abstract wreath product $A\mathrm{wr}B$.

Definition 1.2.6. *Let $A \leq \mathrm{Sym}(m)$ and $B \leq \mathrm{Sym}(n)$ be two permutation groups.*

*The **permutational wreath action** of $A\mathrm{wr}B$ is the action on the set $\boldsymbol{m} \times \boldsymbol{n}$ defined by: for $(x, y) \in \boldsymbol{m} \times \boldsymbol{n}$ and $(a_1, \ldots, a_n)b \in A\mathrm{wr}B$*

$$(x, y)^{(a_1,\ldots,a_n)b} = (x^{a_y}, y^b). \tag{1.2.2}$$

*The permutation group $A\mathrm{wr}B \leq \mathrm{Sym}(mn)$ with the action (1.2.2) is called the **permutational wreath product** of A by B and it is denoted by $A \wr B$.*

*The **product action** of $A\mathrm{wr}B$ is the action on the set $\boldsymbol{m}^n$ defined by: for $(x_1, \ldots, x_n) \in \boldsymbol{m}^n$ and $(a_1, \ldots, a_n)b \in A\mathrm{wr}B$*

$$\begin{cases} (x_1, \ldots, x_n)^{(a_1,\ldots,a_n)} = (x_1^{a_1}, \ldots, x_n^{a_n}) \\ (x_1, \ldots, x_n)^{b^{-1}} = (x_{1^b}, \ldots, x_{n^b}) \end{cases}. \tag{1.2.3}$$

The permutation group $A\mathrm{wr}B \leq \mathrm{Sym}(m^n)$ *with the action* (1.2.3) *is called the* ***exponentiation*** *of* A *by* B *and it is denoted by* $A \textcircled{\wr} B$.

We will briefly discuss the main features of the permutation groups just defined. The permutational wreath product $A \wr B$ is an imprimitive permutation group with blocks $B_j = \{(i,j) | i \in \boldsymbol{m}\}$ for $j \in \boldsymbol{n}$. Moreover, $A \wr B$ is transitive if and only A and B are.

On the other hand, $A \textcircled{\wr} B$ is transitive if and only if A is, and the exponentiation $A \textcircled{\wr} B$ is primitive for "reasonable" permutation groups A and B as the following theorem shows.

Theorem 1.2.7. ([7, Theorem 2.7A]) *Let* $A \leq \mathrm{Sym}(m)$ *and* $B \leq \mathrm{Sym}(n)$ *be non-trivial permutation groups. Then the exponentiation* $A \textcircled{\wr} B$ *is primitive if and only if it satisfies both of the following conditions:*

1. *A is primitive but not regular;*

2. *B is transitive.*

1.3 Topological notions

1.3.1 General Topology

For the convenience of the reader we include here some basic topological definition.

Definition 1.3.1. *Let* X *be a topological space. The space* X *is said to be* ***discrete*** *if every subset of* X *is open. Equivalently,* X *is discrete if for every* $x \in X$ *the singleton subset* $\{x\}$ *of* X *is open.*

The space X *is said* ***pre-compact*** *if, for every collection* $\{A_i\}_{i \in I}$ *of open subsets* A_i *of* X *such that* $X = \bigcup_{i \in I} A_i$, *there exist* $n \in \mathbb{N}$ *and* $i_1, \ldots, i_n \in I$ *for which* $X = A_{i_1} \cup \ldots \cup A_{i_n}$.

Remark 1.3.2. Let X be a topological space. We say that a collection of subsets $\{A_i\}_{i \in I}$ of X satisfies the **finite intersection property** if for any finite subcollection J of I the intersection $\bigcap_{j \in J} A_j$ is non-empty.

Taking complements we see that the topological space X is pre-compact if and only if any collection of closed subsets with the finite intersection property in X has non-trivial intersection.

Definition 1.3.3. *Let X be topological space. The space X is called **Hausdorff** if for every $x, y \in X$ there exist A and B open neighbourhoods of x and y respectively such that $A \cap B = \varnothing$.*

*The space X is said to be **compact** if it is pre-compact and Hausdorff.*

Definition 1.3.4. *Let X be a topological space. The space X is said to be **disconnected** if there exist non-empty open subsets A and B of X such that $X = A \cup B$ and $A \cap B = \varnothing$. The space X is said **connected** if it is not disconnected. The maximal connected subsets (ordered by set-theoretic inclusion) of X are called **connected components**. If the connected components of X consist of the singletons $\{x\}$, for $x \in X$, we say that X is **totally disconnected**.*

Definition 1.3.5. *Let X be a topological space and let Y be a subset of X. A point y in Y is said to be **interior** in Y if there exists an open subset U of Y such that $y \in U$. The **interior** of Y, denoted by $\mathring{Y}$, is the set of all interior points of Y. The **closure** of Y, denoted by $\overline{Y}$, is the smallest closed subset of X that contains Y.*

The closure of a subspace Y in the topological space X can be obtained as the intersection of all closed subspaces of X that contain Y.

The next theorem is a fundamental result in the theory of compact topological spaces and has many application in different parts of mathematics.

Theorem 1.3.6. (Baire Category Theorem) *Let X be a compact topological* *space. Let $\{C_i\}_{i\in\mathbb{N}}$ be a countable collection of closed subsets C_i of X such that each C_i has empty interior. Then the union $\bigcup_{i\in\mathbb{N}} C_i$ has empty interior.*

1.3.2 Topological groups

Definition 1.3.7. *A* ***topological group*** *is a group G equipped with a topology such that the maps*

$$\begin{array}{rccc} \mathrm{mult}: & G\times G & \to & G \\ & (g,h) & \mapsto & gh \end{array} \quad \textit{and} \quad \begin{array}{rccc} \mathrm{inv}: & G & \to & G \\ & g & \mapsto & g^{-1} \end{array}$$

are continuous.

Since a topological group has both a group structure and a topology built in, we can consider objects with both group-theoretical and topological properties. For example, we can consider *closed subgroups* or *open normal subgroups*. For a topological group G, we will write $H \leq_o G$ when H is a subgroup of G which is open with respect to the topology of G. We will use a similar convention for closed, closed normal, open and open normal subgroups.

We will now describe a property of topological groups that will be used many times in the introductory chapters. Let G be a topological group and choose $g \in G$. Then the map $\varphi_g : G \to G$ defined by $\varphi_g(x) = gx$ is a homeomorphism of G. In fact, multiplication by g and by g^{-1} in G are continuous by the definition of a topological group and they are the inverse of each other. This means that a topological group is a *homogeneous* topological space, in the sense that the topology near a point "looks like" the topology near any other point. We will use this fact in the proof of the next lemma.

Lemma 1.3.8. *Let G be a finite topological group. If $\{x\}$ is closed in G for all $x \in G$, then G is discrete.*

Proof. We have that $G \setminus \{e\}$ is the union of $\{x\}$ for $x \in G$ such that $x \neq e$. By hypothesis this is a closed set and therefore $\{e\}$ is open. For every $g \in G$, the map φ_g from G to G defined by $h \mapsto gh$ is a homeomorphism, thus $\{g\} = \varphi_g(\{e\})$ is open and G is discrete. $\square$

In particular, a finite Hausdorff topological group satisfies the hypothesis of the previous lemma. Hence the only Hausdorff topology on a finite topological group is the discrete topology.

1.3.3 A little measure theory

Definition 1.3.9. *Let S be a set and Σ a subset of the set 2^S (the power set of S). We say that Σ is a **σ-algebra** if it satisfies the following properties:*

1) *$X \in \Sigma$;*

2) *if $A \in \Sigma$, then $S \setminus A \in \Sigma$;*

3) *if $A_i \in \Sigma$ for i in a countable set I, then $\bigcup_{i \in I} A_i \in \Sigma$.*

Definition 1.3.10. *Let S be a set and let Σ be a σ-algebra. A function $\mu : \Sigma \to [0, +\infty]$ is called a **measure** on Σ if it satisfies the following properties:*

1) *$\mu(\varnothing) = 0$;*

2) *for all $\{E_i\}_{i \in \mathbb{N}}$ such that $E_i \cap E_j = \varnothing$ for $i \neq j$,*
$$\mu\left(\bigcup_{i \in \mathbb{N}} E_i\right) = \sum_{i \in \mathbb{N}} \mu(E_i).$$

Let G be a compact topological group. Let $\mathcal{B}$ be the smallest σ-algebra that contains all open subsets of G; this is called the **Borel algebra**. Let $g \in G$ and S a subset of G. Define $gS = \{gs \mid s \in S\}$ and $Sg = \{sg \mid s \in S\}$. We say that a measure μ on $\mathcal{B}$ is **left-invariant** if $\mu(gB) = \mu(B)$ for every $B \in \mathcal{B}$ and $g \in G$. We say that a measure μ on $\mathcal{B}$ is **right-invariant** if $\mu(Bg) = \mu(B)$ for every $B \in \mathcal{B}$ and $g \in G$.

The next theorem is of fundamental importance in the theory of locally compact topological groups, but we will state and use it only in the instance of compact topological groups.

Theorem 1.3.11. (Haar's Theorem) *Let G be a compact topological group and let $\mathcal{B}$ be the Borel algebra of G. Then there exists a unique measure μ on $\mathcal{B}$ such that $\mu(G) = 1$ and μ is left-invariant and right-invariant.*

The unique measure on the Borel algebra of a compact topological group G described in Haar's Theorem will be called the **Haar measure** of G.

1.3.4 Metric on topological groups

Definition 1.3.12. *Let X be a topological space. A **metric** (or distance) on X is a function $d : X \times X \to [0, +\infty)$ such that for every $x, y, z \in X$*

1) $d(x, y) = 0$ *if and only* $x = y$*;*

2) $d(x, y) = d(y, x)$*;*

3) $d(x, z) \leq d(x, y) + d(y, z)$.

*A **metric space** is a couple (X, d) where X is a topological space and d is a metric on X. The **ball** of radius $r \in [0, +\infty)$ around $x \in X$ in the metric space (X, d) is the subset*

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

Definition 1.3.13. *Let X be a topological space. A collection $\{B_i\}_{i\in I}$ of open subsets B_i of X is said to be a **base** of X if for every open subset U of X there exists $J \subseteq I$ such that $U = \bigcup_{j\in J} B_j$.*

*The topological space X is said to be **countably based** if there exists a base $\{B_n\}_{n\in\mathbb{N}}$ of X with only countably many elements.*

Remark 1.3.14. Let G be a topological group and suppose that there exists a countable descending chain of open normal subgroups $\{G_n\}_{n\in\mathbb{N}}$ of G with $G_1 = G$ and $\bigcap_{n\in\mathbb{N}} G_n = \{e\}$. Then we can define a distance on G in the following way: for $x, y \in G$ set

$$d(x, y) = \inf \left\{ |G : G_i|^{-1} \mid xy^{-1} \in G_i \right\}. \tag{1.3.1}$$

Therefore every countably based topological group admits a metric. In particular, every countably based profinite group is a metric space with the metric (1.3.1).

It is interesting to notice that the converse of the previous remark holds by the following deep topological theorem.

Theorem 1.3.15. ([20, 7.3]) *Let G be a topological group. Then G admits a metric if and only if G is countably based.*

1.4 Inverse limits and profinite groups

Profinite groups have wide application in different parts of mathematics. For example, they come up naturally in Group Theory, Number Theory and Analysis. Here we are going to recall the basic definitions and fix the notation that we will need in the forthcoming chapters. Our exposition follows closely the fundamental [31, Chapter 1].

A **directed set** is a partially ordered set $(I, \leq)$ such that for all $i, j \in I$ there exists $k \in I$ for which $i \leq k$ and $j \leq k$.

Definition 1.4.1. *An* ***inverse system*** $(\{X_i\}_{i\in I}, \varphi_{ij})$ *of topological groups indexed by a directed set I consists of a family $\{X_i\}_{i\in I}$ of topological groups X_i, with $i \in I$, and a family $\{\varphi_{ij} : X_j \to X_i | i, j \in I,\ i \leq j\}$ of continuous homomorphisms such that φ_{ii} is the identity map on X_i for each $i \in I$ and $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ whenever $i \leq j \leq k$.*

In the next chapters we will consider inverse limits of finite groups. Remember that the only possible way of defining a Hausdorff topology on a finite set is with the discrete topology.

Let $(\{X_i\}_{i\in I}, \varphi_{ij})$ be an inverse system of topological groups and let Y be a topological group. We shall call a family $\{\psi_i : Y \to X_i | i \in I\}$ of continuous homomorphisms **compatible** with $(\{X_i\}_{i\in I}, \varphi_{ij})$ if $\varphi_{ij}\psi_j = \psi_i$ whenever $i \leq j$, or equivalently if the following diagrams

$$\begin{tikzcd}
 & Y \arrow[dl, "\psi_j"'] \arrow[dr, "\psi_i"] & \\
X_j \arrow[rr, "\varphi_{ij}"'] & & X_i
\end{tikzcd}$$

Figure 1.1: Diagram definition of compatible maps.

are commutative for every $i \leq j$.

Definition 1.4.2. *An* ***inverse limit*** (X, φ_i) *of an inverse system of topological groups* $(\{X_i\}_{i\in I}, \varphi_{ij})$ *is a topological group* X *together with a family* $\{\varphi_i : X \to X_i\}$ *of continuous homomorphisms compatible with* $(\{X_i\}_{i\in I}, \varphi_{ij})$ *and with the following universal property: whenever* $\{\psi_i : Y \to X_i\}$ *is a family of continuous homomorphisms compatible with* $(\{X_i\}_{i\in I}, \varphi_{ij})$ *from a topological group* Y*, there is a unique continuous homomorphism* $\psi : Y \to X$ *such that* $\varphi_i\psi = \psi_i$ *for each* $i \in I$*.*

Thus we require that there is a unique ψ such that each of the following diagrams are commutative.

$$\begin{tikzcd}
 & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi_i"] & \\
X \arrow[rr, "\varphi_i"'] & & X_i
\end{tikzcd}$$

Figure 1.2: Diagram definition of inverse limit.

It is possible to show that inverse limits are unique up to isomorphism. We will denote *the* inverse limit of the inverse system $(\{X_i\}_{i\in I}, \varphi_{ij})$ by $\varprojlim X_i$, as it is customary to drop the reference to the maps φ_{ij}. Furthermore, inverse limits inherit many properties from the associated inverse system. We are just going to list below some basic facts that will be relevant to us in the forthcoming chapters.

Proposition 1.4.3. ([31, Proposition 1.1.5(d)-(e)]) *Let* $(\{X_i\}_{i\in I}, \varphi_{ij})$ *be an inverse system of topological groups. If each* X_i *is a compact Hausdorff topological group, so is* $\varprojlim X_i$.

In view of the previous proposition, all the finite groups considered in this thesis will be equipped with the discrete topology (see Lemma 1.3.8).

Let $\mathcal{C}$ be a class of finite groups. We call a group F a **$\mathcal{C}$-group** if $F \in \mathcal{C}$, and we call G a **pro-$\mathcal{C}$ group** if it is an inverse limit of $\mathcal{C}$-groups.

Lemma 1.4.4. *Let* $\mathcal{C}$ *be a class of finite groups. Then every* $\mathcal{C}$*-group is a pro-*$\mathcal{C}$ *group.*

Proof. Let $F \in \mathcal{C}$ and consider the trivial poset $\mathbf{1} = \{1\}$. Then $\mathbf{1}$ is a directed set and $(\{F\}, \mathrm{id}_F)$ is clearly an inverse system of $\mathcal{C}$-groups indexed by $\mathbf{1}$. Now, (F, id_F) satisfies the requirements of the inverse limit of $(\{F\}, \mathrm{id}_F)$ and, by uniqueness, we have $\varprojlim F = F$. □

We say that $\mathcal{C}$ is **closed for subgroups** (respectively **quotients**) if every subgroup (respectively quotient) of a $\mathcal{C}$-group is a $\mathcal{C}$-group, and we say that $\mathcal{C}$ is **closed for finite direct products** if $F_1 \times F_2 \in \mathcal{C}$ whenever $F_1 \in \mathcal{C}$ and $F_2 \in \mathcal{C}$. Some important classes are:

- the class of all finite groups,
- the class of finite p-groups where p is a fixed prime,
- the class of finite nilpotent groups,
- the class of finite solvable groups.

All the classes listed above are closed for subgroups, finite direct products and quotients. An inverse limit of finite p-groups is called a **pro-p group** and an inverse limit of finite nilpotent groups is called **pronilpotent group**.

Theorem 1.4.5. ([31, Theorem 1.2.3]) *Let $\mathcal{C}$ be a class of finite groups which is closed for subgroups, finite direct products and quotients, and let G be a topological group. The following are equivalent:*

1. *G is a pro-$\mathcal{C}$ group;*

2. *G is compact and $\bigcap\{N | N \lhd_O G,\ G/N \in \mathcal{C}\} = \{e\}$;*

3. *G is compact and totally disconnected, and $G/N \in \mathcal{C}$ for every $N \lhd_O G$.*

Corollary 1.4.6. ([31, Corollary 1.2.4]) *Let G be a topological group. The following are equivalent:*

1. *G is profinite;*

2. *G is compact and $\bigcap\{N | N \lhd_O G\} = \{e\}$;*

3. *G is compact and totally disconnected.*

We call a family I of normal subgroups of a group G a **filter base** if for all $K_1, K_2 \in I$ there is a subgroup $K_3 \in I$ which is contained in $K_1 \cap K_2$.

Theorem 1.4.7. ([31, Theorem 1.2.5(a)]) *Let G be a profinite group. If I is a filter base of closed normal subgroups of G such that $\bigcap\{N | N \in I\} = \{e\}$ then*

$$G \cong \varprojlim_{N \in I} G/N.$$

Moreover

$$H \cong \varprojlim_{N \in I} H/(H \cap N)$$

for each closed subgroup H and

$$G/K \cong \varprojlim_{N \in I} G/KN$$

for each closed normal subgroup K.

From the previous theorem, closed subgroups of profinite groups and quotients modulo closed normal subgroups of profinite groups are themselves profinite groups.

The next lemma will come in handy in many occasions during the rest of this work.

Proposition 1.4.8. ([31, Exercise 1.6.8]) *Let I be a directed set.*

Let $(\{G_i\}_{i\in I}, u_{ij})$ and $(\{H_i\}_{i\in I}, v_{ij})$ be inverse systems of finite groups and let (G, u_i) and (H, v_i) be their respective inverse limits. Consider a set of homomorphisms $\{a_i : G_i \to H_i | i \in I\}$ such that $v_{ij}a_j = a_iu_{ij}$ for all $i \leq j$. Then exists a continuous homomorphism $a : G \to H$ such that $v_ia = a_iu_i$. Moreover, if the a_i's are isomorphisms, then a is a continuous isomorphism.

Proof. For $i \leq j$ we have the commutative diagrams

$$\begin{tikzcd} & G \arrow[dl, "u_j"'] \arrow[dr, "u_i"] & \\ G_j \arrow[rr, "u_{ij}"'] & & G_i \end{tikzcd} \qquad \begin{tikzcd} & H \arrow[dl, "v_j"'] \arrow[dr, "v_i"] & \\ H_j \arrow[rr, "v_{ij}"'] & & H_i \end{tikzcd}$$

Figure 1.3: Inverse limits of $(\{G_i\}_{i\in I}, u_{ij})$ and $(\{H_i\}_{i\in I}, v_{ij})$.

so $u_{ij}u_j = u_i$ and $v_{ij}v_j = v_i$. Consider the homomorphisms $a_iu_i : G \to H_i$; then we have the following diagram

$$
\begin{tikzcd}
 & & G \arrow[dl, "u_j"'] \arrow[dr, "u_i"] & & \\
 & G_j \arrow[rr, "u_{ij}"'] \arrow[dl, "a_j"'] & & G_i \arrow[dr, "a_i"] & \\
H_j \arrow[rrrr, "v_{ij}"'] & & & & H_i
\end{tikzcd}
$$

Figure 1.4: $(G, a_i u_i)$ is compatible with (H_i, v_{ij}).

where the big triangle commutes because $v_{ij} a_j u_j = a_i u_{ij} u_j = a_i u_i$ and so $(G, a_i u_i)$ is compatible with the inverse system (H_i, v_{ij}). By the universal property of (H, v_i) there exist a continuous homomorphism $a : G \to H$ that completes the previous diagram

$$
\begin{tikzcd}
 & & H \arrow[dddll, dashed, bend right, "v_j"'] \arrow[dddrr, dashed, bend left, "v_i"] & & \\
 & & G \arrow[u, "a"'] \arrow[dl, "u_j"'] \arrow[dr, "u_i"] & & \\
 & G_j \arrow[rr, "u_{ij}"'] \arrow[dl, "a_j"'] & & G_i \arrow[dr, "a_i"] & \\
H_j \arrow[rrrr, "v_{ij}"'] & & & & H_i
\end{tikzcd}
$$

Figure 1.5: Existence of the homomorphism $a : G \to H$.

as required.

For the last part, it is not too difficult to see that $(\{\ker\ a_i\}_{i \in I}, u_{ij_{|\ker\ a_j}})$ is an inverse system of topological groups and the continuous homomorphisms

$u_{i|\ker\ a}$, $i \in I$, are compatible with it. Since (G, u_i) is the inverse limit of $(\{G_i\}_{i\in I}, u_{ij})$, it follows that $\ker\ a = \varprojlim_{i\in I} \ker\ a_i$. The proof of the fact that $\mathrm{Im}\ a = \varprojlim_{i\in I} \mathrm{Im}\ a_i$ is very similar and will be left out. In particular, if all a_i's are isomorphisms then $\ker\ a$ is trivial and $\mathrm{Im}\ a$ is the whole of H; therefore a is a continuous isomorphism. $\square$

Chapter 2

Main definitions and state of the art

In this chapter we introduce the main concepts and definitions that we will consider in this thesis and we will also give a short introduction and basic properties of the profinite properties that will be used. As in the previous chapter, we will include the proof of a few elementary lemmas to contribute to the clarity of the exposition.

2.1 Hereditarily just infinite profinite groups

The next definition is central in this thesis and all the profinite groups in the next chapters will have this property.

Definition 2.1.1. *We say that a profinite group G is* ***just infinite*** *if it is infinite, and every non-trivial closed normal subgroup of G is open. We say G is* ***hereditarily just infinite*** *if in addition H is just infinite for every open subgroup H of G.*

Hereditarily just infinite groups appear naturally in the classification of just infinite profinite groups with no non-trivial abelian subnormal subgroups obtained by J. Wilson in [30].

Theorem 2.1.2. ([30]) *Let G be a just infinite profinite group with no non-trivial abelian normal subgroup. Then either G is a branch group, or G contains an open normal subgroup which is isomorphic to the direct product of a finite number of copies of some hereditarily just infinite profinite group.*

Branch groups are certain subgroups of the automorphism group of a rooted tree and they have received a considerable amount of attention in the past years, while hereditarily just infinite groups remained a little in the background.

For some time after the appearance of [30], there was the feeling that it would be possible to characterize hereditarily just infinite groups. In particular, all hereditarily just infinite groups known in the beginning of the millennium were virtually pro-p. The construction of a family of hereditarily just infinite profinite groups which are not virtually pro-p was carried out in [32, Construction A] and the aim of this thesis is a better understanding of the properties of this family. We will give the construction of these groups in Section 2.2.

In this thesis we will deal mostly with the profinite groups obtained from a generalisation of [32, Construction A], so we will not use general results regarding hereditarily just infinite profinite groups. On the other hand we would like to mention a few results of C. Reid about the general structure of hereditarily just infinite profinite groups which are not virtually pronilpotent. We need a few accessory definitions.

Definition 2.1.3. *A finite group G is said to have a* ***central decomposition*** *if there exist an integer n and proper subgroups $H_1, \ldots, H_n$ of G such that G is generated by $H_1, \ldots, H_n$ and whenever $i \neq j$, then $[H_i, H_j] = 1$. The finite group G is said to be* ***centrally indecomposable*** *if G admits no central decomposition.*

Definition 2.1.4. *Let H be a finite perfect group and fix a surjective homomorphism $F \to H$ with kernel R from an appropriate free group F. The* ***Schur multiplier*** *of H is the finite group $R/[F,R]$. We denote the Schur multiplier of the finite perfect group H by $M(H)$.*

Definition 2.1.5. *Let $\mathcal{C}$ be a class of finite groups. We say that $\mathcal{C}$* ***satisfies*** *$(*)$ if $\mathcal{C}$ satisfies the following property:*

$(*)$ The class $\mathcal{C}$ consists of characteristically simple groups. For each prime p, if $\mathcal{C}$ contains some elementary abelian p-group, then within the class of finite groups, $\mathcal{C}$ contains all elementary abelian p-groups and all direct powers of non-abelian simple groups S such that p divides the order of the Schur multiplier of S.

We remind the reader that it is easy to prove that a characteristically simple group is isomorphic to the direct power of a finite simple group.

Definition 2.1.6. *Let G be a finite group. Let $1 < A \trianglelefteq G$ and define $M_G(A)$ to be the intersection of all maximal G-invariant subgroups of A. We say that A is* ***narrow*** *in G and write $A \trianglelefteq_{\mathrm{nar}} G$ if there is a unique maximal G-invariant subgroup of A, in other words $M_G(A)$ is the maximal G-invariant subgroup of A.*

In [23] the author obtained a classification of hereditarily just infinite profinite groups which are not virtually-pronilpotent via their inverse limits and it can be found in the next theorem.

Theorem 2.1.7. ([23, Theorem 5.4]) *Let G be a hereditarily just infinite profinite group that is not virtually pronilpotent. For each $n \in \mathbb{N}$, let $\mathcal{C}_n$ be a class of finite groups such that G has infinitely many chief factors in $\mathcal{C}_n$, and suppose also that $\mathcal{C}_n$ satisfies condition $(*)$. Then G is the inverse limit of an inverse system $\Lambda = \{(G_n)_{n\in\mathbb{N}}, \rho_n : G_{n+1} \to G_n\}$ satisfying the following description:*

Each G_n has a specified subgroup A_n such that, letting $P_n = \rho_n(A_{n+1})$:

(*i*) $A_n > P_n > \{e\}$;

(*ii*) $A_n \trianglelefteq_{\mathrm{nar}} G_n$;

(*iii*)′ *any subgroup of* G *normalized by* A_n *either is contained in* $M_{G_n}(A_n)$ *or properly contains* $P_n C_{G_n}(P_n)$ *(or both);*

(*iv*) P_n *is a minimal normal subgroup of* G_n;

(*v*) $P_n \in \mathcal{C}_n$ *for all* n.

Conversely, let Λ *be an inverse system satisfying the above description (ignoring condition* (*v*)*). Then the inverse limit of* Λ *is hereditarily just infinite and not virtually pronilpotent. Indeed the following weaker condition suffices in place of* (*iii*)′*:*

(*iii*)″ $C_{G_n}(P_n) < A_n$ *for all* n*, and for infinitely many* n*, every normal subgroup* U *of* G_n *containing* A_n *is centrally indecomposable.*

The classes $\mathcal{C}_n$ that appear in the previous theorem are just an expedient used in the proof and they will not play a role in the rest of this thesis. We will only apply [23, Theorem 5.4] in the next section to show that certain inverse limits of finite groups are hereditarily just infinite non-(virtually pro-p) profinite groups. We will never deal with condition (*v*) of the previous theorem.

2.2 Inverse limits of iterated wreath products

The main object of study in this thesis will be a sub-family of a family of profinite groups that received a lot of attention in the past: the *infinitely iterated wreath products.*

Definition 2.2.1. *Let* $\mathcal{S} = (S_n)_{n\in\mathbb{N}}$ *be a sequence of finite groups. An* n***-th iterated wreath product*** *of the sequence* $\mathcal{S}$ *is the finite group* $\widehat{S}_n$ *inductively* *defined by the following process:* $\widehat{S}_1 = S_1$*, then choose a transitive and faithful action of* $\widehat{S}_n$ *on* $\widehat{m}_n$ *points and define* $\widehat{S}_{n+1} = S_{n+1}\mathrm{wr}\widehat{S}_n$*. The new sequence of permutation groups* $(\widehat{S}_n \leq \mathrm{Sym}(\widehat{m}_n))_{n\in\mathbb{N}}$*, together with the standard projections* $\pi_n : \widehat{S}_{n+1} \to \widehat{S}_n$*, forms an inverse system of finite groups. The inverse limit* $\varprojlim_{n\in\mathbb{N}} \widehat{S}_n$ *is called an* ***infinitely iterated wreath product*** *of type* $\mathcal{S}$*. We will write IIWP of type* $\mathcal{S}$ *for short.*

We would like to point out that non-equivalent choices for the action of $\widehat{S}_n$ yield non-isomorphic iterated wreath products. IIWPs were first defined by P. Hall in [12] and then considered in many other papers, e.g. [4, 21, 19, 3], where many of their interesting properties were discovered. IIWPs are interesting profinite groups and they provide counterexamples to many problems in Group Theory. Just to mention one, see [15, Section 13.3] where the authors use IIWPs to solve part of the "Gap Conjecture" for the subgroup growth of a profinite group (for more on subgroup growth of profinite groups see Section A). This thesis will be focused on a particular kind of IIWPs: the *generalised Wilson groups.*

Definition 2.2.2. *Let* $\mathcal{S} = (S_n \leq \mathrm{Sym}(m_n))_{n\in\mathbb{N}}$ *be a sequence of non-abelian simple transitive permutation groups and let* $(k_n)_{n\in\mathbb{N}}$ *be an increasing sequence of positive integers with* $k_1 \geq 2$*. A* ***generalised Wilson group of type*** $(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$ *is an infinitely iterated wreath product of type* $\mathcal{S}$*,* $\varprojlim \widehat{S}_k$*, where for* $n \in \mathbb{N}$ *the action of* $\widehat{S}_{k_n}$ *is chosen to be the product action of the wreath product* $S_{k_n}\mathrm{wr}\widehat{S}_{k_n-1}$*. We will write GW group for short.*

A generalised Wilson group is just an infinitely iterated wreath product where we choose the action of the iterated wreath products to be the product action infinitely many times.

Definition 2.2.3. *The* n***-th iterated exponentiation*** $\widetilde{S}_n$ *of the sequence* $\mathcal{S}$ *is inductively defined as follows:* $\widetilde{m}_1 = m_1$*,* $\widetilde{S}_1 = S_1$ *and* $\widetilde{m}_{n+1} = m_{n+1}^{\widetilde{m}_n}$*,*

$\widetilde{S}_{n+1} = S_{n+1} \wr \widetilde{S}_n \leq \mathrm{Sym}(\widetilde{m}_{n+1})$, *for* $n \in \mathbb{N}$. *The* ***infinitely iterated exponentiation of type*** $\mathcal{S}$ *is the inverse limit* $\varprojlim \widetilde{S}_n$. *We will write IIE of type* $\mathcal{S}$ *for short.*

An infinitely iterated exponentiation is just an infinitely iterated wreath product where we choose the action of the iterated wreath products to be the product action every time. Notice that an infinitely iterated exponentiation of type $\mathcal{S}$ is a generalised Wilson group of type $(\mathcal{S}, (n)_{n\in\mathbb{N}})$.

In [32] the author proves that a GW group of type $(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$ is hereditarily just infinite and not virtually pro-p in the special case where $S_{2n+1} = S_{2n}$, $k_n = 2n+1$ and the action of the group $\widetilde{S}_{k_n}$ is the "standard wreath action" (i.e. with $\widetilde{S}_{k_n-1}$ acting on itself by right multiplication) for all n. We are now going to show that [23, Theorem 5.4] (see Theorem 2.1.7) applies to every generalised Wilson group.

Proposition 2.2.4. *Let* $\mathcal{S} = (S_n \leq \mathrm{Sym}(m_n))_{n\in\mathbb{N}}$ *be a sequence of non-abelian simple transitive permutation groups and let* $(k_n)_{n\in\mathbb{N}}$ *be an increasing sequence of positive integers with* $k_1 \geq 2$. *Let* $G = \varprojlim \widehat{S}_k$ *be a GW group of type* $(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$. *Then* G *is hereditarily just infinite and not virtually pronilpotent.*

Proof. Let $G = \varprojlim \widehat{S}_k$ be as in the hypotheses. We will show that G satisfies conditions $(i) - (iv)$ of Theorem 2.1.7. By Proposition 1.4.8 with the homomorphisms $\{a_n = \mathrm{id}_{\widehat{S}_{k_n+1}}\}_{n\in\mathbb{N}}$, we have that $G \cong \varprojlim \widehat{S}_{k_n+1}$.

Now, let $\pi_{n,m} : \widehat{S}_m \to \widehat{S}_n$ be the standard projection of the wreath product for $m \geq n$. Let $\rho_n = \pi_{k_{n-1}+1,k_n+1}$ be the projection from $\widehat{S}_{k_n+1}$ to $\widehat{S}_{k_{n-1}+1}$. Define the subgroups $A_n = \ker(\pi_{k_{n-1},k_n+1})$ and $P_n = \ker(\pi_{k_n,k_n+1})$ of $\widehat{S}_{k_n+1}$ for all $n \in \mathbb{N}$. Therefore $\rho_n(A_n) = P_{n-1}$ and property (i) of Theorem 2.1.7 is satisfied. By Lemma 1.2.5, P_n is a unique minimal normal subgroup of $\widehat{S}_{k_n+1}$, so (iv) of Theorem 2.1.7 is satisfied.

By the choice of the sequence $\mathcal{S}$, the center of P_n is trivial and the action of $\widehat{S}_{k_n}$ is faithful, thus $C_{\widehat{S}_{k_n+1}}(P_n) = \{e\}$. Moreover, any normal subgroup of $\widehat{S}_{k_n+1}$ containing A_n must contain P_n and therefore it is centrally indecomposable; in fact, if $U_1, U_2 \lhd \widehat{S}_{k_n+1}$ and $A_n \leq U_1, U_2$ then $[U_1, U_2] \geq P_n$. So $(iii)''$ of Theorem 2.1.7 is satisfied.

All that is left to prove is that A_n is narrow in $\widehat{S}_{k_n+1}$. Observe that a maximal $\widehat{S}_{k_n+1}$-invariant subgroup of A_n must contain $K = \ker(\rho_n)$, therefore a maximal $\widehat{S}_{k_n+1}$-invariant subgroup of A_n corresponds to a $\widehat{S}_{k_{n-1}+1}$-invariant subgroup of P_{n-1}. We observe that the group P_{n-1} is isomorphic to $S_{k_{n-1}+1}^{\widehat{m}_{k_{n-1}}}$ and since $\widehat{S}_{k_{n-1}}$ acts on the labels of the components of P_{n-1} with the product action, by hypothesis and by Lemma 1.2.5, the only invariant $\widehat{S}_{k_{n-1}+1}$-subgroup of P_{n-1} is P_{n-1} itself and thus the unique $\widehat{S}_{k_n+1}$-invariant subgroup of A_n is K. Hence property (ii) of Theorem 2.1.7 is met.

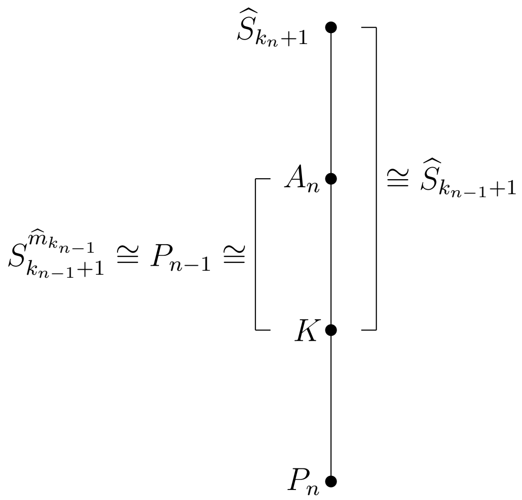

Figure 2.1: GW groups are hereditarily just infinite.

Therefore, by Theorem 2.1.7, generalised Wilson groups are hereditarily just infinite profinite groups and they are not virtually pro-p. $\square$

The existence of hereditarily just infinite and not virtually pro-p profinite groups was unknown before the publication of [32]. We take the opportunity to list here the known results about generalised Wilson groups. To this end we need to state an accessory definition. Let G be a group and let $X = \{x_1, \dots, x_k\}$ be a subset of G, as customary we will denote by $\langle x_1, \dots, x_k \rangle$ the smallest subgroup of G containing X.

Definition 2.2.5. *Let G be a finitely generated profinite group and let k be a positive integer. Let μ be the Haar measure of G^k. The profinite group G is said to be **positively k-generated** if*

$$\mu\left(\{(x_1, \dots, x_k) \in G^k \mid \overline{\langle x_1, \dots, x_k \rangle} = G\}\right) > 0.$$

Remark 2.2.6. Let G be a finitely generated profinite group (see Section 2.3.1 for the definition). Then the set $X = \{(x_1, \dots, x_k) \in G^k \mid \overline{\langle x_1, \dots, x_k \rangle} = G\}$ is in the Borel algebra of G^k and hence measurable. In fact, for $N \lhd_O G$, let π_N be the continuous projection map from G to G/N, then

$$X = \bigcap_{N \lhd_O G} \pi_N^{-1}\left(\{(x_1, \dots, x_k) \in (G/N)^k \mid \overline{\langle x_1, \dots, x_k \rangle N = G/N}\}\right),$$

which is a countable intersection of pre-images of finite (and hence closed) subsets of the finite groups $(G/N)^k$.

Let $\mathcal{B}$ be the Borel σ-algebra of G. Since $(G, \mu, \mathcal{B})$ is a *probability space*, the previous definition can be interpreted in the following sense: in a positively k-generated profinite group "the probability that k randomly chosen elements generate the whole G is positive".

The following theorem by M. Quick generalizes a result by Bhattacharjee about iterated wreath products of alternating groups (see [3]).

Theorem 2.2.7. ([21, Theorem 1]) *Let $\mathcal{S} = (S_n)_{n \in \mathbb{N}}$ be a sequence of finite groups. Suppose we choose a transitive and faithful action of S_1 on at least 35 points. Then the infinitely iterated wreath product $\varprojlim \widehat{S}_k$ is positively 2-generated for any choice of the action of $\widehat{S}_k$, $k \geq 2$.*

The author of [21] is mainly interested in asymptotic results, so hypotheses as the one in the previous theorem arise naturally in the proofs. The proof of [21, Theorem 1] relies on the classification of finite non-abelian simple groups.

Therefore GW groups of type $(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$ with $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ are positively 2-generated provided that m_1 is large enough ($m_1 > 35$).

The following theorem by Lucchini and Menegazzo is one of a long series of results about *monolithic finite groups*. A monolithic finite group is a finite group with a unique minimal normal subgroup.

Theorem 2.2.8. ([16, Theorem A]) *Let G be a finite non-cyclic group with a unique minimal normal subgroup N. Then* $\mathrm{d}(G) = \max\{2, \mathrm{d}(G/N)\}$.

By [16, Theorem A], the infinitely iterated exponentiation of type $\mathcal{S}$ is always 2-generated, and again this result relies on the classification of finite simple groups.

These were the only results about generalised Wilson groups known to the author, other than the ones proved in this thesis.

2.3 Overview of the profinite properties studied

In this thesis we are going to study various properties of profinite groups. Some of these are well known, while others are maybe familiar only to the specialists. We will include here a short state of the art for each property.

2.3.1 Finite topological generation

The study of the number of generators of a group is one of the main trends in Group Theory. Here we present an account of the properties of this invariant and list some results that will be used later on.

Definition 2.3.1. *Let G be a group and let X be a non-empty subset of G.*

*The **subgroup generated by** X **in** G is the smallest subgroup of G that contains X, it is denoted by $\langle X\rangle$.*

Let G be a group and let X be a subset of G. It is an easy exercise to prove that $\langle X\rangle = \{x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n} \mid x_i \in X,\ \varepsilon_i \in \mathbb{Z} \text{ for } n \in \mathbb{N} \text{ and } i \in \boldsymbol{n}\}$.

Definition 2.3.2. *Let G be a group. The **minimal number of generators** of the group G is*

$$\mathrm{d}(G) = \inf\{|X| \mid X \subseteq G,\ \langle X\rangle = G\} \in \mathbb{N} \cup \{\infty\}.$$

*If $\mathrm{d}(G)$ is finite, the group G is said **finitely generated**. If the group G is finitely generated and $d = \mathrm{d}(G)$, sometimes we will say that G is d-**generated**.*

Remark 2.3.3. Let G be a finitely generated group, then G is countable. In fact, let $G = \langle X\rangle$ where X is a finite subset of G, then

$$G = \bigcup_{n,m\in\mathbb{N}} \{x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n} \mid x_i \in X,\ -m \leq \varepsilon_i \leq m \text{ for } i \in \boldsymbol{n}\}$$

is a countable union of finite subsets.

Let G be a countable profinite group; then G is finite. In fact, suppose that G is a countable profinite group. By Corollary 1.4.6, G is compact and totally disconnected as a topological space. In particular G is Hausdorff, therefore for every $x \in X$ the singleton subset $\{x\}$ is closed in G. Since the union $\bigcup_{x\in X}\{x\} = G$ has non-empty interior, by Baire Category Theorem there exists an element $x_0 \in X$ such that $\{x_0\}$ is open in G.

On the other hand, for $g \in G$, the map φ_g from G to G defined by $\varphi_g(x) = gx$ is an homeomorphism. It follows that for every $x \in X$ the subset $\{x\}$ is open, therefore G is a discrete topological space. Since G is compact, G must be finite.

We present a second interesting proof of the fact that a countable profinite group must be finite using the Haar measure. Suppose by contradiction that

G is a countable infinite profinite group and let μ be the Haar measure of G. The subset $\{x\}$ of G is closed for every $x \in G$, so $G \setminus \{x\}$ is open and hence it is in the Borel algebra $\mathcal{B}$, thus its complement $\{x\}$ is in $\mathcal{B}$ for every $x \in G$. Now, μ is left invariant, therefore $\mu(\{x\}) = \mu(\{x \cdot e\}) = \mu(\{e\})$ for all $x \in G$. It follows that

$$1 = \mu(G) = \sum_{x \in G} \mu(\{x\}) = |G| \cdot \mu(\{e\})$$

and either $\mu(\{e\}) = 0$, which implies $\mu(G) = 0$, or $\mu(\{e\}) > 0$, which gives $\mu(G) = \infty$. In both cases we have a contradiction, so G must be finite.

Hence, if G is an infinite profinite group, then G uncountable. Thus, an infinite profinite group cannot be finitely generated in the above sense. Luckily we have a generalisation of the concept of finite generation in the context of infinite profinite groups.

Definition 2.3.4. *Let G be a profinite group. The* ***minimal number of topological generators*** *of the group G is*

$$\mathrm{d}(G) = \inf\{|X| \mid X \subseteq G,\ \overline{\langle X \rangle} = G\} \in \mathbb{N} \cup \{\infty\}.$$

If $\mathrm{d}(G)$ is finite, the profinite group G is said to be ***topologically finitely generated****. If the profinite group G is topologically finitely generated and $d = \mathrm{d}(G)$, sometimes we will say that G is* ***topologically d-generated****.*

Remark 2.3.5. Note that if G is a finite group and we want a Hausdorff topology on G, then G must be discrete. Therefore the two notions of finite generation above coincide for a finite group. In the rest of the thesis we will abuse a little the notation and we will drop the word "topological" when we deal with topological generators of profinite groups. Since there is no room for confusion, we can read the previous definition without the words "topological" and "topologically".

Proposition 2.3.6. ([31, Proposition 4.1.1]) *Let G be a profinite group and let X be a subset of G.*

(a) *If X generates G topologically, then XK/K generates G/K topologically, for each closed normal subgroup K of G.*

(b) *If XN/N generates G/N for each $N \lhd_O G$, then X generates G.*

The next result tells us that a profinite group is finitely generated if and only if the number of generators of its finite continuous images is uniformly bounded. The proof is a standard compactness argument.

Lemma 2.3.7. *Let G be a profinite group and let I be a filter base of open normal subgroups of G. Then* $\mathrm{d}(G) = \sup\{d(G/N) | N \in I\}$.

Proof. Set $d = \sup\{d(G/N)|N \in I\}$. Clearly $\mathrm{d}(G) \geq d$. Thus, if d is infinite, the claim follows. Suppose now that d is finite, we are going to prove that $\mathrm{d}(G) \leq d$. By definition, $\mathrm{d}(G/N) \leq d$ for every $N \in I$. By Corollary 1.4.6 and Theorem 1.4.5, $G = \varprojlim_{N \in I} G/N$ is a compact topological space. Let $\pi_N : G \to G/N$ be the projection maps of the inverse limit, for $N \in I$. For every $N \in I$, put

$$\widetilde{D}_N = \{(g_1, \ldots, g_d) \in G^d \mid \langle \pi_N(g_1), \ldots, \pi_N(g_d) \rangle = G/N\},$$

so $\widetilde{D}_N$ is non-empty and $\widetilde{D}_N$ is closed because it is the pre-image of the finite set $\{(h_1N, \ldots, h_dN) \in (G/N)^d \mid \langle h_1N, \ldots, h_dN \rangle = G/N\}$ via the continuous map π_N. Since I is a directed set, for every $N_1, \ldots, N_r \in I$ there exists $K \in I$ such that $N_l \geq K$ for all $l \in \boldsymbol{r}$. Thus, $\widetilde{D}_{N_1} \cap \ldots \cap \widetilde{D}_{N_r}$ contains $\widetilde{D}_K$ and the collection $\{\widetilde{D}_N\}_{N \in I}$ has the finite intersection property. Set $D = \bigcap_{N \in I} \widetilde{D}_N$, since the group G^d is compact, it follows that D is non-empty. Hence, there exists $(x_1, \ldots, x_d) \in D$ and, by construction, the set $X = \{x_1, \ldots, x_d\}$ generates the profinite group G topologically. This concludes the proof. $\square$

The definition of number of generators for a profinite group is in many ways a good generalisation of the minimal number of generators of an abstract group. We list some nice properties in the next lemma. These properties will be used without any special mention in the next chapters.

Lemma 2.3.8. *Let G be a finitely generated profinite group. Then*

1. *every open subgroup of G is finitely generated. In particular,*

$$\mathrm{d}(H) - 1 \leq (\mathrm{d}(G) - 1)|G : H|$$

   for all $H \leq_o G$;

2. *if K is a normal closed subgroup of G, we have $\mathrm{d}(G/K) \leq \mathrm{d}(G)$;*

3. *for $n \in \mathbb{N}$, there are only finitely many open subgroups of G of index less than n. In particular, the number of open subgroups of index n in G is at most $n(n!)^{\mathrm{d}(G)-1}$.*

Proof. Let $\mathrm{d}(G) = d$ and choose a finite subset $X = \{x_1, \ldots, x_d\}$ of G such that $G = \overline{\langle X \rangle}$. Let I be a filter base of open normal subgroups of G.

1. Let H be an open subgroup of G and set $n = |G : H|$. By [31, Theorem 1.2.5(a)], H is isomorphic to the inverse limit of the finite groups $H/(H \cap N) \cong HN/N$, for $N \in I$. For every $N \in I$, HN/N is a subgroup of G/N of index $|G/N : HN/N| = |G : HN| \leq n$. By the Reidemeister-Schreier index formula for abstract groups, the number of generators of HN/N is at most $(d-1)n + 1$. By Lemma 2.3.7, it follows that $\mathrm{d}(H) \leq (d-1)n + 1$.

2. This follows readily from [31, Proposition 4.1.1].

3. Let H be an open subgroup of G such that $|G : H| = n$. The group G acts by right multiplication on the right cosets of H in G. Any of the $n!$ labellings of the right cosets of H in G defines a continuous homomorphism $\psi : G \to \mathrm{Sym}(n)$ such that $\psi(G)$ is transitive and such that there exists $i \in \boldsymbol{n}$ for which $H = \mathrm{St}_G(i)$. On the other hand, for every continuous homomorphism $\psi : G \to \mathrm{Sym}(n)$ such that $\psi(G)$ is transitive, there are n subgroups of G of index n arising from $\psi^{-1}(\mathrm{St}_G(i))$,

for $i \in \boldsymbol{n}$. Therefore, the number of open subgroups of index n in G is exactly

$$\frac{|\{\psi : G \to \mathrm{Sym}(n) \mid \psi(G) \text{ is transitive}\}| \cdot n}{n!}.$$

Since G is d-generated, the total number of homomorphisms from G to $\mathrm{Sym}(n)$ can be bounded above by $(n!)^d$. Thus, the number of open subgroups of index n in G is at most $n(n!)^d/n! = n(n!)^{\mathrm{d}(G)-1}$.

□

2.3.2 Lower rank

This subsection follows closely the excellent account on lower rank given in [25] by A. Shalev.

Definition 2.3.9. *Let G be a profinite group. The* ***lower rank*** *of G is the minimal positive integer r such that G has a base of open neighbourhoods of the identity made of r-generated open subgroups.*

Remark 2.3.10. The previous definition can also be stated in terms of inf and sup as follows. The lower rank of a profinite group G is the integer $\mathrm{lr}(G)$ defined by:

$$\mathrm{lr}(G) = \inf\{\sup\{\inf\{\mathrm{d}(H_i) \mid i \geq N\} \mid N \in \mathbb{N}\} \mid \{H_i\}_{i\in\mathbb{N}} \in \mathcal{C}\}$$

where $\mathcal{C} = \{\{H_i\}_{i\in\mathbb{N}} | H_i \leq_o G,\ G = H_1 \geq H_2 \geq \dots \text{ and } \bigcap_{i\in\mathbb{N}} H_i = \{e\}\}$ is the collection of all descending chains of open subgroups that form a base for the topology of G. Sometimes this definition of lower rank is stated in terms of the "liminf" of a net: $\mathrm{lr}(G) = \liminf\{\mathrm{d}(H) | H \leq_o G\}$. This is just a short form for the above definition. In this thesis we will use interchangeably the previous equivalent definitions.

The lower rank comes up naturally when studying some probabilistic properties of profinite groups.

Definition 2.3.11. *Let G be a profinite group and k be a positive integer. Let μ be the Haar measure of G^k. Define*

$$Q(G,k) = \mu\left(\{(x_1,\dots,x_k) \in G^k \mid \overline{\langle x_1,\dots,x_k\rangle} \textit{ is open in } G\}\right).$$

The number $Q(G,k)$ is the probability that k "randomly chosen" elements in G generate an open subgroup of G.

What can we say about the profinite groups G that satisfy $Q(G,k) = 1$ for some $k \in \mathbb{N}$? This problem has been considered a few times in the literature. It is proved in [17] that every profinite group with polynomial subgroup growth satisfies $Q(G,k) = 1$ for some integer k. In particular, p-adic analytic pro-p groups satisfy the property. In [1] the authors prove that a compact open subgroup G of a simply connected, semisimple algebraic group over a nonarchimedean local field satisfies $Q(G,2) = 1$; in particular, $Q(\mathrm{SL}_d^1(\mathbb{F}_p[\![t]\!]),2) = 1$ for $p \geq 5$. The number $Q(G,k)$ is relevant to us because of the following lemma.

Lemma 2.3.12. *Let k be an integer. Let G be a profinite group such that $Q(G,k) = 1$. Then the lower rank of G is less than k.*

Proof. Let μ be the Haar measure of G^k. Let $H \leq_o G$. We will prove that $Q(H,k) = 1$. First, let ν be the Haar measure of H^k and notice that $\nu(K) = \mu(K)/\mu(H^k)$ for every $K \leq_o H^k$. In fact, the function $\overline{\nu}(K) = \mu(K)/\mu(H^k)$ satisfies the requirements of the Haar measure of H^k and by Haar's Theorem we must have $\overline{\nu} = \nu$.

Now, the set $Y = \{(y_1,\dots,y_k) \in H^k \mid \overline{\langle y_1,\dots,y_k\rangle} \leq_o H\}$ clearly contains the intersection of $X = \{(x_1,\dots,x_k) \in G^k \mid \overline{\langle x_1,\dots,x_k\rangle} \leq_o G\}$ and H^k. Notice that $\mu(X \cup H^k) \geq \mu(X) = 1$. It follows that

$$\nu(Y) \geq \nu(X \cap H^k) = \frac{\mu(X \cap H^k)}{\mu(H^k)} = \frac{\mu(X) + \mu(H^k) - \mu(X \cup H^k)}{\mu(H^k)} = $$
$$= \frac{1 + \mu(H^k) - 1}{\mu(H^k)} = 1$$

and thus $Q(H,k) = 1$. Therefore, for every $H \leq_o G$ there exist an open k-generated subgroup of G contained in H and this proves the lemma. $\square$

It is not known if the reverse of the previous lemma holds. For example, by [5, Theorem 7], the Nottingham group $\mathcal{N}(\mathbb{F}_p)$ over $\mathbb{F}_p$ has lower rank 2 for $p \geq 5$. Many experts believe that $Q(\mathcal{N}(\mathbb{F}_p), 2) = 1$ for $p \geq 5$, but this is not yet been proven.

Turning to the other end of the spectrum, it is easy to see that free profinite groups have infinite lower rank by the Reidemeister-Schreier index formula. Moreover, new examples of hereditarily just infinite pro-p groups with infinite lower rank have been constructed in the recent work [9].

Calculating explicitly the lower rank of a profinite group is proved to be a challenging problem. In fact, apart from compact p-adic analytic pro-p groups (where the lower rank coincides with the number of generators of the *p-adic Lie algebra*), no profinite groups are known to have finite lower rank strictly greater than 2. In Section 4.2 we will show that some IIEs have finite lower rank and we strongly suspect that these are likely to have lower rank strictly greater than 2.

2.3.3 Hausdorff dimension

In p-adic analytic pro-p groups we have a natural concept of "dimension". In fact, a p-adic analytic group G has the structure of a manifold over $\mathbb{Z}_p$ and we say that the dimension of G is the unique natural number $\dim(G)$ such that G is locally homeomorphic to $\mathbb{Z}_p^{\dim(G)}$. It would be useful to have such a concept of dimension in general profinite groups. The next theorem/definition is an attempt to define such a concept.

The notion of Hausdorff dimension is well-known in Analysis. We will not state here the full definition of Hausdorff dimension, but we will quote a theorem by Barnea and Shalev that allows us to calculate the Hausdorff dimension of a closed subgroup of a countably based profinite group. We remind the reader that for every descending chain $\mathcal{G} = \{G_n\}_{n\in\mathbb{N}}$ of open normal subgroups of a profinite group G we can define an invariant metric by (1.3.1). We will denote by $\dim_{H,\mathcal{G}}$ the Hausdorff dimension function of G with respect to the metric induced by the chain $\mathcal{G}$.

Theorem/Definition 2.3.1. ([2, Theorem 2.4]) *Let G be an infinite countably based profinite group and fix a descending chain $\mathcal{G} = \{G_n\}_{n\in\mathbb{N}}$ of open normal subgroups of G such that $\bigcap_{n\in\mathbb{N}} G_n = \{e\}$ and $G_1 = G$. Let H be a closed subgroup of G. Then the* ***Hausdorff dimension*** *of H (with respect to $\mathcal{G}$) is*

$$\dim_{H,\mathcal{G}}(H) = \liminf_{n\to\infty} \frac{\log|HG_n : G_n|}{\log|G : G_n|}.$$

The previous limit defines a real number in $[0, 1]$.

Remark 2.3.13. ([2, Example 2.5]) Unfortunately the previous definition really depends on the chain $\mathcal{G}$. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ and $H = \{0\} \times \mathbb{Z}_p$. Consider the chains $\mathcal{G}_1 = \{p^{n-1}\mathbb{Z}_p \times p^{n-1}\mathbb{Z}_p\}_{n\in\mathbb{N}}$ and $\mathcal{G}_2 = \{p^{2(n-1)}\mathbb{Z}_p \times p^{n-1}\mathbb{Z}_p\}_{n\in\mathbb{N}}$. Then the Hausdorff dimension of H with respect to $\mathcal{G}_1$ is

$$\begin{aligned}\dim_{H,\mathcal{G}_1}(H) = \liminf_{n\to\infty} & \frac{\log|p^{n-1}\mathbb{Z}_p \times \mathbb{Z}_p : p^{n-1}\mathbb{Z}_p \times p^{n-1}\mathbb{Z}_p|}{\log|\mathbb{Z}_p \times \mathbb{Z}_p : p^{n-1}\mathbb{Z}_p \times p^{n-1}\mathbb{Z}_p|} \\ & = \liminf_{n\to\infty} \frac{n-1}{2(n-1)} = \frac{1}{2},\end{aligned}$$

while the Hausdorff dimension of H with respect to $\mathcal{G}_2$ is

$$\begin{aligned}\dim_{H,\mathcal{G}_1}(H) = \liminf_{n\to\infty} & \frac{\log|p^{2(n-1)}\mathbb{Z}_p \times \mathbb{Z}_p : p^{2(n-1)}\mathbb{Z}_p \times p^{n-1}\mathbb{Z}_p|}{\log|\mathbb{Z}_p \times \mathbb{Z}_p : p^{2(n-1)}\mathbb{Z}_p \times p^{n-1}\mathbb{Z}_p|} \\ & = \liminf_{n\to\infty} \frac{n-1}{3(n-1)} = \frac{1}{3}.\end{aligned}$$

Definition 2.3.14. *Let G be an infinite countably based profinite group. Fix a descending chain $\mathcal{G} = \{G_n\}_{n\in\mathbb{N}}$ of open subgroups such that $\bigcap_{n\in\mathbb{N}} G_n = \{e\}$ and $G_1 = G$. Let $\dim_{H,\mathcal{G}}$ be the Hausdorff dimension of G with respect to $\mathcal{G}$. The* ***Hausdorff dimension spectrum*** *of G is the set* $\mathrm{Spec}_{\mathcal{G}}(G) = \{\dim_{H,\mathcal{G}}(H) \mid H \leq_c G\}$.

Remark 2.3.15. The Hausdorff dimension spectrum of any profinite group G (independently from the chain $\mathcal{G}$) satisfies

$$\{0,1\} \subseteq \mathrm{Spec}_{\mathcal{G}}(G) \subseteq [0,1].$$

In fact, the Hausdorff dimension of any open subgroup of G is 1 and finite subgroups of G have Hausdorff dimension 0.

Of course, to have a "good" generalisation of dimension we would like the Hausdorff dimension to agree with the usual dimension when this is available. It turns out that this is the case for p-adic analytic pro-p groups, if we consider the Hausdorff dimension with respect to a "natural" chain.

Theorem 2.3.16. ([2, Theorem 1.1]) *Let G be a p-adic analytic pro-p group and consider the chain $\mathcal{G} = \{G^{p^{n-1}}\}_{n\in\mathbb{N}}$. Then, for every $H \leq_c G$, we have*

$$\dim_{H,\mathcal{G}}(H) = \frac{\dim(H)}{\dim(G)}.$$

Corollary 2.3.17. ([2, Corollary 1.2]) *Let G be a p-adic analytic pro-p group and consider the chain $\mathcal{G} = \{G^{p^{n-1}}\}_{n\in\mathbb{N}}$. Then $\mathrm{Spec}_{\mathcal{G}}(G) \subseteq \{0, 1/d, \ldots, 1\}$ where $d = \dim G$.*

The converse of the previous corollary is false.

Remark 2.3.18. ([10, Lemma 5.4.1]) Consider the finitely generated non-(p-adic analytic) pro-p group $G = (C_p \wr \mathbb{Z}_p) \times \mathbb{Z}_p$. Put $\mathcal{G} = \{(C_p \wr \mathbb{Z}_p)^{p^n} \times p^{p^{2n}}\mathbb{Z}_p\}_{n\geq 0}$. Then $\mathrm{Spec}_{\mathcal{G}} = \{0,1\}$.

The converse of [2, Corollary 1.2] might still be true if we restrict to the chain $\{G^{p^{n-1}}\}_{n\in\mathbb{N}}$ in pro-p groups. This has not yet been proved.

By [2, Theorem 1.1], the Hausdorff dimension of pro-p groups with a manifold structure over $\mathbb{Z}_p$ is well-behaved. What about pro-p groups with a manifold structure over $\mathbb{F}_p[\![t]\!]$?

Theorem 2.3.19. ([2, Theorem 1.4]) *Let p be a prime number and let $d \geq 2$. Consider the chain $\mathcal{G} = \{\ker(\mathrm{SL}_d(\mathbb{F}_p[\![t]\!]) \to \mathrm{SL}_d(\mathbb{F}_p[\![t]\!]/t^n\mathbb{F}_p[\![t]\!]))\}_{n\in\mathbb{N}}$. Then* (*i*) $\mathrm{Spec}_{\mathcal{G}}(\mathrm{SL}_d(\mathbb{F}_p[\![t]\!]))$ *contains intervals; in fact*

$$\mathrm{Spec}_{\mathcal{G}}(\mathrm{SL}_d(\mathbb{F}_p[\![t]\!])) \supseteq [0, \frac{d(d+1)-2}{2(d^2-1)}].$$

(*ii*) *If* $p > 2$ *then* 1 *is an isolated point in* $\mathrm{Spec}_{\mathcal{G}}(\mathrm{SL}_d(\mathbb{F}_p[\![t]\!]))$*; in fact*

$$\mathrm{Spec}_{\mathcal{G}}(\mathrm{SL}_d(\mathbb{F}_p[\![t]\!])) \cap (1 - \frac{1}{d+1}, 1) = \emptyset.$$

Corollary 2.3.20. ([2, Corollary 1.5]) *Let* $p > 2$ *be a prime number. Consider the chain* $\mathcal{G} = \{\ker(\mathrm{SL}_2(\mathbb{F}_p[\![t]\!]) \to \mathrm{SL}_2(\mathbb{F}_p[\![t]\!]/t^n\mathbb{F}_p[\![t]\!]))\}_{n\in\mathbb{N}}$*. Then*

$$\mathrm{Spec}_{\mathcal{G}}(\mathrm{SL}_2(\mathbb{F}_p[\![t]\!])) = [0, 2/3] \cup \{1\}.$$

We now turn to another question. Do there exist a profinite group G and a descending chain $\mathcal{G} = \{G_n\}_{n\in\mathbb{N}}$ of open subgroups such that $\bigcap_{n\in\mathbb{N}} G_n = \{e\}$ and $G_1 = G$ such that $\mathrm{Spec}_{\mathcal{G}}(G) = [0, 1]$? In Giannelli's thesis [10] and in an unpublished work by Levài the following is shown.

Proposition 2.3.21. ([10, Esempio 5.3.1]) *Let* $G = \varprojlim_{n\in\mathbb{N}} C_p \mathrm{\ wr\ } C_{p^n}$ *and* $\varprojlim_{n\in\mathbb{N}} C_{p^n} = \overline{\langle a\rangle}$*. Consider the chain* $\mathcal{G} = \left\{\overline{\langle a^{p^{n-1}}\rangle^G}\right\}_{n\in\mathbb{N}}$*. Then the Hausdorff dimension spectrum of* G*, with respect to* $\mathcal{G}$*, is the whole interval* $[0, 1]$*.*

The previous result is of interest because it is not even known if the spectrum of the free pro-p group on d generators is $[0, 1]$ with respect to some chain $\mathcal{G}$. In Chapter 6 we will prove that the Hausdorff dimension spectrum of an infinitely iterated exponentiation is $[0, 1]$ with respect to its unique maximal chain of open normal subgroups.

Chapter 3

Generators of iterated exponentiations of perfect groups

The main objective of this chapter is the study of infinitely iterated exponentiations. In particular we will deal with the topological number of generators of special IIEs. Throughout this section d will denote a positive integer. The content of this chapter has been published in [28].

3.1 Introduction

In [4] the following result is proved.

Theorem 3.1.1. ([4, Theorem 1]) *Let $\mathcal{S} = (G_n \leq \mathrm{Sym}(X_n))_{n\in\mathbb{N}}$ be a sequence of finite transitive permutation groups with uniformly bounded number of generators. Then the infinitely iterated permutational wreath product of type $\mathcal{S}$ is finitely generated if and only if the profinite abelian group $\prod_{n\geq 1} G_n/G_n'$ is finitely generated.*

The group $\prod_{n\geq 1} G_n/G_n'$ is a continuous image of the infinitely iterated permutational wreath product above (since it is its abelianization), therefore the “only if” direction of the theorem is trivial.

In this chapter we will prove two parallel results to [4, Theorem 1]. We will show that an infinitely iterated exponentiation and an *infinitely iterated "mixed" wreath product of stride at most m* of a sequence of finite d-generated perfect transitive permutation groups are topologically finitely generated under certain conditions. For the formal definition of "mixed" wreath product and "stride" see Definition 3.1.3 in this section.

In Section 3.2 we prove our first main result of the chapter.

Theorem 3.1.2. *Let d be an integer. Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a sequence of finite transitive permutation groups such that each S_k is perfect and at most d-generated as an abstract group. Suppose that for every $k \in \mathbb{N}$ there exist elements $i, j \in \boldsymbol{m}_k$ such that $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then the infinitely iterated exponentiation of the groups in $\mathcal{S}$ is topologically finitely generated.*

The proof of Theorem 3.1.2 gives an explicit set of $d + \mathrm{d}(S_1)$ generators for $\varprojlim \widetilde{S}_k$ and this bound is asymptotically best possible (see Lemma 3.2.2). The groups under study here are very different from the ones in [4]. We cannot rely on the tree-like structure of iterated wreath products and the iterated exponentiation of permutation groups is in general not associative. The non-associativity of the exponentiation of permutation groups is the main reason why we need to ask that the groups in the sequence $\mathcal{S}$ have non-regular actions.

Using the same methods of the proof of Theorem 3.1.2 we can improve our bound for a sequence $\mathcal{S}$ of perfect 2-generated perfect groups (see Corollary 3.2.8).

In Section 3.3 we use [13, Theorem 3.1] to exchange the exponentiation and the permutational wreath product to obtain the finite generation of iterated wreath products with "mixed" action.

Definition 3.1.3. *Let $(k_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers. Define the sequence $(G_n)_{n\in\mathbb{N}}$ of perfect transitive subgroups of $\mathrm{Sym}(r_n)$ starting* *from the groups in $\mathcal{S}$ in the following way: $G_0 = \{e\}$ and for $k \geq 1$*

$$G_k = \begin{cases} S_k \circledwr G_{k-1} & \text{if } k \in \{k_1, k_2, \ldots\}, \\ S_k \wr G_{k-1} & \text{otherwise.} \end{cases}$$

The permutation groups G_n are called ***iterated mixed wreath products of type*** *$(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$.*

Let m be an integer. If the sequence $(k_n)_{n\in\mathbb{N}}$ is such that $k_{n+1} - k_n \leq m$ for every $n \in \mathbb{N}$, we say that the iterated mixed wreath product G_n of type $(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$ has ***stride at most*** *m.*

The groups G_n, together with the projections $G_n \to G_{n-1}$, form an inverse system of finite groups. We say that the profinite group $\varprojlim G_n$ is an ***infinitely iterated mixed wreath product of type*** *$(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$. If the groups G_n have stride at most m we say that $\varprojlim G_n$ has stride at most m.*

We remark that an infinitely iterated exponentiation is an infinitely iterated mixed wreath product of stride at most one.

The second main result of this section is the following. It will be proved in Section 3.3.

Theorem 3.1.4. *Let d be an integer. Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a sequence of finite transitive permutation groups such that each S_k is perfect and at most d-generated as an abstract group. Suppose that for every $k \in \mathbb{N}$ there exist elements $i, j \in \boldsymbol{m}_k$ such that $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Let $G = \varprojlim G_n$ be an infinitely iterated mixed wreath product of type $(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$ of stride at most m. Then G is topologically finitely generated.*

The hypotheses of Theorem 3.1.4 can be weakened in two ways (see Remark 3.3.5). We conclude with Section 3.4 where we use the techniques of this chapter to find the minimal number of generators of infinitely iterated exponentiations and infinitely iterated mixed wreath products of particular sequences $\mathcal{S}$ (see Corollary 3.4.2).

3.2 Finite generation of IIEs of perfect groups

First we find a lower bound for the minimal number of generators of a wreath product of perfect non-simple groups. This shows that the bound given by Theorem 3.1.2 can be improved only by a multiplicative and an additive constant. We will denote by ${}^{\mathrm{t}}x$ the transpose of a vector $x \in \mathbb{Z}^n$.

In the next lemma we will use the following.

Theorem 3.2.1. ([29, Lemma 2]) *Let B be a non-trivial finite perfect group and s the order of a smallest simple image of B. Then* $\mathrm{d}(B^{s^n}) \leq \mathrm{d}(B) + n$ *for all integers $n \geq 0$; and hence* $\mathrm{d}(B^n) < \mathrm{d}(B) + 1 + \log_s n$ *for all $n \geq 1$.*

Lemma 3.2.2. *Let N be a natural number. Let A be a finite simple group and let $B \leq \mathrm{Sym}(n)$ be a finite permutation group. Then*

$$\mathrm{d}(A^N \wr B) \geq \max\left\{\frac{1}{n}\left(\mathrm{d}\left(A^N\right) - \mathrm{d}(A) - 1\right), \mathrm{d}(B)\right\}.$$

Proof. Set $G = A^N \wr B = (A^N)^n \rtimes B$ and $\mathrm{d}(G) = d$. It is clear that $d \geq \mathrm{d}(B)$, since B is a quotient of G. Let

$$\gamma_j = ((x_{11}^{(j)}, \ldots, x_{1N}^{(j)}), \cdots, (x_{n1}^{(j)}, \ldots, x_{nN}^{(j)}))\sigma_j = (\gamma_1^{(j)}, \cdots, \gamma_n^{(j)})\sigma_j \in A^N \wr B,$$

for $j = 1, \ldots, d$, be generators for G. Form the $N \times nd$ matrix M with entries $M_{l,n(j-1)+i} = x_{il}^{(j)}$ for $i = 1, \ldots, n$, $j = 1, \ldots, d$ and $l = 1, \ldots, N$. For every number $m \in \{1, \ldots, nd\}$ there exist unique $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$ such that $m = n(j-1) + i$ and the $(n(j-1)+i)$-th column of M is the vector ${}^{\mathrm{t}}\gamma_i^{(j)}$:

$$M = \left({}^{\mathrm{t}}\gamma_1^{(1)}, \ldots, {}^{\mathrm{t}}\gamma_n^{(1)}, \ldots\ldots, {}^{\mathrm{t}}\gamma_1^{(d)}, \ldots, {}^{\mathrm{t}}\gamma_n^{(d)}\right).$$

Our goal is to show that $N \leq |A|^{nd}$. Suppose by contradiction that $N > |A|^{nd}$. Then, since the $x_{il}^{(j)}$ are elements of A, we would have that two rows of M are equal. Without loss of generality we can suppose that the first and the second rows are equal. In particular it follows that $x_{i1}^{(j)} = x_{i2}^{(j)}$ for every $i = 1, \ldots, n$

and for every $j = 1, \ldots, d$. Since the action of B permutes as blocks for its action the n N-tuples of $(A^N)^n$, any element $((y_{11}, \ldots, y_{1N}), \cdots, (y_{n1}, \ldots, y_{nN}))\tau$ of the subgroup generated by the γ_j's satisfies $y_{11} = y_{12}$. This is a contradiction with our assumption that the γ_j's generate G.

Therefore $N \leq |A|^{nd}$ and applying logarithms on both sides of the inequality we have

$$d \geq \frac{1}{n \log |A|} \log N = \frac{1}{n} \log_{|A|} N > \frac{1}{n} \left(\mathrm{d}\left(A^N\right) - \mathrm{d}(A) - 1 \right),$$

where the last inequality holds by Theorem 3.2.1. □

Before proving Theorem 3.1.2 we fix some notation.

Notation 1. An $\widetilde{m}_k$-tuple of elements in $\{1, \ldots, m_{k+1}\}$ will be denoted as $(i_1, \ldots, i_{\widetilde{m}_k})_{\widetilde{m}_k}$. This notation will be convenient in particular when we will have to deal with $\widetilde{m}_k$-tuples where all the coordinates are equal, for example $(1, \ldots, 1)_{\widetilde{m}_k}$. We will denote an element of the group $S_{k+1}^{\widetilde{m}_k}$ as $(\sigma_1, \ldots, \sigma_{\widetilde{m}_k})_{\widetilde{m}_k}$.

Definition 3.2.3. *For $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \boldsymbol{m}^n$ we say that $(i_1, \ldots, i_n)$ **precedes** $(j_1, \ldots, j_n)$, if and only if there exists $1 \leq l \leq n$ such that $i_k = j_k$ for $1 \leq k \leq l - 1$ and $i_l < j_l$. The relation "precedes" defines a total order on $\{1, \ldots, m\}^n$ that is called the **lexicographic order**.*

The following straightforward lemma is one of the key tricks to prove Theorem 3.1.2.

Lemma 3.2.4. *Let $G \leq \mathrm{Sym}(m)$ and $H \leq \mathrm{Sym}(n)$ be permutation groups. Then the subgroup H of the exponentiation $G \wr H$ acts trivially on the subset*

$$\{(i, \ldots, i) \mid i \in \boldsymbol{m}\}.$$

Theorem 3.1.2 will now follow from an application of the next lemma.

Lemma 3.2.5. *Let $\mathcal{S} = \{S_k\}_{k \in \boldsymbol{n}}$ be a sequence of transitive subgroups of $\mathrm{Sym}(m_k)$ and let d be an integer. Suppose that S_k is perfect and at most d-generated for $k = 2, \ldots, n$. Suppose that for every $k \in \boldsymbol{n}$ there exist elements*

$i, j \in \boldsymbol{m}_k$ such that $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then the iterated exponentiation $\widetilde{S}_n$ of the sequence $\mathcal{S}$ satisfies $\mathrm{d}(\widetilde{S}_n) \leq d + \mathrm{d}(S_1)$.

Proof. Let $S_1 = \langle \alpha_1(1), \dots, \alpha_{\mathrm{d}(S_1)}(1) \rangle$, $S_k = \langle \alpha_1(k), \dots, \alpha_d(k) \rangle$, $k = 2, \dots, n$ and order the elements of $\{1, \dots, m_{k+1}\}^{\widetilde{m}_k}$ with respect to the lexicographic order. Without loss of generality we can suppose that for every $k \in \boldsymbol{n}$ we have

$$\mathrm{St}_{S_k}(1) \neq \mathrm{St}_{S_k}(2). \tag{3.2.1}$$

We will now define d elements of $\widetilde{S}_n$ that together with the generators of S_1 will generate $\widetilde{S}_n$. Define the elements $\beta_1, \dots, \beta_d \in \widetilde{S}_n$ as

$$\beta_j = (\alpha_j(n), e, \dots, e)_{\widetilde{m}_{n-1}} \cdot (\alpha_j(n-1), e, \dots, e)_{\widetilde{m}_{n-2}} \cdots (\alpha_j(2), e, \dots, e)_{\widetilde{m}_1}$$

for $j = 1, \dots, d$. Note that the $\alpha_j(k)$'s are in the first place of the $\widetilde{m}_{k-1}$-tuples, which corresponds to the element $(1, \dots, 1)_{\widetilde{m}_{k-1}} \in \{1, \dots, m_k\}^{\widetilde{m}_{k-1}}$.

Let $A = \langle \alpha_1(1), \dots, \alpha_{\mathrm{d}(S_1)}(1), \beta_1, \dots, \beta_d \rangle \leq \widetilde{S}_n$. We claim that $A = \widetilde{S}_n$. We will prove by induction on k that $\widetilde{S}_k \leq A$ for $k = 1, \dots, n$. Trivially $\widetilde{S}_1 = S_1 \leq A$. Supposing by the inductive hypothesis that $\widetilde{S}_k \leq A$, we have to show that we can write any element of $\widetilde{S}_{k+1}$ as a product of the generators in A. Because $\widetilde{S}_{k+1} = S_{k+1}^{\widetilde{m}_k} \cdot \widetilde{S}_k$, it will suffice to show that $S_{k+1}^{\widetilde{m}_k} \leq A$.

By the transitivity of S_k there is an element $t \in S_k$ such that $1^t = 2$ and by the inductive hypothesis the element $\sigma = (e, \dots, e, t)_{\widetilde{m}_{k-1}} \in S_k^{\widetilde{m}_{k-1}}$ belongs to A. By Lemma 3.2.4 it follows that for $j = k, \dots, n$

$$(1, \dots, 1)^{\sigma}_{\widetilde{m}_j} = (1, \dots, 1)_{\widetilde{m}_j} \tag{3.2.2}$$

and from the definition of lexicographic order and exponentiation

$$(1, \dots, 1)^{\sigma}_{\widetilde{m}_{k-1}} = (1^e, \dots, 1^e, 1^t)_{\widetilde{m}_{k-1}} = (1, \dots, 1, 2)_{\widetilde{m}_{k-1}}. \tag{3.2.3}$$

We remind the reader that the element $(1, \dots, 1, 2)_{\widetilde{m}_{k-1}}$ is the second element in the set $\{1, \dots, m_k\}^{\widetilde{m}_{k-1}}$ with respect to the lexicographic order. Moreover, since $\widetilde{S}_k \leq A$, $\beta'_j = (\alpha_j(n), e, \dots, e)_{\widetilde{m}_{n-1}} \cdots (\alpha_j(k+1), e, \dots, e)_{\widetilde{m}_k}$ belongs to A. Set $\gamma_j = [\sigma, \beta'_j]$, then $\gamma_j \in A$. By (3.2.2), $(\alpha_j(l), e, \dots, e)^{\sigma}_{\widetilde{m}_{l-1}} =$

$(\alpha_j(l), e, \ldots, e)_{\widetilde{m}_{l-1}}$ for $l = k+2, \ldots, n$ and, by (3.2.3), $(\alpha_j(k+1), e, \ldots, e)^{\sigma}_{\widetilde{m}_k} = (e, \alpha_j(k+1), e, \ldots, e)_{\widetilde{m}_k}$. Therefore $(\beta'_j)^{\sigma} = (\alpha_j(n), e, \ldots, e)_{\widetilde{m}_{n-1}} \cdots (e, \alpha_j(k+1), e, \ldots, e)_{\widetilde{m}_k}$ with $\alpha_j(k+1)$ in second position in the last $\widetilde{m}_k$-tuple and so $\gamma_j = ((\beta'_j)^{\sigma})^{-1}\beta'_j = (\alpha_j(k+1), \alpha_j(k+1)^{-1}, e, \ldots, e)_{\widetilde{m}_k}$.

By inductive hypothesis $\widetilde{S}_k \leq A$, therefore A is transitive on $\boldsymbol{m}_k^{\widetilde{m}_{k-1}}$. To conclude the proof it is sufficient to show that we can write any element of the form $(\lambda, e, \ldots, e)_{\widetilde{m}_k}$ in $S_{k+1}^{\widetilde{m}_k}$ as a word in the γ_j's. We can then move λ around, using the transitive action of $\widetilde{S}_k$.

As S_{k+1} is perfect it is sufficient to prove that we can write any commutator $([\lambda_1, \lambda_2], e, \ldots, e)_{\widetilde{m}_k}$ as a word in the γ_j's. By (3.2.1) there are $s \in S_k$ and $r \in \boldsymbol{m}_k$, $r \neq 2$, such that $1^s = 1$ and $2^s = r$. By the inductive hypothesis $\mu = (e, \ldots, e, s)_{\widetilde{m}_{k-1}}$ belongs to A. Let $\lambda_1, \lambda_2 \in S_{k+1}$. Since the $\alpha_j(k+1)$'s generate S_{k+1}, there exist two d-variables words w_1 and w_2 such that $\lambda_1 = w_1(\alpha_1(k+1), \ldots, \alpha_d(k+1))$ and $\lambda_2 = w_2(\alpha_1(k+1), \ldots, \alpha_d(k+1))$. Thus, if we set $\delta_i = w_i(\alpha_1(k+1)^{-1}, \ldots, \alpha_d(k+1)^{-1})$ for $i = 1, 2$, the elements $w_1(\gamma_1, \ldots, \gamma_d) = (\lambda_1, \delta_1, e, \ldots, e)_{\widetilde{m}_k}$ and $w_2(\gamma_1, \ldots, \gamma_d) = (\lambda_2, \delta_2, e, \ldots, e)_{\widetilde{m}_k}$ belong to A. The definition of μ and an easy calculation now yield

$$\left[(\lambda_1, \delta_1, e, \ldots, e)_{\widetilde{m}_k}, (\lambda_2, \delta_2, e, \ldots, e)^{\mu}_{\widetilde{m}_k}\right] = ([\lambda_1, \lambda_2], e, \ldots, e)_{\widetilde{m}_k}.$$

Thus for every $\lambda \in S_{k+1}$ the $\widetilde{m}_k$-tuple $(\lambda, e, \ldots, e)_{\widetilde{m}_k}$ is in A. It follows that $S_{k+1}^{\widetilde{m}_k} \leq A$ and $\widetilde{S}_{k+1} = S_{k+1}^{\widetilde{m}_k} \cdot \widetilde{S}_k \leq A$. The result follows by induction. $\square$

We would like to point out that in the previous proof we exhibited an explicit set of $d + \mathrm{d}(S_1)$ generators for $\widetilde{S}_n$. We are now ready for the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. For every $n \in \mathbb{N}$, Lemma 3.2.5 gives us $d + \mathrm{d}(S_1)$ generators of $\widetilde{S}_n$, of the form described at the beginning of the proof of Lemma 3.2.5, $\alpha_1(1), \ldots, \alpha_{\mathrm{d}(S_1)}(1), \beta_1^{(n)}, \ldots, \beta_d^{(n)}$. For $n \in \mathbb{N}$, let π_n be the inverse limit projection from $\varprojlim \widetilde{S}_k$ to $\widetilde{S}_n$. Let $a_1(1), \ldots, a_{\mathrm{d}(S_1)}(1), b_1, \ldots, b_d$

be the unique elements of $\varprojlim \widetilde{S}_k$ such that $\pi_n(a_i(1)) = \alpha_i(1)$ and $\pi_n(b_j) = \beta_j^{(n)}$ for all $i \in \mathbf{d(S_1)}$, $j \in \boldsymbol{d}$ and $n \in \mathbb{N}$. So $a_1(1), \ldots, a_{\mathrm{d}(S_1)}(1), b_1, \ldots, b_d$ generate $\varprojlim \widetilde{S}_k$ by [31, Proposition 4.1.1]. $\square$

We will use the following result on perfect permutation groups.

Lemma 3.2.6. ([24, Lemma 2]) *Let $S \leq \mathrm{Sym}(n)$ be a perfect permutation group. Suppose that for all $i, j \in \boldsymbol{n}$ we have $\mathrm{St}_S(i) \neq \mathrm{St}_S(j)$. Then there is $\mu \in S$ and $r \in \boldsymbol{n}$ such that*

$$n^\mu = n \quad \textit{and} \quad r^{\mu^2} \neq r.$$

Using [24, Lemma 2] it is possible to improve the previous bound for 2-generated groups with the same method.

Lemma 3.2.7. *Let $\mathcal{S} = \{S_k\}_{k \in \boldsymbol{n}}$ be a sequence of perfect 2-generated transitive subgroups of $\mathrm{Sym}(m_k)$ such that for every $k \in \boldsymbol{n}$ and all $i, j \in \boldsymbol{m}_k$ $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then $\widetilde{S}_n$ is generated by the generators of S_1 together with another suitable element.*

Proof. Let $S_k = \langle \alpha_1(k), \alpha_2(k) \rangle$. By [24, Lemma 2], for $k \in \boldsymbol{n}$, there exist $\sigma_k \in S_k$ and $1 \leq r_k \leq m_k$ such that $r_k^{\sigma_k^2} \neq r_k$. Let

$$\beta = (\ldots, \alpha_1(n), \ldots, \alpha_2(n), \ldots)_{\widetilde{m}_{n-1}} \cdot \ldots \cdot (\ldots, \alpha_1(2), \ldots, \alpha_2(2), \ldots)_{\widetilde{m}_1}$$

where the element $\alpha_1(2)$ is in position $r_1^{\sigma_1}$, $\alpha_2(2)$ is in position r_1, $\alpha_1(k+1)$ is in position $(r_k^{\sigma_k}, \ldots, r_k^{\sigma_k})_{\widetilde{m}_{k-1}}$, $\alpha_2(k+1)$ is in position $(r_k, \ldots, r_k)_{\widetilde{m}_{k-1}}$ for $k = 2, \ldots, n-1$ and the identity in all the unspecified positions. Set $A = \langle \alpha_1(1), \alpha_2(1), \beta \rangle$ and proceed exactly as in the proof of Lemma 3.2.5, with β instead of β_i and $(\sigma_k, \ldots, \sigma_k)_{\widetilde{m}_{k-1}}$ instead of σ, to show that $A = \widetilde{S}_n$. $\square$

Using Lemma 3.2.7 in place of Lemma 3.2.5 in the proof of Theorem 3.1.2 yields the following corollary.

Corollary 3.2.8. *Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of perfect 2-generated transitive subgroups of* $\mathrm{Sym}(m_k)$*. Suppose that for every $k \in \mathbb{N}$ and all $i, j \in \boldsymbol{m}_k$ we have $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. Then the infinitely iterated exponentiation $\varprojlim \widetilde{S}_n$ of $\mathcal{S}$ satisfies* $\mathrm{d}\left(\varprojlim \widetilde{S}_n\right) \leq 3$.

Again we would like to point out that in Corollary 3.2.8 we can find an explicit set of three generators for $\varprojlim \widetilde{S}_n$.

As a consequence of [21], the minimal number of generators of the infinitely iterated exponentiation of a sequence $\mathcal{S}$ of finite non-abelian simple transitive permutation groups is two. However, perfect groups can be "far" from simple and we conjecture that in the case of perfect non-simple groups Lemma 3.2.7 is best possible but we do not have an explicit example to confirm this.

3.3 Finite generation of mixed wreath products of perfect groups

We now proceed to the proof of Theorem 3.1.4. We will use the following.

Theorem 3.3.1. ([13, Theorem 3.1]) *Let n_1, n_2 and n_3 be integers and let $A \leq \mathrm{Sym}(n_1), B \leq \mathrm{Sym}(n_2)$ and $C \leq \mathrm{Sym}(n_3)$ be permutation groups. Then $A \circledwr (B \wr C)$ and $(A \circledwr B) \circledwr C$ are isomorphic as permutation groups.*

The next lemma is an application of [13, Theorem 3.1] and it will be used in the proof of Theorem 3.1.4. Remember that we denote by $\widetilde{H}_n$ the iterated exponentiation of the sequence of permutation groups $\{H_k\}_{k\in\boldsymbol{n}}$.

Lemma 3.3.2. *Let $\mathcal{S} = \{S_k\}_{k\in\mathbb{N}}$ be a sequence of subgroups of* $\mathrm{Sym}(m_k)$*. Let $\{k_n\}_{n\in\mathbb{N}}$ be an increasing sequence of integers and let G_n be an iterated mixed wreath product of type $(\mathcal{S}, \{k_n\}_{n\in\mathbb{N}})$. Set $k_0 = 0$ and define the permutation groups $\widehat{S}^{(i)}_{k_n}$ for $n \in \mathbb{N}$ and $i \in \boldsymbol{k}_n \smallsetminus \boldsymbol{k}_{n-1}$ as follows: $\widehat{S}^{(k_n)}_{k_n} = S_{k_n}$ and $\widehat{S}^{(i)}_{k_n} = \widehat{S}^{(i+1)}_{k_n} \circledwr S_i$. Define $H_n = \widehat{S}^{(k_{n-1}+1)}_{k_n}$ for $n \in \mathbb{N}$. Then G_{k_n} is isomorphic to $\widetilde{H}_n$ as a permutation group, for every $n \in \mathbb{N}$.*

Proof. The proof is by induction on n. If $n = 1$ and $k_1 = 1$ the claim is trivial. If $k_1 > 1$ repeated applications of [13, Theorem 3.1] yield $G_{k_1} \cong H_1$.

Suppose that $G_{k_{n-1}} \cong \widetilde{H}_{n-1}$. By construction $G_{k_n} \cong S_{k_n} \textcircled{\wr} G_{k_n - 1}$ and $G_i \cong S_i \wr G_{i-1}$ for $i \in \boldsymbol{k}_n \smallsetminus \boldsymbol{k}_{n-1}$. Therefore repeated applications of [13, Theorem 3.1] yield $G_{k_n} \cong (\widehat{S}_{k_n}^{(i+1)} \textcircled{\wr} S_i) \textcircled{\wr} G_{i-1}$ for $i \in \boldsymbol{k}_n \smallsetminus \boldsymbol{k}_{n-1}$. Thus $G_{k_n} \cong \widehat{S}_{k_n}^{(k_{n-1}+1)} \textcircled{\wr} G_{k_{n-1}}$ and, by the inductive hypothesis, we conclude $G_{k_n} \cong H_n \textcircled{\wr} \widetilde{H}_{n-1} \cong \widetilde{H}_n$. The claim follows by induction. □

Lemma 3.3.3. *Let $A \leq \mathrm{Sym}(m)$ and $B \leq \mathrm{Sym}(n)$ be permutation groups and set $G = A \textcircled{\wr} B$. Suppose that $m, n \geq 2$ and B is transitive. Then there exist $x, y \in \boldsymbol{m}^n$ such that $\mathrm{St}_G(x) \neq \mathrm{St}_G(y)$.*

Proof. Consider the elements $x = (1, \dots, 1)_n$ and $y = (2, 1, \dots, 1)_n$ in $\boldsymbol{m}^n$. Because B is transitive there exists $b \in B$ such that $1^b = 2$, so $x^b = x$ and $y^b = (1, 2, 1, \dots, 1)_n \neq y$. So b is in the stabiliser of x but not in the stabiliser of y. □

The following lemma follows directly from the definition of the exponentiation of permutation groups.

Lemma 3.3.4. *Let $A \leq \mathrm{Sym}(m)$ and $B \leq \mathrm{Sym}(n)$ be permutation groups and suppose that A is transitive. Then $A \textcircled{\wr} B$ is transitive.*

Finally we use Lemma 3.2.5, Lemma 3.3.2, Lemma 3.3.3 and Lemma 3.3.4 to prove Theorem 3.1.4.

Proof of Theorem 3.1.4. Let $G = \varprojlim G_n$ be an infinitely iterated mixed wreath product of type $(\mathcal{S}, \{k_n\}_{n \in \mathbb{N}})$ and of stride at most m. We will use the same setup and notation as in Lemma 3.3.2. We have that G_{k_n} is isomorphic to $\widetilde{H}_n$ for every $n \in \mathbb{N}$, hence it is sufficient to show that the sequence $\{H_n\}_{n \in \mathbb{N}}$ satisfies the hypotheses of Lemma 3.2.5. It is clear that every H_n is perfect and it can be generated by md elements because it is an iterated wreath product of length at most m made of d-generated groups. Since each S_k is transitive, the permutation group $H_n = \widehat{S}_{k_n}^{(k_{n-1}+2)} \wr S_{k_{n-1}+1}$ is transitive by iterated applications of Lemma 3.3.4. Moreover, by Lemma 3.3.3, H_n satisfies the hypothesis on the stabilisers in Lemma 3.2.5. The proof is completed by applying Lemma 3.2.5 and [31, Proposition 4.1.1]. $\square$

If the "inverse" iterated exponentiations $\widehat{S}_n$ in Lemma 3.3.2 had a uniform bound on the number of generators, it would be possible to prove that infinitely iterated mixed wreath products of arbitrarily large stride are topologically finitely generated. We do not know if this is the case.

Remark 3.3.5. We can weaken the hypothesis of Theorem 3.1.4 in the following ways. Let $\{k_n\}_{n\in\mathbb{N}}$ be an increasing sequence of integers and $\mathcal{S}$ a sequence of finite perfect, at most d-generated, transitive permutation groups such that:

- for every k_n satisfying $k_n = k_{n-1} + 1$ there exist elements $i, j \in \boldsymbol{m}_{k_n}$ that have different stabilisers for the action of S_{k_n}.
- $k_{n+1} - k_n \geq 2$ for every $n \in \mathbb{N}$.

The proof of Theorem 3.1.4 with these hypotheses remains the same.

3.4 An application

In this section we find explicitly two generators for the infinitely iterated exponentiation of particular sequences $\mathcal{S}$. We start with a lemma.

Lemma 3.4.1. *Let $\mathcal{S} = \{S_k\}_{k\in\boldsymbol{n}}$, be a sequence of 2-generated perfect transitive subgroups of $\mathrm{Sym}(m_k)$. Suppose that for every $k \in \boldsymbol{n}$ there exist two generators a_k, b_k of S_k such that:*

- $\mathrm{fix}(a_k)$ *and* $\mathrm{fix}(b_k)$ *are non-empty,*
- $(|a_1|, |b_j|) = 1$ *and* $(|b_1|, |a_j|) = 1$ *for* $j = 2, \ldots, n$.

Then $\mathrm{d}(\widetilde{S}_n) = 2$.

Proof. Let $u_k \in \mathrm{fix}(a_k)$, $v_k \in \mathrm{fix}(b_k)$. In the spirit of Lemma 3.2.5 define the following elements of $\boldsymbol{m}_i^{\widetilde{m}_{i-1}}$

$$\underline{u}_i = (u_i, \dots, u_i)_{\widetilde{m}_{i-1}} \quad \text{and} \quad \underline{v}_i = (v_i, \dots, v_i)_{\widetilde{m}_{i-1}}$$

for $i = 2, \dots, n-1$. By the transitivity of S_k there is $\sigma \in S_k$ such that $u_k^\sigma = v_k$ and, by Lemma 3.2.4, $\mu = (\sigma, \dots, \sigma)_{\widetilde{m}_k}$ is such that

$$\underline{u}_j^{\mu} = \underline{u}_j \quad \text{and} \quad \underline{v}_j^{\mu^{-1}} = \underline{v}_j \tag{3.4.1}$$

for every $j \geq k+1$ and by definition of exponentiation we have

$$\underline{u}_k^{\mu} = (u_k^\sigma, \dots, u_k^\sigma)_{\widetilde{m}_k} = \underline{v}_k. \tag{3.4.2}$$

For the rest of the proof we will write the position of an element in a tuple below the element itself. We claim that the elements

$$\beta_1 = (e, \dots, e, \underset{\underline{u}_{n-1}}{a_n}, e \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \dots, e, \underset{\underline{u}_2}{a_3}, e \dots, e)_{\widetilde{m}_2} \cdot \\ \cdot (e, \dots, e, \underset{v_1}{a_2}, e \dots, e)_{\widetilde{m}_1} b_1$$

and

$$\beta_2 = (e, \dots, e, \underset{\underline{v}_{n-1}}{b_n}, e \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \dots, e, \underset{\underline{v}_2}{b_3}, e \dots, e)_{\widetilde{m}_2} \cdot \\ \cdot (e, \dots, e, \underset{u_1}{b_2}, e \dots, e)_{\widetilde{m}_1} a_1$$

generate the group $\widetilde{S}_n$. Let $A = \langle \beta_1, \beta_2 \rangle$, we will prove by induction that $\widetilde{S}_k \leq A$ for $k = 1, \dots, n$. It follows from Lemma 3.2.4 and the definition of u_i and v_i that $(e, \dots, e, a_i, e \dots, e)_{\widetilde{m}_i}$ commutes with $(e, \dots, e, a_j, e \dots, e)_{\widetilde{m}_j}$ for $i \neq j$. Set $p = \prod_{i=2}^n |a_i|$ and $q = \prod_{i=2}^n |b_i|$, then $\beta_1^p = b_1^p$ and $\beta_2^q = a_1^q$, so $S_1 \leq A$.

By the inductive hypothesis the group $\widetilde{S}_k$ is contained in A. Our goal is to write any element of $S_{k+1}^{\widetilde{m}_k}$ as a word in β_1, β_2. Clearly the elements

$$\beta_1' = (e, \dots, e, \underset{\underline{u}_{n-1}}{a_n}, e \dots, e)_{\widetilde{m}_{n-1}} \cdots (e, \dots, e, \underset{\underline{u}_k}{a_{k+1}}, e \dots, e)_{\widetilde{m}_k}$$

and

$$\beta_2' = (e, \ldots, e, \underset{v_{n-1}}{b_n}, e \ldots, e)_{\widetilde{m}_{n-1}} \cdots (e, \ldots, e, \underset{v_k}{b_{k+1}}, e \ldots, e)_{\widetilde{m}_k}$$

belong to A.

Let us now consider the commutators $\gamma_i = [\mu_i, \beta_i']$ for $i = 1, 2$. Following exactly the steps of Lemma 3.2.5 we can use (3.4.1), (3.4.2) (instead of (3.2.2) and (3.2.3)) and Lemma 3.2.4 to show $S_{k+1}^{\widetilde{m}_k} \leq A$. Therefore $\widetilde{S}_{k+1} = S_{k+1}^{\widetilde{m}_k} \cdot \widetilde{S}_k \leq A$. The result follows by induction. □

Since none of the groups $\widetilde{S}_k$ is cyclic, the proof that $\varprojlim \widetilde{S}_k$ is topologically 2-generated is now the same as the proof of Theorem 3.1.2 using Lemma 3.4.1 instead of Lemma 3.2.5. We have proved the following.

Corollary 3.4.2. *Let $\mathcal{S} = \{S_k\}_{k \in \mathbb{N}}$, be a sequence of 2-generated perfect transitive subgroups of $\mathrm{Sym}(m_k)$. Suppose that for every $k \in \mathbb{N}$ there exist two generators a_k, b_k of S_k such that:*

- *$\mathrm{fix}(a_k)$ and $\mathrm{fix}(b_k)$ are non-empty,*
- *$(|a_1|, |b_j|) = 1$ and $(|b_1|, |a_j|) = 1$ for $j \geq 2$.*

Then the infinitely iterated exponentiation $\varprojlim \widetilde{S}_k$ is topologically 2-generated and we produce explicitly two generators for the group.

Reproducing the steps of the proof of Corollary 3.4.2 we obtain the following result.

Corollary 3.4.3. *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k \in \mathbb{N}}$, be a sequence of finite 2-generated perfect transitive permutation groups. Suppose that for every $k \in \mathbb{N}$ there exist two generators a_k, b_k of S_k such that:*

- *$\mathrm{fix}(a_k)$ and $\mathrm{fix}(b_k)$ are non-empty,*
- *$(|a_1|, |b_j|) = 1$ and $(|b_1|, |a_j|) = 1$ for $j \geq 2$,*
- *$(|a_{2j+1}|, |a_{2j}|) = 1$ and $(|b_{2j+1}|, |b_{2j}|) = 1$ for $j \in \boldsymbol{n}$.*

Then the infinitely iterated mixed wreath product of type $(\mathcal{S}, (2n+1)_{n\in\mathbb{N}})$, $\varprojlim \widetilde{S}_k$, *is topologically* 2*-generated and we produce explicitly two generators for the group.*

We conclude with two examples of sequences $\mathcal{S}$ that satisfy the hypotheses of Corollary 3.4.2 and Corollary 3.4.3. In [18], the following fundamental result is proved.

Theorem 3.4.4. (Main Theorem of [18]) *Let* n *be a natural number. Suppose that* l_1 *and* l_2 *are odd natural numbers such that* $l_1, l_2 < n$ *and* $l_1 + l_2 > n$. *Then there exist two cycles* a *and* b *in* $\mathrm{Alt}(n)$ *of order* l_1 *and* l_2 *respectively such that* $\langle a, b\rangle = \mathrm{Alt}(n)$.

Remark 3.4.5. Using the main result of [18] we can produce sequences of perfect groups that satisfy the hypotheses of Corollary 3.4.2 and Corollary 3.4.3. Let $(m_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of integers satisfying

there exist two primes p_k, q_k *with* $m_{k-1} \leq p_k, q_k < m_k$ *such that* $p_k + q_k > m_k$

and consider the alternating group $\mathrm{Alt}(m_k) \leq \mathrm{Sym}(m_k)$. By [18] we can find two cycles $a_k, b_k \in \mathrm{Alt}(m_k)$ of length p_k and q_k respectively such that $\mathrm{Alt}(m_k) = \langle a_k, b_k\rangle$ and by our assumption a_k, b_k satisfy the hypotheses of Corollary 3.4.2 and Corollary 3.4.3 for the sequence $(\mathrm{Alt}(m_k) \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$.

Remark 3.4.6. Let $m \geq 48$ be an integer. Consider the alternating groups $\mathrm{Alt}(m)$. Then the elements $a_1 = (1\ 2\ 3)$ and

$$b_1 = \begin{cases} (2\ 3 \ldots m) & \text{if } m \text{ is even} \\ (3\ 4 \ldots m) & \text{if } m \text{ is odd} \end{cases}$$

generate $\mathrm{Alt}(m)$. Notice that $|b_1|$ is always odd.

By Bertrand's Postulate (proved by Chebyshev in 1850, see [8] for an elementary proof by Erdös), there exists a prime p such that $\lfloor m/2 \rfloor < p \leq m$. Since $m \geq 48$, either $p+4$, $p+8$ and $p+12$ lie in the interval $(\lfloor m/2 \rfloor, m)$ or

$p-4$, $p-8$ and $p-12$ do. Suppose that $t_1 = p+4$ and $t_2 = p+8$ and $p+12$ lie in the interval. Again by [18], we can find two cycles a_2 and b_2 of length t_2 and t_1 respectively such that $\mathrm{Alt}(m) = \langle a_2, b_2 \rangle$. We claim that there are two odd integers l_1 and l_2, $\lfloor m/2 \rfloor < l_1, l_2 \leq m$ such that $(l_1, 3) = (l_1, t_1) = 1$ and $(l_2, |b_1|) = (l_2, p) = 1$.

It is easy to see that $(n, n+4) = 1$ for every odd integer n. We can choose $l_1 = t_2$, in fact $(p+12, p+8) = 1$ and clearly $(p+12, 3) = 1$. Remember that $|b_1|$ is odd and $|p - |b_1|| < m/2$, since $p, |b_1| \in (\lfloor m/2 \rfloor, m)$. If $p \neq |b_1| - 2$ we choose $l_2 = |b_1| - 2$, while if $p = |b_1| - 2$ we choose $l_2 = |b_1| - 4$. The proof of the case where $p-4$, $p-8$ and $p-12$ lie in the interval $(\lfloor m/2 \rfloor, m)$ follows the same lines and it will be omitted.

Let a_3 and b_3 be the cycles of length l_2 and l_1 respectively which generate $\mathrm{Alt}(m)$ by the main theorem of [18]. Let $a_{2k+1} = a_3$, $a_{2k} = a_2$, $b_{2k+1} = b_3$ and $b_{2k} = b_2$ for $k \geq 2$. Then by construction a_k, b_k satisfy the hypotheses of Corollary 3.4.2 and Corollary 3.4.3 for the sequence $\mathcal{S} = (\mathrm{Alt}(m) \leq \mathrm{Sym}(m))_{k \in \mathbb{N}}$.

Chapter 4

On generation properties of generalised Wilson groups

The goal of this chapter is to study the behaviour of two profinite generation properties in the family of generalised Wilson groups: lower rank and profinite presentability. First, we will prove that there exist GW groups of finite lower rank and that an arbitrary direct product of certain hereditarily just infinite profinite groups with finite lower rank has again finite lower rank. Secondly, we will show that many generalised Wilson groups are not topologically finitely presentable. Most of the content of this chapter will appear as a research paper in the future. The result in Section 4.3 is joint work with Benjamin Klopsch.

4.1 Introduction

Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of positive integers. Throughout this section we denote by $\mathcal{S}$ a sequence of finite non-abelian simple transitive permutation groups $(S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$.

In Section 4.2 we study the lower rank of an infinitely iterated exponentiation (see Section 2.3.2 for the definition and some basic properties). Only very few profinite groups are known to have finite lower rank; amongst them are: p-adic analytic pro-p groups for every prime p, $SL_d(\mathbb{F}_p[\![t]\!])$ and the Nottingham group $\mathcal{N}(\mathbb{F}_p)$ for $p \geq 5$. In view of the scarcity of known profinite groups with finite lower rank, it is natural to search for new examples. Our first result of this chapter is that certain infinitely iterated exponentiations are examples of new profinite groups with finite lower rank.

Remember the notation for IIEs: for a sequence of integers $(m_k)_{k\in\mathbb{N}}$ we write $\widetilde{m}_1 = m_1$ and $\widetilde{m}_{k+1} = m_{k+1}^{\widetilde{m}_k}$. Recall that $\mathrm{d}(G)$ denotes the minimal number of generators of the finite group G.

Theorem 4.1.1. *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a sequence of finite non-abelian simple transitive permutation groups. Suppose there is a fixed $r \in \mathbb{N}$ such that $\mathrm{d}(S_n^{\widetilde{m}_{n-1}}) \leq r$ for infinitely many $n \in \mathbb{N}$. Then the infinitely iterated exponentiation of type $\mathcal{S}$ has lower rank at most r.*

In Section 4.3 we prove that the direct product of non-(virtually abelian) hereditarily just infinite profinite groups is very well behaved with respect to the lower rank.

Theorem 4.1.2. *Let $G_1, \ldots, G_n$ be non-(virtually abelian) hereditarily just infinite profinite groups with finite lower rank. Set $r = \max\{\mathrm{lr}(G_i) | i = 1, \ldots, n\}$. Suppose that for every open neighbourhood of the identity $\prod_{i=1}^n U_i$ in $G_1 \times \ldots \times G_n$ there exist $H_i \leq_o G_i$ with $H_i \leq U_i$, $\mathrm{d}(H_i) \leq r$ and $H_i \ncong H_j$ for $j < i$ and $i, j = 1, \ldots, n$. Then $\mathrm{lr}(G_1 \times \ldots \times G_n) \leq r$.*

In particular, the IIEs of type $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ satisfying the hypotheses of Theorem 4.1.1 and with $S_i \ncong S_j$ for $i \neq j$ satisfy the hypotheses of Theorem 4.1.2. Hence, taking direct products of certain infinitely iterated exponentiations with finite lower rank produces new examples of profinite groups with finite lower rank.

In Section 4.4 we work on the topological finite presentability of infinitely iterated wreath products (see Section 4.4). As a consequence of [21, Theorem A], any infinitely iterated wreath product of type $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$

is 2-generated, provided that $m_1 > 35$. It is then natural to ask whether infinitely iterated wreath products are finitely presentable. The following definition is well-known.

Definition 4.1.3. *Let H be a finite perfect group and fix a surjective homomorphism $F \to H$ with kernel R from an appropriate free group F. The* ***Schur multiplier*** *of H is the finite group $R/[F, R]$. We denote the Schur multiplier of the finite perfect group H by $M(H)$.*

It is possible to show that in the previous definition $M(H)$ does not depend on the choice of the homomorphism $F \to H$, but only on the perfect group H. The last result of this chapter is the following.

Theorem 4.1.4. *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k \in \mathbb{N}}$ be a sequence of finite non-abelian simple transitive permutation groups. Suppose that the profinite group $\prod_{n \in \mathbb{N}} M(S_n)$ is not topologically finitely generated, then any infinitely iterated wreath product of type $\mathcal{S}$ is not topologically finitely presentable.*

As a corollary of Theorem 4.1.4 we obtain a sufficient condition for the non-presentability of a generalised Wilson group.

Corollary 4.1.5. *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k \in \mathbb{N}}$ be a sequence of finite non-abelian simple transitive permutation groups and let $(k_n)_{n \in \mathbb{N}}$ an increasing sequence of integers. Suppose that the profinite group $\prod_{n \in \mathbb{N}} M(S_n)$ is not topologically finitely generated, then a generalised Wilson group of type $(\mathcal{S}, (k_n)_{n \in \mathbb{N}})$ is not topologically finitely presentable.*

See Remark 4.4.4 for a number of cases where the hypothesis of Theorem 4.1.4 holds.

4.2 IIEs with finite lower rank

In this section we discuss the lower rank of the infinitely iterated exponentiation of type $\mathcal{S}$ for special sequences $\mathcal{S}$. Remember that, for a sequence

$(m_k)_{k\in\mathbb{N}}$, we write $\widetilde{m}_1 = m_1$ and $\widetilde{m}_{k+1} = m_{k+1}^{\widetilde{m}_k}$.

Proof of Theorem 4.1.1. Set $G = \varprojlim \widetilde{S}_k$. Consider the subgroups $N_k = \ker(G \to \widetilde{S}_k)$ for $k \in \mathbb{N}$. It is clear that these subgroups form a base for the topology at the identity and, by Lemma 1.2.5, they are the only open normal subgroups of G. Moreover, N_k/N_{k+1} is isomorphic to $S_{k+1}^{\widetilde{m}_k}$ for every $k \geq 1$.

By definition of product action and by Lemma 1.2.5, N_k/N_{k+1} is the unique minimal normal subgroup of N_i/N_{k+1} every $k \in \mathbb{N}$ and for every $i = 1, \ldots, k$. Repeated applications of Theorem 2.2.8 yield $\mathrm{d}(N_{k-1}) = \mathrm{d}(N_{k-1}/N_k)$. By hypothesis, $\mathrm{d}(S_k^{\widetilde{m}_{k-1}}) \leq r$ for infinitely many k and $\{N_k | \mathrm{d}(N_k) \leq r\}$ is the required base for the topology of G. $\square$

Remark 4.2.1. The hypotheses of Theorem 4.1.1 are satisfied, with $r = 2$, by the sequences $\mathcal{S} = (\mathrm{PSL}_2(p_n) \leq \mathrm{Sym}(p_n + 1))_{n\in\mathbb{N}}$ where $\mathrm{PSL}_2(p_n)$ acts on the projective line over $\mathbb{F}_{p_n}$ and $(p_n)_{n\in\mathbb{N}}$ is any sequence of primes satisfying

$$p_n \geq \frac{1}{4}(p_{n-1} + 1)(p_{n-1}^2 - 2p_{n-1} - 1) - 2.$$

This follows from the calculation of the *Eulerian function* for $\mathrm{PSL}_2(p)$ (see [11]). In particular, infinitely iterated exponentiations of these sequences have lower rank 2.

We conjecture that there exists a GW group with infinite lower rank. One strong candidate for this is the infinitely iterated exponentiation of a constant sequence, i.e. $S_k = S$ for $k \in \mathbb{N}$.

The following straight-forward lemma provides a sufficient condition for a profinite group to have infinite lower rank.

Lemma 4.2.2. *Let G be a profinite group. Suppose there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f(n) = \infty$ and, for all $\Gamma \leq_o G$ with $|G : \Gamma| = n$, $\mathrm{d}(\Gamma) \geq f(n)$. Then G has infinite lower rank.*

Any free profinite group satisfies the hypotheses of Lemma 4.2.2 by the

Schereier index formula. Moreover, the converse of Lemma 4.2.2 does not hold as shown in the next example.

Example 4.2.1. Let F_3 be the free profinite group with 3 generators x, y, z and let $\mathbb{Z}_p$ be the pro-cyclic pro-p group with generator a. Set $G = F_3 \times \mathbb{Z}_p$. It is easy to see that the lower rank of G is at least the lower rank of F_3 (see Lemma 4.3.1) and, by the Schereier index formula, the lower rank of F_3 is infinite. Thus G has infinite lower rank. On the other hand, for every $n \in \mathbb{N}$, $H_n = \overline{\langle a^{p^n}, x, y, z \rangle}$ is an open 4-generated subgroup of G with index p^n.

We are also convinced that there are GW groups of arbitrary finite lower rank, but a proof of this result has to involve an accurate study of the subgroup structure of GW groups.

Conjecture 1. There exists a generalised Wilson group of lower rank r for every $r \in \mathbb{N} \cup \{\infty\}$.

A positive answer to this conjecture would produce interesting examples as the only known family of profinite groups of arbitrary finite lower rank are p-adic analytic pro-p groups, where the lower rank coincides with the number of generators of the associated p-adic Lie algebra (see [14]).

4.3 Lower rank of direct products of hereditarily just infinite profinite groups

The content of this section is joint work with Benjamin Klopsch.

In the previous section we proved that some profinite groups have finite lower rank. Looking at the other end of the spectrum, namely profinite groups with infinite lower rank, we are again in shortage of examples. By the Reidemeister-Schereier index formula we have that any free profinite (or pro-p) group has infinite lower rank. Moreover, Ershov and Jaikin-Zaipirain constructed new hereditarily just infinite pro-p groups with infinite lower rank.

These groups are a generalisation of the so called *Tarski monsters* (see [9, Theorem 8.7]).

Since the number of generators of a profinite group G might grow taking direct powers of G, a natural candidate to search for profinite groups with large lower rank is the direct product of profinite groups. Next we prove a basic lemma that justifies this intuition.

Lemma 4.3.1. *Let G be a profinite group and let N be a closed normal subgroup of G. Then $\mathrm{lr}(G/N) \leq \mathrm{lr}(G)$.*

Proof. If the lower rank of G is infinite there is nothing to prove. Let $r = \mathrm{lr}(G) < \infty$. The projection $G \to G/N$ is a continuous surjective homomorphism. Let $\{H_i\}_{i \in I}$ be a collection of open subgroups of G that forms a base of open neighbourhoods for the identity in G with $\mathrm{d}(H_i) \leq r$ for $i \in I$. Now, $\{H_i N/N\}_{i \in \mathbb{N}}$ is a collection of open subgroups of G/N that forms a base of open neighbourhoods for the identity in G/N and $\mathrm{d}(H_i N/N) \leq \mathrm{d}(H_i) \leq r$ for $i \in I$. Therefore $\mathrm{lr}(G/N) \leq r$. □

Thus $\mathrm{lr}(G) \leq \mathrm{lr}(G \times G)$ for every profinite group. In this light Theorem 4.1.2 is rather unexpected. Before the proof of Theorem 4.1.2 we need to prove a couple of lemmas.

Lemma 4.3.2. *Let H be a non-(virtually abelian) hereditarily just infinite profinite group and let L be an open subgroup of H. Then there exists $a \in H$ such that L is not contained in $C_H(a)$.*

Proof. Suppose by contradiction that the subgroup L is contained in $C_H(a)$ for all $a \in H$, then L is contained in the intersection of all $C_H(a)$ for $a \in H$, which is $Z(H)$. By hypothesis $Z(H)$ is trivial, a contradiction with the assumption that L was open in H. □

Definition 4.3.3. *Let $H_1, \ldots, H_n$ be topological groups and let $H = H_1 \times \ldots \times H_n$. For $i \in \boldsymbol{n}$, let π_i be the continuous projection of H onto the i-th factor.*

We say that a subgroup K of H is a ***subdirect product*** *of H if K is closed and $\pi_i(K) = H_i$ for $i \in \boldsymbol{n}$.*

Let us now remark a couple of basic topological facts that will be used in the next proofs.

Remark 4.3.4. Let G and H be profinite groups. Then the continuous projection π from $G \times H$ onto G is continuous and closed. Moreover, a function f between topological spaces X and Y is continuous if and only if $f(\overline{A})$ is contained in $\overline{f(A)}$ for every subset A of X.

Let G be a group and let H be a subgroup of G. We will denote by H^G the **normal closure** of H in G, i.e. the smallest (by inclusion) normal subgroup of G that contains H. It is easy to see that $H^G = \langle h^g \mid h \in H,\ g \in G \rangle$.

Lemma 4.3.5. *Let $H_1, \ldots, H_n$ be profinite groups and let $H = H_1 \times \ldots \times H_n$. Let π_1 be the continuous projection of H onto the first factor. Let K be a subdirect product of H and let $(k_1, \ldots, k_n) \in K$. Define the subgroup*

$$L = \langle (k_1, \ldots, k_n)^{(h_1, t_2, \ldots, t_n)} \mid h_1 \in H_1,\ (h_1, t_2, \ldots, t_n) \in K \rangle.$$

Then $\overline{L}$ is contained in K and $\pi_1(\overline{L}) = \overline{\langle k_1 \rangle^{H_1}}$.

Proof. By its definition, L is contained in K. Since K is closed, $\overline{L}$ is contained in K. By Remark 4.3.4, π_1 is a closed map, thus the closed subgroup $\pi_1(\overline{L})$ of H_1 contains $\langle k_1 \rangle^{H_1}$, it follows that $\pi_1(\overline{L})$ contains $\overline{\langle k_1 \rangle^{H_1}}$. On the other hand, again by Remark 4.3.4, π_1 is continuous and $\pi_1(L) = \langle k_1 \rangle^{H_1}$, it follows that $\pi_1(\overline{L})$ is contained in $\overline{\langle k_1 \rangle^{H_1}}$. □

The previous lemma holds in a similar fashion for every other projection π_j and an appropriate modification of L.

Lemma 4.3.6. *Let G and H be profinite groups and let π be the continuous projection from $G \times H$ onto H. Suppose that S is a abstract subgroup of $G \times H$ such that $\pi(S) = \{e\}$, then $\pi(\overline{S}) = \{e\}$.*

Proof. By Remark 4.3.4, $\pi(\overline{S}) \subseteq \overline{\pi(S)} = \overline{\{e\}} = \{e\}$. $\square$

We are now ready for the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. Let $F = F_d$ be the free profinite group on d generators $x_1, \ldots, x_d$. By the hypothesis, to show that $\mathrm{lr}(G_1 \times \ldots \times G_n) \leq d$, it is sufficient to construct an open d-generated subgroup $K \leq_o H_1 \times \ldots \times H_n$ for every H_i as in the hypotheses. Let $H_i = \overline{\langle h_{i1}, \ldots, h_{id} \rangle}$ and choose profinite presentations $R_i \to F \xrightarrow{\varphi_i} H_i$ for H_i, $i = 1, \ldots, n$. Set $\underline{h}_k = (h_{1k}, \ldots, h_{nk})$, for $k \in \boldsymbol{d}$, and $K = \overline{\langle \underline{h}_k | 1 \leq k \leq d \rangle}$, clearly K is a d-generated closed subgroup of $H = H_1 \times \ldots \times H_n$ and K is a subdirect product of H. The rest of the proof is devoted to show that K is open in H. To this end, it sufficient to produce an open subgroup K_i of H_i such that $\{e\} \times \ldots \times \{e\} \times K_i \times \{e\} \times \ldots \times \{e\}$ is contained in K for every coordinate i. Without loss of generality, modulo renaming the coordinates, we can suppose $i = 1$. In the rest of this proof we will use repeatedly the following fact.
<u>Claim:</u> For every $2 \leq j \leq d$, the image of $R_1 R_j$ in H_1 is an open normal subgroup of H_1.
<u>Proof of Claim:</u> By hypothesis, $R_1 \neq R_j$. Moreover, $F/R_1 \cong H_1$ and $F/R_j \cong H_j$ are just infinite groups and therefore R_1 is not contained and does not contain R_j. Hence $R_1 R_j / R_1$ is a non-trivial closed normal subgroup of $F/R_1 \cong H_1$ and, since H_1 is just infinite, the Claim follows. $\square$

For $i \in \boldsymbol{n}$, let π_i be the continuous projection of H onto H_i. By the Claim and by Lemma 4.3.2, there exists $a_1 \in H_1$ such that the image of $R_1 R_2$ in H_1 is not contained in $C_{H_1}(a_1)$. Since K is a subdirect product of H, there exist $l_i^{(1)} \in H_i$, for $i = 2, \ldots, n$, such that the n-tuple $(a_1, l_2^{(1)}, \ldots, l_n^{(1)})$ belongs to K.

By the Claim and by the choice of the element a_1, there exists a word $w_2 = w_2(x_1, \ldots, x_d) \in R_1 R_2$ such that $w_2 \notin R_1$ and $w_2(h_{11}, \ldots, h_{1d})$ does not belong to $C_{H_1}(a_1)$. By the choice of w_2, the commutator

$$c_1 = [w_2(\underline{h}_1, \ldots, \underline{h}_d), (a_1, l_2^{(1)}, \ldots, l_n^{(1)})] = ([w_2(h_{11}, \ldots, h_{1d}), a_1], e, l'_3, \ldots, l'_n)$$

belongs to K, where $l'_j = [w_2(h_{j1}, \ldots, h_{jd}), l_j^{(1)}]$ for $j = 3, \ldots, n$ and the first coordinate in non-trivial. Set

$$K^{(2)} = \overline{\langle c_1^{(h_1, t_2, \ldots, t_n)} \mid h_1 \in H_1, (h_1, t_2, \ldots, t_n) \in K \rangle}.$$

By Lemma 4.3.5 and the choice of w_2 and a_1, $K^{(2)}$ is contained in K and $\pi_1(K^{(2)}) = \overline{\langle [w_2(h_{11}, \ldots, h_{1d}), a_1] \rangle^{H_1}}$ is a non-trivial closed normal subgroup of the just infinite group H_1. Therefore $\pi_1(K^{(2)})$ is open in H_1 and, by Lemma 4.3.6, we have $\pi_2(K^{(2)}) = \{e\}$.

Let $2 \leq i < n$ and suppose we defined a subgroup $K^{(i)}$ of H contained in K such that $\pi_1(K^{(i)})$ is open in H_1 and $\pi_j(K^{(i)}) = \{e\}$, for $j = 2, \ldots, i$. We are going to produce a subgroup $K^{(i+1)}$ of H contained in K such that $\pi_1(K^{(i+1)})$ is open in H_1 and $\pi_j(K^{(i+1)}) = \{e\}$, for $j = 2, \ldots, i+1$.

We remind the reader that $\varphi_1 : F \to H_1$ is the epimorphism used in the chosen presentation of H_1. Notice that $\varphi_1(R_1 R_{i+1}) \cap \pi_1(K^{(i)})$ is a non-trivial open subgroup of $\pi_1(K^{(i)})$. By Lemma 4.3.2, there exists $a_i \in \pi_1(K^{(i)})$ such that $\varphi_1(R_1 R_{i+1}) \cap \pi_1(K^{(i)})$ is not contained in $C_{\pi_1(K^{(i)})}(a_i)$ and there exist $l_j^{(i)} \in H_j$, for $j = i+1, \ldots, n$, such that the n-tuple $(a_i, e, \ldots, e, l_{i+1}^{(i)}, \ldots, l_n^{(i)})$ belongs to $K^{(i)} \leq K$.

By the Claim and by the choice of the element a_i, there exists a word $w_{i+1} = w_{i+1}(x_1, \ldots, x_d) \in R_1 R_{i+1}$ such that $w_{i+1} \notin R_1$ and $w_{i+1}(h_{11}, \ldots, h_{1d})$ does not belong to $C_{\pi_1(K^{(i)})}(a_i)$. By the choice of w_{i+1}, the commutator

$$\begin{aligned} c_i = [w_{i+1}(\underline{h}_1, \ldots, \underline{h}_d), (a_i, e, \ldots, e, l_{i+1}^{(i)}, \ldots, l_n^{(i)})] = \\ = ([w_{i+1}(h_{11}, \ldots, h_{1d}), a_i], e, \ldots, e, l'_{i+2}, \ldots, l'_n) \end{aligned}$$

belongs to K, where $l'_j = [w_{i+1}(h_{j1}, \ldots, h_{jd}), l_j^{(i)}]$ for $j = i+2, \ldots, n$ and the first coordinate in non-trivial. Define the subgroup

$$K^{(i+1)} = \overline{\langle c_i^{(h_1, t_2, \ldots, t_n)} \mid h_1 \in H_1, (h_1, t_2, \ldots, t_n) \in K \rangle}.$$

By Lemma 4.3.5 and the choice of w_{i+1} and a_i, $K^{(i+1)}$ is contained in K. Moreover, $\pi_1(K^{(i+1)})$ is open in H_1 and, by construction and Lemma 4.3.6, $\pi_j(K^{(i+1)}) = \{e\}$ for $j = 2, \ldots, i+1$.

By inductive process, the subgroup $K^{(n)}$ is contained in K, $\pi_1(K^{(n)})$ is open in H_1 and $\pi_j(K^{(n)}) = \{e\}$ for $j = 2, \ldots, n$.

It follows that K is open. This concludes the proof. $\square$

Theorem 4.1.2 can be applied to infinitely iterated exponentiations of type $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ where, for $i, j \in \mathbb{N}$, $i \neq j$, the permutation groups S_i and S_j are non-isomorphic and the sequence $\mathcal{S}$ satisfies the hypotheses of Theorem 4.1.1.

4.4 Finite presentability of IIEs

We start with by recalling the definition of finite topological presentability. Let G be a profinite group, recall that $\mathrm{d}(G)$ denotes the minimal size of a subset of G which generates a dense subgroup of G.

Definition 4.4.1. *Let N be a closed normal subgroup of the profinite group G and let $\mathcal{R}$ be a subset of N, we say that N is **(topologically) normally generated in G by $\mathcal{R}$** if the G-conjugates of the elements of $\mathcal{R}$ generate a dense subgroup of N.*

Definition 4.4.2. *Let G be a topologically d-generated profinite group. We can define a continuous epimorphism $F \to G$ from the free profinite group on d generators onto G; the kernel of this epimorphism is a closed normal subgroup R of F. Let $\mathcal{S}$ be a set of topological generators for F and let R be topologically normally generated by a subset $\mathcal{R}$ of R, then these give us a **profinite presentation** of G and we write $G = \langle \mathcal{S} | \mathcal{R} \rangle$. We say that a finitely generated profinite group G is **topologically finitely presentable** if there exists a presentation $G = \langle \mathcal{S} | \mathcal{R} \rangle$ of G such that $\mathcal{S}$ and $\mathcal{R}$ are finite.*

The following lemma is an application of [22, Theorem 3] and it will be used in the proof of Theorem 4.1.4.

Lemma 4.4.3. *Let $\mathcal{S} = (S_k)_{k\in\mathbb{N}}$ be a sequence of finite perfect groups. Suppose that $\prod_{n\in\mathbb{N}} M(S_n)$ is not topologically finitely generated. Then the sequence $(\mathrm{d}(M(\widehat{S}_n)))_{n\in\mathbb{N}}$ is unbounded.*

Proof. By [22, Theorem 3], if A is a perfect group and B is a perfect permutation group, then $M(A \wr B) \cong M(A) \times M(B)$. Therefore $M(\widehat{S}_n) = \prod_{k=1}^{n} M(S_k)$, the claim follows. $\square$

We will now use Lemma 4.4.3 to prove Theorem 4.1.4.

Proof of Theorem 4.1.4. Let G be an infinitely iterated wreath product group of type $\mathcal{S}$. If G is not topologically finitely generated, then G is not finitely presentable. Now suppose that G is d-generated. Let $F = \widehat{F}_d$ be the profinite free group of rank d, $\varphi : F \to G$ a continuous epimorphism, $R = \ker\varphi$ and let N be an open normal subgroup of F. Now, NR is an open normal subgroup of F that contains R, thus F/NR is isomorphic to a continuous quotient of G. By Lemma 1.2.5, the only open normal subgroups of G are kernels of the inverse limit projections from G to G_n for some integer n. Therefore, any continuous quotient of G is isomorphic, as an abstract group, to an iterated wreath product.

The number of relations of G in the chosen presentation is the (possibly infinite) number $r(G)$ of normal generators for R as a subgroup of F. The quotient $R/[R,R]$ of R is abelian and $R/[F,R]$ is a quotient of the latter, hence $r(G) \geq \mathrm{d}(R/[R,R]) \geq \mathrm{d}(R/[F,R])$.

Set $M = [F, NR]$. The group $R/(M \cap R)$ is a quotient of $R/[F,R]$, hence $\mathrm{d}(R/[F,R])$ is at greater than $\mathrm{d}(R/(M \cap R))$.

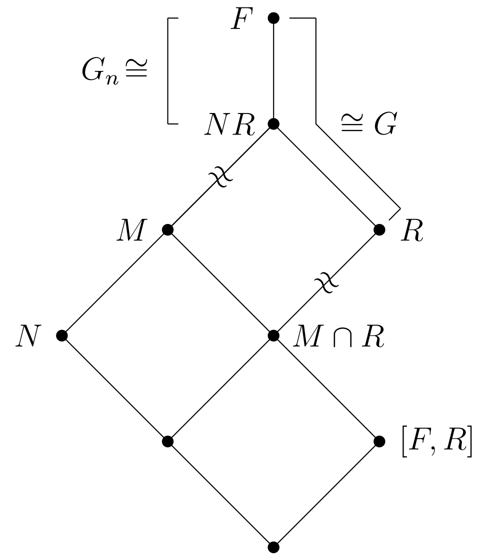

Figure 4.1: Subgroups in Theorem 4.1.4.

On the other hand, the group $R/(M \cap R)$ is isomorphic to NR/M and $NR/M = NR/[F, NR]$ is the Schur multiplier of an iterated wreath product. Therefore $r(G) \geq \mathrm{d}(NR/[F, NR])$. By Lemma 4.4.3 and by hypothesis, the last quantity is unbounded as N ranges between all open normal subgroups of F. Thus G cannot be finitely presentable. □

Remark 4.4.4. A sequence of finite non-abelian simple groups $(S_k)_{k \in \mathbb{N}}$ such that a fixed prime p divides $M(S_n)$ for infinitely many n satisfies the hypotheses of Theorem 4.1.4. Let $\mathcal{C}$ be the constant sequence $(\mathrm{Alt}(36) \leq \mathrm{Sym}(36))_{k \in \mathbb{N}}$, then every infinitely iterated wreath product of type $\mathcal{C}$ is finitely generated by [21, Theorem 1], but it is not finitely presentable by Theorem 4.1.4. In fact $M(\mathrm{Alt}(36))$ has order two.

Looking at the tables of the ATLAS ([6]), we can see that roughly half of the finite non-abelian simple groups have trivial Schur multiplier. On the other hand, if S is a finite non-abelian simple group with non-trivial Schur multiplier and S is not a Chevalley group $\mathrm{PSL}_n(q)$ or a Steinberg group $\mathrm{PSU}_n(q)$, with $n \geq 2$ and q power of a prime, then $|M(S)|$ is divisible by 2 or 3 (possibly both)[1]. Therefore we can apply Theorem 4.1.4 to all sequences $\mathcal{S}$ which contain infinitely many finite non-abelian simple groups with non-trivial Schur multiplier and not isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$.

We believe that no generalised Wilson group is finitely presentable, but we do not have a proof of this general statement.

Conjecture 2. Every generalised Wilson group is not finitely presentable.

[1]The Schur multiplier of $\mathrm{PSL}_n(q)$ is cyclic of order $\gcd(n, q-1)$ and the Schur multiplier of $\mathrm{PSU}_n(q)$ is cyclic of order $\gcd(n, q+1)$. Thus there are cases where the Schur multiplier of $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$ is divisible by 2 or 3, but we omitted these to keep the exposition simple.

Chapter 5

Embedding theorems for IIEs

The next chapter is devoted to the study of two properties: self-similarity and embeddability of specific profinite groups in infinitely iterated exponentiations. The results in Section 5.3 are joint work with Benjamin Klopsch.

5.1 Introduction

Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of positive integers such that $m_k \geq 2$ for all k. Throughout this chapter we denote by $\mathcal{S}$ a sequence of finite transitive permutation groups $(S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ such that each S_k is a non-abelian simple as an abstract group.

In Section 5.2 we study embedding of finitely generated profinite groups with specified continuous composition factors in IIEs. The following is the first main result of [32].

Theorem 5.1.1. ([32, Theorem A]) *There exists a hereditarily just infinite profinite group in which every countably based profinite group can be embedded.*

Our first result is a generalisation of the previous theorem to finitely generated profinite groups with restricted *composition factors*.

Definition 5.1.2. *Let G be a profinite group. A* ***composition series*** *of G is a descending chain of closed subnormal subgroups $(G_n)_{n\in\mathbb{N}}$, with $G_{n+1} \lhd G_n$*

for $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} G_n = \{e\}$ *and such that* G_n/G_{n+1} *is finite and simple for every* n. *The quotients* G_n/G_{n+1} *are called the* ***composition factors*** *of* G.

For a finitely generated profinite group G it is possible to use the Jordan-Hölder Theorem for finite groups to show that the *multiset* of composition factors of G is countable and "unique" (intended in the same sense as for finite groups).

Theorem 5.1.3. *Let* G *be a finitely generated profinite group. Let* $\mathcal{X} = \{T_n | n \in \mathbb{N}\}$ *be the set of composition factors of* G. *Then there exists a sequence* $\mathcal{S} = \{S_k \leq \mathrm{Sym}(|S_k|)\}_{k\in\mathbb{N}}$ *of permutation groups* $S_k \in \mathcal{X}$, *each* S_k *acting on itself by right multiplication, such that the infinitely iterated exponentiation of type* $\mathcal{S}$ *contains a closed subgroup isomorphic to* G.

Notice that the IIE constructed in the proof of Theorem 5.1.3 depends on the profinite group G we start with. It would be interesting, for any fixed set of composition factors $\mathcal{X}$, to construct a "universal" infinitely iterated exponentiation E with set of composition factors $\mathcal{X}$ such that every finitely generated profinite group with composition factors in $\mathcal{X}$ can be embedded in E. We do not know if this is the case.

Incidentally, to prove Theorem 5.1.3, we found a new way to embed an iterated permutational wreath product in an iterated exponentiation of permutation groups.

In Section 5.3 we find sufficient conditions on a sequence $\mathcal{S}$ of finite non-abelian simple permutation groups for the infinitely iterated exponentiation of type $\mathcal{S}$ to have a proper closed subgroup isomorphic to itself, i.e. *self-embeddable*. In the proofs of Section 5.3 it will be handy to have a way of saying when a permutation group is "embedded" in another.

Definition 5.1.4. *Let* $G \leq \mathrm{Sym}(m)$ *and* $H \leq \mathrm{Sym}(n)$ *be permutation groups with* $n \leq m$. *We say that* H *a* ***sub-permutation group*** *of* G *if there exist*

$I \subseteq \boldsymbol{m}$ and $H' \leq G$ such that $|I| = n$, H' fixes I setwise and $H \leq \mathrm{Sym}(n)$ and $H' \leq \mathrm{Sym}(I)$ are equivalent as permutation groups.

We would like to point out that, in the previous definition, H' does not indicate the derived subgroup of H. Note that in the previous definition H' does not have to fix the points outside the subset I.

Example 5.1.1. With the notation of the previous definition, the group $H = \langle (1\ 2) \rangle \leq \mathrm{Sym}(2)$ is a sub-permutation group of $G = \langle (1\ 2)(3\ 4) \rangle \leq \mathrm{Sym}(4)$ with $I = \{1, 2\} \subset \{1, 2, 3, 4\}$ and $H' = G$.

Our second theorem of this chapter is the following.

Theorem 5.1.5. *Let $\mathcal{S} = \{S_k \leq \mathrm{Sym}(m_k)\}_{k \in \mathbb{N}}$ be a sequence of permutation groups. Suppose that there exist a fixed natural number N such that for any $k \geq N$, there are infinitely many $j \in \mathbb{N}$ such that $S_k \leq \mathrm{Sym}(m_k)$ is a sub-permutation group of $S_j \leq \mathrm{Sym}(m_j)$. Let G be the infinitely iterated exponentiation of type $\mathcal{S}$. Then there exists a proper closed subgroup H of G isomorphic to G. In particular, the infinitely iterated exponentiation of type $\mathcal{S}$ is self-similar.*

Moreover, we prove that the conditions of Theorem 5.1.5 are also necessary for the IIE of type a sequence made of finite *minimal* simple permutation groups to admit a proper closed subgroup isomorphic to the whole group (see Proposition 5.3.4).

5.2 Embedding of finitely generated profinite groups with restricted composition factors in IIEs

Let G be a finitely generated profinite group and let $\mathcal{X} = \{T_n | n \in \mathbb{N}\}$ be the countable set of composition factors of G. Repeated applications of Theorem 1.2.3 and a standard compactness argument yield that there exist a function $\sigma_G : \mathbb{N} \to \mathbb{N}$ and a continuous injective homomorphism

$$G \hookrightarrow W(\sigma_G, \mathcal{X}) = \varprojlim_{n \in \mathbb{N}} T_{\sigma_G(n)} \wr \ldots \wr T_{\sigma_G(1)}, \tag{5.2.1}$$

where each $T_{\sigma_G(n)}$ acts on itself by right multiplication.

The group $W(\sigma_G, \mathcal{X})$ is the infinitely iterated permutational wreath product of type $\{T_{\sigma_G(n)} \leq \mathrm{Sym}(|T_{\sigma_G(n)}|)\}_{n \in \mathbb{N}}$. The function σ_G "keeps track" of the order in which the groups of $\mathcal{X}$ appear in $W(\sigma_G, \mathcal{X})$.

By (5.2.1), to prove that a finitely generated profinite group with set of composition factors $\mathcal{X}$ can be embedded as a closed subgroup of an infinitely iterated exponentiation of a sequence of $\mathcal{X}$-groups, it will suffice to show that, for any function $\sigma : \mathbb{N} \to \mathbb{N}$, $W(\sigma, \mathcal{X})$ is isomorphic to a closed subgroup of the infinitely iterated exponentiation of a sequence of $\mathcal{X}$-groups.

Notation 2. Let X be a set and n be a integer. If $\underline{x} \in X^n$ we will write $\underline{x}(i)$ for the i-th component of the vector $\underline{x}$ and if $I = \{i_1, \ldots, i_m\} \subset \mathbf{n}$ with $i_1 < \ldots < i_m$ is a subset of indices we will write

$$\underline{x}(I) = (x_{i_1}, \ldots, x_{i_m})$$

for the m-tuple of elements of the vector $\underline{x}$ in the I-positions. Let T be a group and $I = \{i_1, \ldots, i_m\}$ be a subset of $\mathbf{n}$ with $i_1 < \ldots < i_m$. We will write

$$T^n(I) = \{(x_1, \ldots, x_n) \in T^n \mid \forall i \notin I \ x_i = e \text{ and } \forall j \in I \ x_j \in T\}$$

for the subgroup of T^n with elements of T in the I-positions and the identity element everywhere else. We will also write

$$\mathrm{diag}(T^n(I)) = \{(x_1, \ldots, x_n) \in T^n(I) \mid \forall j, k \in I \ x_j = x_k\}.$$

Notice that if $T \leq \mathrm{Sym}(l)$ is a permutation group and $I = \{i_1, \ldots, i_m\} \subset \mathbf{n}$ is a subset of indices, then $T^n(I)$ acts faithfully on $\boldsymbol{l}^m$. Furthermore, if we set $L = \{\underline{x} \in \boldsymbol{l}^n \mid \forall i, j \in I \ \underline{x}(i) = \underline{x}(j) \text{ and } \forall k \notin I \ \underline{x}(k) = 1\}$, then $T \leq \mathrm{Sym}(l)$ is equivalent to $\mathrm{diag}(T^n(I)) \leq \mathrm{Sym}(L)$.

Before the proof of Theorem 5.1.3 we need to state an ancillary definition. Let $G \leq \mathrm{Sym}(n)$ be a permutation group and fix $r \in \boldsymbol{n}$. We can define an action of G on the subsets of $\boldsymbol{n}$ of size r by setting, for $J \subset \boldsymbol{n}$ with $|J| = r$ and $g \in G$, $J^g = \{j^g \mid j \in J\}$. It is easy to check that this is an action of G and in general this is not faithful.

Definition 5.2.1. *Let $G \leq \mathrm{Sym}(m)$ and $H \leq \mathrm{Sym}(n)$ be a permutation groups with $n \leq m$. Consider the action of G on the subsets of $\{1, \ldots, m\}$ defined for $J \subseteq \{1, \ldots, m\}$ and $g \in G$ by $J^g = \{j^g \mid j \in J\}$. We say that H is **P-embedded** in G if there exists a collection of subsets I of $\{1, \ldots, m\}$ such that*

(1) *$I = n$ and*

(2) *for all $J_1, J_2 \in I$, $|J_1| = |J_2|$ and $J_1 \cap J_2 = \varnothing$;*

and there exists a subgroup and H' of G such that

(3) *I is H'-invariant,*

(4) *the action of H' on I is faithful;*

(5) *$H'_{|I} \leq \mathrm{Sym}(I)$ is equivalent to $H \leq \mathrm{Sym}(n)$.*

The first four conditions of the previous definition ensure that the action defined by $J^g = \{j^g \mid j \in J\}$ for $J \in I$ and $h \in H'$ makes $H'_{|I} \leq \mathrm{Sym}(I)$ a permutation group. If we take $r = 1$ in the previous definition we obtain the definition of sub-permutation group given in the introduction of this chapter. The previous definition seems rather technical, but the fact that $H \leq \mathrm{Sym}(n)$ is P-embedded in $G \leq \mathrm{Sym}(m)$ just means that H acts on some subsets of $\boldsymbol{m}$ the same way that it acts on $\boldsymbol{n}$.

Example 5.2.1. Consider the permutation group $V = \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$ acting on $\{1, 2, 3, 4\}$ (the Klein group). Then $H = \langle (1\ 2)(3\ 4) \rangle \leq \mathrm{Sym}(4)$ is a sub-permutation group of $V \leq \mathrm{Sym}(4)$, in fact $H' = \langle(1\ 2)(3\ 4)\rangle \leq V$ is isomorphic to H and this together with $I = \{1, 2, 3, 4\}$ satisfy the required properties.

On the other hand $H \leq \mathrm{Sym}(4)$ is also P-embedded in $V \leq \mathrm{Sym}(4)$. Consider the set $I = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\} \subset 2^{\{1,2,3,4\}}$ and the subgroup H' as before, then $H'_{|I} \leq \mathrm{Sym}(I)$ is equivalent to $H \leq \mathrm{Sym}(4)$.

We would like to point out that if a permutation group $H \leq \mathrm{Sym}(n)$ is P-embedded in another permutation group $G \leq \mathrm{Sym}(m)$, then there exists an embedding of H into G.

Notation 3. Let $\mathcal{S} = \{S_k \leq \mathrm{Sym}(m_k)\}_{k\in\mathbb{N}}$ be a sequence of permutation groups and let $S_0 \leq \mathrm{Sym}(m_0)$ be a fixed permutation group. Define inductively the following sets $A_1 = \boldsymbol{m}_1$ and $A_{n+1} = \boldsymbol{m_{n+1}} \times A_n$ for $n \in \mathbb{N}$. Put

$$\begin{cases} W_1 = S_1 \leq \mathrm{Sym}(A_1) \\ W_{n+1} = S_{n+1} \wr W_n \leq \mathrm{Sym}(A_{n+1}) \quad \text{for } n \in \mathbb{N} \end{cases}$$

The permutation group $W_n \leq \mathrm{Sym}(A_n)$ is the n-th iterated permutational wreath product of type $\mathcal{S}$. Define inductively the following sets $B_0 = \boldsymbol{m_0}$ and $B_n = \boldsymbol{m_n}^{b_{n-1}}$ for $n \in \mathbb{N}$. Put

$$\begin{cases} E_0 = S_1 \leq \mathrm{Sym}(b_0) \\ E_n = S_n \textcircled{\wr} E_{n-1} \leq \mathrm{Sym}(B_n) \quad \text{for } n \in \mathbb{N} \end{cases}$$

The permutation group $E_n \leq \mathrm{Sym}(B_n)$ is the $(n+1)$-th iterated exponentiation of type $\{S_0 \leq \mathrm{Sym}(m_0), S_1 \leq \mathrm{Sym}(m_1), S_2 \leq \mathrm{Sym}(m_2), \ldots\}$.

The proof of Theorem 5.1.3 is based on the following proposition.

Proposition 5.2.2. *Let $\mathcal{S} = \{S_k \leq \mathrm{Sym}(m_k)\}_{k\in\mathbb{N}}$ be a sequence of permutation groups and let $S_0 \leq \mathrm{Sym}(m_0)$ be a fixed permutation group. Then the n-th iterated permutational wreath product of type $\mathcal{S}$ is P-embedded in the $(n+1)$-th iterated exponentiation of type $\{S_0 \leq \mathrm{Sym}(m_0), S_1 \leq \mathrm{Sym}(m_1), S_2 \leq \mathrm{Sym}(m_2), \ldots\}$ for all $n \in \mathbb{N}$.*

Proof. We will use Notation 3. Set $|A_n| = a_n$ and $|B_n| = b_n$ for all n. We will prove the proposition by induction on n.

For $i \in A_1$, consider the following subset of $B_1 = \boldsymbol{m_1}^{m_0}$,

$$Z(i) = \{\underline{x} \in B_1 \mid \underline{x}(1) = i\}.$$

It is clear that $|Z(i)| = |Z(j)|$ and $Z(i) \cap Z(j) = \varnothing$ for all $i, j \in A_1$. Set $I_1 = \{Z(i) \mid i \in A_1\}$, then $|I_1| = a_1$. Define a subgroup of E_1 by setting $W_1' = \{(\sigma, e, \ldots, e)_{b_0} \mid \sigma \in S_1\}$. It is clear I_1 is W_1'-invariant. Consider the maps

$$\begin{array}{rccc} \varphi_1 : & W_1' & \longrightarrow & W_1 \\ & (\sigma, e \ldots, e)_{b_0} & \longmapsto & \sigma \end{array} \quad \text{and} \quad \begin{array}{rccc} f_1 : & A_1 & \longrightarrow & I_1 \\ & i & \longmapsto & Z(i) \end{array}.$$

Then clearly φ_1 is an isomorphism and f_1 is a bijection. Moreover, $f_1(i^\sigma) = f_1(i)^{\varphi_1(\sigma)}$ for all $i \in A_1$ and $\sigma \in W_1$. Therefore $(W_1')_{|I_1} \leq \mathrm{Sym}(I_1)$ is equivalent to $W_1 \leq \mathrm{Sym}(A_1)$ and W_1 is P-embedded in E_1.

Suppose by inductive hypothesis that there exist a subgroup W_n' of E_n, a subset I_n of 2^{B_n} such that $|I_n| = a_n$, $|Z_1| = |Z_2|$ and $Z_1 \cap Z_2 = \varnothing$ for all $Z_1, Z_2 \in I_n$, I_n is W_n'-invariant and $(W_n')_{|I_n} \leq \mathrm{Sym}(I_n)$ is equivalent to $W_n \leq \mathrm{Sym}(A_n)$ via an isomorphism $\varphi_n : W_n' \to W_n$ and a bijection $f_n : A_n \to I_n$ satisfying $f_n(a^{\varphi_n(g)}) = f_n(a)^g$ for all $a \in A_n$ and $g \in W_n'$.

We are now going to prove that W_{n+1} is P-embedded in E_{n+1}. Remember Notation 2 and define the following subgroup of $S_{n+1}^{b_n}$:

$$(S_{n+1}^{a_n})' = \prod_{Z \in I_n} \mathrm{diag}(S^{b_n}(Z)).$$

Define the subgroup $W_{n+1}' = (S_{n+1}^{a_n})' \cdot W_n'$ of E_{n+1}. We will now show that W_{n+1}' is isomorphic to W_{n+1}. Remember that $I_n = \{f_n(1), \ldots, f_n(a_n)\}$. For all $i \in \boldsymbol{a_n}$, fix an element $z_i \in f_n(i)$. Then there exists a permutation ρ of $\{1, \ldots, a_n\}$ such that $z_{\rho(1)} < \ldots < z_{\rho(n)}$. Set $\mathcal{Z} = \{z_{\rho(1)}, \ldots, z_{\rho(n)}\}$. Remembering Notation 2, define the following map

$$\begin{array}{rccc} \varphi_{n+1} : & W_{n+1}' = (S_{n+1}^{a_n})' \cdot W_n' & \longrightarrow & W_{n+1} = S_{n+1}^{a_n} \cdot W_n \\ & \underline{g} \cdot h & \longmapsto & \underline{g}(\mathcal{Z}) \cdot \varphi_n(h) \end{array}.$$

The restriction of φ_{n+1} to $(S_{n+1}^{a_n})'$ is clearly an isomorphism from $(S_{n+1}^{a_n})'$ to $S_{n+1}^{a_n}$ by the definition of $(S_{n+1}^{a_n})'$ and it does not depend on the choice of the z_i's. Furthermore, by induction $|W_n| = |W_n'|$ and the size of the image of W_{n+1}' via φ_{n+1} is at least $|(S_{n+1}^{a_n})'|\cdot|W_n'|$, which is equal to the size of W_{n+1}. To prove that φ_{n+1} is an isomorphism it is now sufficient to show that

$$\varphi_{n+1}(\underline{g}_1 \cdot h_1 \cdot \underline{g}_2 \cdot h_2) = \varphi_{n+1}(\underline{g}_1 \cdot h_1) \cdot \varphi_{n+1}(\underline{g}_2 \cdot h_2) \tag{5.2.2}$$

for all $\underline{g}_1, \underline{g}_2 \in (S_{n+1}^{a_n})'$ and all $h_1, h_2 \in W_n'$. Simplifying (5.2.2) yields that we just have to show that

$$\left(\underline{g}^h\right)(\mathcal{Z}) = \underline{g}(\mathcal{Z})^{\varphi_n(h)}$$

for all $\underline{g} \in (S_{n+1}^{a_n})'$ and $h \in W_n'$. By inductive hypothesis $W_n \leq \mathrm{Sym}(A_n)$ is equivalent to $(W_n')_{|I_n} \leq \mathrm{Sym}(I_n)$, hence the previous equality is satisfied and the map φ_{n+1} is an isomorphism.

We will now define a set $I_{n+1} \subseteq 2^{B_{n+1}}$. For $x \in \boldsymbol{m_{n+1}}$ and $y \in A_n$, define the following subset of $B_{n+1} = \boldsymbol{m_{n+1}}^{b_n}$:

$$Z(x,y) = \left\{ \underline{x} \in \boldsymbol{m_{n+1}}^{b_n} \;\middle|\; \begin{array}{c} \forall i \in f_n(y), \underline{x}(i) = x \text{ and } \forall y \neq l \in A_n, \\ \exists u, v \in f_n(l) \text{ such that } \underline{x}(u) \neq \underline{x}(v) \end{array} \right\}.$$

The element $y \in A_n$ corresponds to the set $f_n(y) \subset B_n$ and $Z(x,y)$ is the set of the b_n-tuples of $\boldsymbol{m_{n+1}}^{b_n}$ that have the $f_n(y)$-components all equal to x and the $f_n(y')$-components not all equal for $y' \neq y$. By inductive hypothesis the elements of I_n are disjoint, hence $Z(x,y)$ is well-defined for all x and y. Again by inductive hypothesis, $|f_n(i)| = |f_n(j)|$ for $i, j \in A_n$ and it follows that $|Z(x,y)|$ is constant for all x and y. By definition the $Z(x,y)$'s are also pair-wise disjoint. Set $I_{n+1} = \{Z(x,y) \mid x \in \boldsymbol{m_{n+1}}, y \in A_n\}$. The fact that the $Z(x,y)$'s are disjoint yields that $|I_{n+1}| = a_n \cdot m_{n+1} = a_{n+1}$.

Consider elements $\underline{g} \in (S_{n+1}^{a_n})'$ and $h \in W_n'$. By construction, for every $i \in \boldsymbol{a_n}$ and every $j \in f_n(i)$ there exists $g_i \in S_{n+1}$ such that $\underline{g}(j) = g_i$. Therefore I_{n+1} is $(S_{n+1}^{a_n})'$-invariant. Moreover, I_n is W_n'-invariant by induction and, by definition of product action, I_{n+1} is also W_n'-invariant.

All that is left to prove is that $(W'_{n+1})_{|I_{n+1}} \leq \mathrm{Sym}(I_{n+1})$ is equivalent to $W_{n+1} \leq \mathrm{Sym}(A_{n+1})$. Define the function f_{n+1} from $A_{n+1} = \boldsymbol{m_{n+1}} \times A_n$ to I_{n+1} by $f_{n+1}((x,y)) = Z(x,y)$ for all $x \in \boldsymbol{m_{n+1}}$ and $y \in A_n$. We are going to show that

$$f_{n+1}((x,y)^{\varphi_{n+1}(\underline{g}\cdot h)}) = f_{n+1}((x,y))^{\underline{g}\cdot h} \tag{5.2.3}$$

for every $\underline{g}h \in W'_{n+1}$ and $x \in \boldsymbol{m_{n+1}}$ and $y \in A_n$. Fix $\underline{g} \in (S_{n+1}^{a_n})'$ and $h \in W'_n$. By construction of $(S_{n+1}^{a_n})'$, for every $i \in \boldsymbol{a_n}$ and every $j \in f_n(i)$ there exists $g_i \in S_{n+1}$ such that $\underline{g}(j) = g_i$. In particular, $\underline{g}(z_i) = g_i$.

By definition of φ_{n+1} and the definition of permutational wreath action, the left-hand side of (5.2.3) is equal to $f_{n+1}((x^{\underline{g}(z_y)}, y^{\varphi_n(h)})) = Z(x^{\underline{g}(z_y)}, y^{\varphi_n(h)})$.

Pick $\underline{x} \in Z(x,y)$. By definition of $Z(x,y)$, for every $i \in f_n(y)$ we have $\underline{x}(i) = x$ and thus, for every $j \in f_n(y)^h = f_n(y^{\varphi_n(h)})$ we have $\underline{x}^{\underline{g}h}(j) = x^{g_y} = x^{\underline{g}(z_y)}$. Again by definition of $Z(x,y)$, for every $y \neq l \in A_n$ there exist $u, v \in f_n(l)$ such that $\underline{x}(u) \neq \underline{x}(v)$, therefore $\underline{x}^{\underline{g}h}(u^h) \neq \underline{x}^{\underline{g}h}(v^h)$ with $u^h, v^h \in f_n(l)^h = f_n(l^{\varphi_n(h)})$ and $y^{\varphi_n(h)} \neq l^{\varphi_n(h)}$. This shows that $\underline{x}^{\underline{g}h}$ belongs to $Z(x^{\underline{g}(z_y)}, y^{\varphi_n(h)})$ and $Z(x,y)^{\underline{g}h} \subseteq Z(x^{\underline{g}(z_y)}, y^{\varphi_n(h)})$. The proof of the opposite inclusion is exactly the same and will be omitted. Thus $Z(x,y)^{\underline{g}h} = Z(x^{\underline{g}(z_y)}, y^{\varphi_n(h)})$. This concludes the proof of the proposition. $\square$

In Proposition 5.2.2 we proved more that what is needed for the proof of Theorem 5.1.3. Furthermore, Proposition 5.2.2 is interesting in itself because it provides an embedding of iterated permutational wreath products into iterated exponentiations, a fact that did not seem to be known before. We are now ready for the proof of Theorem 5.1.3.

Proof of Theorem 5.1.3. Let G be a finitely generated profinite group and let $\mathcal{X} = \{T_n \mid n \in \mathbb{N}\}$ be the set of composition factors of G. By (5.2.1), there exists a function $\sigma : \mathbb{N} \to \mathbb{N}$ such that G can be embedded in the infinitely iterated permutational wreath product $W(\sigma, \mathcal{X})$. To prove the theorem it is then sufficient to find an infinitely iterated permutational wreath product in which $W(\sigma,\mathcal{X})$ can be embedded.

Consider the new sequence of finite simple permutation groups $\mathcal{S} = (S_n \leq \mathrm{Sym}(|S_n|))_{n\in\mathbb{N}}$ defined by $S_n = T_{\sigma(n)}$ for $n \geq 1$ with each group acting on itself by right multiplication. Put $S_0 = T_1$ and consider the permutation group $S_0 \leq \mathrm{Sym}(|S_0|)$ with S_0 acting on itself by right multiplication. Let $E(\sigma,\mathcal{X}) = \varprojlim_{n\geq 0} \widetilde{S}_n$ be the infinitely iterated exponentiation of type $\{S_0 \leq \mathrm{Sym}(|S_0|), S_1 \leq \mathrm{Sym}(|S_1|), S_2 \leq \mathrm{Sym}(|S_2|), \ldots\}$. By Proposition 5.2.2 together with Notation 3, the group W_n can be embedded into E_n for $n \in \mathbb{N}$. Therefore, by Proposition 1.4.8, the profinite group $W = \varprojlim W_n = W(\sigma,\mathcal{X})$ can be embedded in $E(\sigma,\mathcal{X}) = \varprojlim E_n$. $\square$

Remark 5.2.3. In Theorem 5.1.3 we did not exclude the possibility of some composition factors in $\mathcal{X}$ being cyclic. In this case Theorem 5.1.3 still holds, but the infinitely iterated exponentiation $\varprojlim \widetilde{S}_n$ might be virtually pro-p.

5.3 Self-similarity of IIEs

The content of this section is joint work with Benjamin Klopsch. Since IIEs are finitely generated, the results in Section 5.2 yield that the infinitely iterated exponentiation of a sequence $\mathcal{S} = \{S_k \leq \mathrm{Sym}(m_k)\}_{k\in\mathbb{N}}$ can be embedded in another IIE of type the sequence $\mathcal{S}' = \{S_0 \leq \mathrm{Sym}(m_0), S_1 \leq \mathrm{Sym}(m_1), \ldots\}$ where $S_0 \leq \mathrm{Sym}(m_0)$ is an arbitrary finite simple permutation group. In this section we are going to improve the results of the previous section showing that certain infinitely iterated exponentiation are self-similar.

In this section we will show that certain infinitely iterated exponentiations satisfy the definition of self-similar with $n = 1$. The proof of Theorem 5.1.5 relies on the following two core lemmas.

Notation 4. Let $f : X \to Y$ be a function and n a natural number. We will write $f^{(n)}$ to denote the function from X^n to Y^n defined by

$$f^{(n)}(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$$

for $x_1, \dots, x_n \in X$. Notice that if f is a bijection, so is $f^{(n)}$. If φ is an isomorphism from the group G to the group H, then $\varphi^{(n)}$ is an isomorphism from the group G^n to the group H^n.

Lemma 5.3.1. *Let $A \leq \mathrm{Sym}(m)$, $A_1 \leq \mathrm{Sym}(m_1)$, $B \leq \mathrm{Sym}(n)$ and $B_1 \leq \mathrm{Sym}(n_1)$ be permutation groups. Suppose that $A_1 \leq \mathrm{Sym}(m_1)$ is a sub-permutation group of $A \leq \mathrm{Sym}(m)$ and $B_1 \leq \mathrm{Sym}(n_1)$ is a sub-permutation group of $B \leq \mathrm{Sym}(n)$. Then $A_1 \wr B_1 \leq \mathrm{Sym}(m_1^{n_1})$ is a sub-permutation group of $A \wr B \leq \mathrm{Sym}(m^n)$.*

Proof. By hypothesis, there exist subgroups $A_2 \leq A$ and $B_2 \leq B$, subsets $I \subseteq \boldsymbol{m}$ and $J \subseteq \boldsymbol{n}$ fixed set-wise by A_2 and B_2 respectively, bijections $f : I \to \boldsymbol{m}_1$ and $g : J \to \boldsymbol{n}_1$ and isomorphisms $\varphi : A_2 \to A_1$ and $\psi : B_2 \to B_1$ such that

$$f(x^a) = f(x)^{\varphi(a)} \quad \text{and} \quad g(y^b) = g(y)^{\psi(b)}$$

for all $x \in I$, $y \in J$, $a \in A_2$ and $b \in B_2$. We can suppose without loss of generality that $I = \boldsymbol{m_1} \subset \boldsymbol{m}$, $J = \boldsymbol{n_1} \subset \boldsymbol{n}$ and $f(x) = x$ for $x \in \boldsymbol{m_1}$ and $g(y) = y$ for $y \in \boldsymbol{n_1}$. Consider the subset of $\boldsymbol{m}^n$ defined by

$$\Gamma = \{\underline{x} \in \boldsymbol{m}^n \mid \forall j \in \boldsymbol{m} \setminus \boldsymbol{m_1}, \underline{x}(j) = 1 \text{ and } \forall j \in \boldsymbol{m_1}, \underline{x}(j) \in \boldsymbol{m_1}\}.$$

Remember Notation 2 and define the function $\gamma : \Gamma \to \boldsymbol{m_1}^{n_1}$ by $\gamma(\underline{x}) = \underline{x}(\boldsymbol{n_1})$, for $\underline{x} \in \Gamma$. It is clear that γ is a bijection. Set

$$H = \{\underline{a} \in A^n \mid \forall j \in \boldsymbol{m} \setminus \boldsymbol{m_1}, \underline{a}(j) = e \text{ and } \forall j \in \boldsymbol{m_1}, \underline{a}(j) \in A_2\},$$

then H is normalized by B_2 in $A \wr B$ and we can form the semidirect product $H \rtimes B_2$. Define the homomorphism

$$\begin{array}{rccc} \delta : & H \rtimes B_2 & \longrightarrow & A_1 \wr B_1 \\ & \underline{a} \cdot b & \longmapsto & (\varphi^{(n)}(\underline{a}))(\boldsymbol{n_1}) \cdot \psi(b) \end{array}$$

It is now easy to check that δ is an isomorphism. All that is left to prove is that $A_1 \wr B_1 \leq \mathrm{Sym}(m_1^{n_1})$ and $H \rtimes B_2 \leq \mathrm{Sym}(\Gamma)$ are equivalent. Then, by all our assumptions,

$$\gamma(\underline{x}^{\underline{a}b}) = \underline{x}^{\underline{a}b}(\boldsymbol{n_1}) = (x_{1^b}^{a_{1^b}}, \ldots, x_{n_1{}^b}^{a_{n_1{}^b}}) = (x_1^{a_1}, \ldots, x_{n_1}^{a_{n_1}})^{\psi(b)} =$$
$$= (\underline{x})(\boldsymbol{n_1})^{(\varphi^{(n)}(\underline{a}))(\boldsymbol{n_1})\psi(b)} = \gamma(\underline{x})^{\delta(\underline{a}b)}$$

for all $\underline{x} \in \Gamma$, $\underline{a} \in H$ and $b \in B_2$. $\square$

Lemma 5.3.2. *Let $A \leq \mathrm{Sym}(m)$, $B \leq \mathrm{Sym}(n)$ and $C \leq \mathrm{Sym}(k)$ be permutation groups. Then the permutation group $A \wr C \leq \mathrm{Sym}(m^k)$ is a subpermutation group of $A \wr (B \wr C) \leq \mathrm{Sym}(m^{(n^k)})$.*

Proof. Set $G = A \wr (B \wr C) \leq \mathrm{Sym}(m^{(n^k)})$. We will first prove that C is P-embedded in $B \wr C$ (see Definition 5.2.1). First of all, C is isomorphic to the subgroup $C' = \{(e, \ldots, e)_k c \mid c \in C\}$ of $B \wr C$. We now need to exhibit a subset $I \subset 2^{\boldsymbol{n}^k}$ with the required properties. For $l \in \boldsymbol{k}$, put

$$J_l = \{\underline{x} \in \boldsymbol{n}^k \mid \forall i, j \in \boldsymbol{k} \setminus \{l\} \;\; x_i = x_j \text{ and } x_i \neq x_l\}$$

and $I = \{J_l \mid l \in \boldsymbol{k}\}$. It is easy to check that the J_l's satisfy the required properties and that $(C')_{|I} \leq \mathrm{Sym}(I)$ is equivalent to $C \leq \mathrm{Sym}(k)$.

We are now going to define a subgroup $(A^k)'$ of $A^{(n^k)}$ isomorphic to A^k. Remember Notation 2 and consider the subgroup $(A^k)'$ of G given by

$$(A^k)' = \prod_{J \in I} \mathrm{diag}(A^{n^k}(J)).$$

Remember that $I = \{J_l \mid l \in \boldsymbol{k}\}$. For $l \in \boldsymbol{k}$ fix an element $j_l \in J_l$. Then there exists a permutation ρ of $\{1, \ldots, k\}$ such that $j_{\rho(1)} < \ldots < j_{\rho(k)}$. Set $\mathcal{J} = \{j_{\rho(1)}, \ldots, j_{\rho(k)}\}$ and define the map

$$\begin{array}{rccc} \varphi: & (A^k)' \cdot C' & \longrightarrow & A \wr C \\ & \underline{a} \cdot c & \longmapsto & \underline{a}(\mathcal{J}) \cdot c \end{array}.$$

Since C is P-embedded in $B \wr C$ and by the definition of the J_l's, it follows that φ is an isomorphism and that $(A^k)'$ is C-invariant.

We are now going to define a subset S of $\boldsymbol{m}^{(n^k)}$. For $\underline{z} \in \boldsymbol{m}^k$, define the element $I(\underline{z})$ in $\boldsymbol{m}^{(n^k)}$ by

$$(I(\underline{z}))(i) = \begin{cases} \underline{z}(l) & \text{for } i \in J_l \\ 1 & \text{otherwise} \end{cases}.$$

Set $S = \{I(\underline{z}) \mid \underline{z} \in \boldsymbol{m}^k\} \subseteq \boldsymbol{m}^{(n^k)}$. Then the map f from S to $\boldsymbol{m}^o$ given by $I(\underline{z}) \mapsto \underline{z}$ is clearly a bijection. By definition of $(A^k)'$ and $I(\underline{z})$, we have that S is $((A^k)' \cdot C')$-invariant and

$$f\left((I(\underline{z}))^{\underline{a} \cdot c}\right) = f(I(\underline{z}))^{\varphi(\underline{a} \cdot c)}$$

for all $\underline{a} \in (A^k)'$, $c \in C$ and every $\underline{z} \in \boldsymbol{m}^k$. This concludes the proof. $\square$

We will now use Lemma 5.3.1 and Lemma 5.3.2 to prove Theorem 5.1.5.

Proof of Theorem 5.1.5. By [31, Exercise 1.6.8], it is sufficient to prove that, for every $n \in \mathbb{N}$, $\widetilde{S}_n$ is a sub-permutation group of $\widetilde{S}_m$ for some $m = m(n)$ depending on n. We will prove this by induction on n.

For the base of the induction just observe that $\widetilde{S}_{N-1}$ is trivially a sub-permutation group of itself. Let $n \geq N$ and suppose by inductive hypothesis that there exists an integer $m(n)$ such that $\widetilde{S}_n$ is a sub-permutation group of $\widetilde{S}_{m(n)}$. By hypothesis, there exists $j > m(n)$ such that S_{n+1} is a sub-permutation group of S_j. By Lemma 5.3.1, $S_{n+1} \wr \widetilde{S}_{j-1}$ is a sub-permutation group of $\widetilde{S}_j$. Moreover, by iterated applications of Lemma 5.3.2, we have that $S_{n+1} \wr \widetilde{S}_{j-i}$ is a sub-permutation group of $\widetilde{S}_j$ for every $i = 1, \ldots, j - m(n)$. Now, again by Lemma 5.3.1 and by inductive hypothesis, $\widetilde{S}_{n+1} = S_{n+1} \wr \widetilde{S}_n$ is a sub-permutation group of $\widetilde{S}_j$. If we set $j = m(n+1)$ the claim follows by induction. $\square$

Let $S \leq \mathrm{Sym}(m)$ be a fixed finite simple permutation group. Theorem 5.1.5 yields that the infinitely iterated exponentiation of type $\{S \leq \mathrm{Sym}(m)\}_{k \in \mathbb{N}}$ is self-similar. Another sequence that satisfies the hypotheses of Theorem 5.1.5 is $\mathcal{A} = \{\mathrm{Alt}(n+4) \leq \mathrm{Sym}(n+4)\}_{n\in\mathbb{N}}$ and notice that there is no pair of isomorphic groups in the sequence $\mathcal{A}$. It seems likely that a "general" sequence of finite simple permutation groups will satisfy the hypotheses of Theorem 5.1.5.

Two questions are now natural: does there exist an infinitely iterated exponentiation which cannot be embedded continuously as a closed subgroup of itself? Are the hypotheses of Theorem 5.1.5 also necessary? The next lemma shows that the answer to both questions is positive if we restrict ourselves to *finite minimal simple permutation groups.*

Definition 5.3.3. *A finite non-abelian simple group S is said to be* ***minimal*** *if every proper subgroup of S is solvable.*

In particular, a finite non-abelian minimal simple group has no proper subgroup isomorphic to a finite non-abelian simple group.

Proposition 5.3.4. *Let $\mathcal{S} = \{S_k \leq \mathrm{Sym}(m_k)\}_{k\in\mathbb{N}}$ be a sequence of finite minimal simple permutation groups. Then the infinitely iterated exponentiation of type $\mathcal{S}$ can be embedded continuously as a closed subgroup of itself if and only if there exist a natural number N such that for all $k \geq N$, the permutation group $S_k \leq \mathrm{Sym}(m_k)$ is equivalent to $S_j \leq \mathrm{Sym}(m_j)$ for infinitely many $j \in \mathbb{N}$.*

Proof. Let G be the infinitely iterated exponentiation of type $\mathcal{S}$. The "if" implication is the content of Theorem 5.1.5. For the converse, suppose that for every $N \in \mathbb{N}$ there exist $k \geq N$ such that $S_k \leq \mathrm{Sym}(m_k)$ is equivalent to $S_j \leq \mathrm{Sym}(m_j)$ for only finitely many $j \in \mathbb{N}$. Moreover, assume that there exists a closed subgroup H of G such that $H \cong G$. We are going to show that $H = G$. Put $N_l = \ker(G \to \widetilde{S}_l)$ and $M_l = \ker(H \to \widetilde{S}_l)$, for $l \in \mathbb{N}$. By Lemma 1.2.5, the N_l's and the M_l's are the unique open normal subgroups of G and H respectively. Hence, for every $n \in \mathbb{N}$ there exists $m = m(n) \in \mathbb{N}$ such that $H \cap N_n = M_m$. We are going to show that $n = m$ for infinitely many n.

By definition of N_n and M_m, $H/M_m = HN_n/N_n \leq G/N_n$, thus $m \leq n$ by comparing sizes. Now, fix $k \in \mathbb{N}$. By hypothesis there exists a label n such that the permutation group $S_k \leq \mathrm{Sym}(m_k)$ is equivalent to $S_n \leq \mathrm{Sym}(m_n)$ and $S_k \leq \mathrm{Sym}(m_k)$ is not equivalent to $S_l \leq \mathrm{Sym}(m_l)$ for every $l > n$. It is easy to see (building explicit composition series) that the set of composition factors of G is $\{S_l \mid l \in \mathbb{N}\}$ and the set of composition factors of N_n is $\{S_l \mid l > n\}$. Thus, by definition of n, the group N_n does not have any composition factor isomorphic to S_k.

Suppose by contradiction that $m < n$, then $M_m = H \cap N_n \geq M_{n-1}$ and $M_{n-1} \leq N_n$. By the choice of n and the definition of M_n, there exist $A \leq_c M_{n-1}$ and $B \lhd_c A$ such that A/B is isomorphic to S_k. In particular, $A \leq_c N_n$ and thus A/B is a composition factor of N_n isomorphic to S_k, a contradiction. Therefore $n = m$ for infinitely many $n \in \mathbb{N}$, it follows that $HN_n = G$ for infinitely many n. Hence $H = G$. $\square$

Remark 5.3.5. Finite non-abelian minimal simple groups have been classified by Thompson in the long series of papers [26, 27]. We report here the classification for completeness. The only finite non-abelian minimal simple groups are: $\mathrm{PSL}_2(2^p)$ for any prime p, $\mathrm{PSL}_2(3^p)$ for any odd prime p, $\mathrm{PSL}_2(p)$ where $p > 3$ and 5 divides $p^2 + 1$, $\mathrm{Sz}(2^p)$ for any odd prime p and $\mathrm{PSL}_3(3)$.

Let $\mathbb{P} = \{p_n \mid n \in \mathbb{N}\}$ be the set of all prime numbers. In particular, the IIE of type $\mathcal{P} = \{\mathrm{PSL}_2(2^{p_n}) \leq \mathrm{Sym}(2^{p_n}+1)\}_{n\in\mathbb{N}}$, with $\mathrm{PSL}_2(2^{p_n})$ acting on the projective line over $\mathbb{F}_{2^{p_n}}$, cannot be embedded as a closed subgroup of itself.

Chapter 6

Hausdorff dimension of IIEs

In this chapter we will study a quantitative property, the Hausdorff dimension, where we are interested to *estimate the size* of closed subgroups of our profinite group. In particular, we are interested in determining the Hausdorff dimension spectrum of IIEs. We refer the reader to Section 2.3.3 for more general details on Hausdorff dimension. This chapter is the result of joint work with Y. Barnea.

6.1 Introduction

Hausdorff dimension in profinite groups has been widely studied, but mostly in the past only the Hausdorff dimension of pro-p groups was considered (there are exceptions to this). A lot has been done on pro-p groups with "small" Hausdorff dimension spectrum such as p-adic analytic groups. What can we say about groups with "big" Hausdorff dimension spectrum? In an unpublished work of Levài and in Giannelli's Masters thesis ([10]) it is shown that the Hausdorff dimension spectrum of $\varprojlim_{n\in\mathbb{N}} C_p$ wr C_{p^n} is $[0,1]$ with respect to a certain power filtration (see Section 2.3.3 for details), but this is only one of a handful of "natural" examples. Here we show that certain infinitely iterated exponentiations have complete Hausdorff dimension spectrum with respect to their unique maximal descending chain of open normal subgroups.

Theorem 6.1.1. *Suppose that there exists a constant $A > 0$ and a natural number M such that $|S_k| \leq |S_{k+1}|^A$ for all $k \geq M$ and let G be the infinitely iterated exponentiation of type $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$. Set $N_k = \ker(G \to \widetilde{S}_k)$ for $k \in \mathbb{N}$ and $\mathcal{G} = \{N_k\}_{k\in\mathbb{N}}$. Then, for every $\alpha \in [0,1]$ there is a closed subgroup H^α of G such that $\dim_{H,\mathcal{G}}(H^\alpha) = \alpha$. In particular $\mathrm{Spec}_{\mathcal{G}}(G) = [0,1]$.*

6.2 Proof of Theorem 6.1.1

Notation 5. For a number $x \in \mathbb{R}$ we will write

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\} \quad \text{and} \quad \{x\} = x - \lfloor x \rfloor.$$

for the integer part and the fractional part of x, respectively.

Before the proof of Theorem 6.1.1 we give a few lemmas that will be used in the proof. The following lemma is straightforward.

Lemma 6.2.1. *Let $G \leq \mathrm{Sym}(n)$ be a permutation group and set S be a subset of $\boldsymbol{n}$. Then S is G-invariant if and only if the complement of S in $\boldsymbol{n}$ is G-invariant.*

The next two lemmas are of analytical flavour.

Lemma 6.2.2. *Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of positive integers with $m_k \geq 2$ for every k. Let $\widetilde{m}_1 = m_1$ and $\widetilde{m}_{k+1} = m_{k+1}^{\widetilde{m}_k}$ for $k \geq 1$. Then*

1. *for every $n \in \mathbb{N}$, $\widetilde{m}_n \geq n$. In particular, $\lim_{n\to\infty} \widetilde{m}_n = \infty$;*
2. $\lim\limits_{n\to\infty} \widetilde{m}_{n-1}/\widetilde{m}_n = 0$;
3. *for every positive constant C there exists $N = N(C) \in \mathbb{N}$ such that $C \cdot \widetilde{m}_{n-1} \leq \widetilde{m}_n$ for every $n \geq N$.*

Proof. 1. By induction. We have $m_1 \geq 2 \geq 1$. Suppose $\widetilde{m}_{n-1} \geq n-1$ for $n \geq 2$, then $m_n^{\widetilde{m}_{n-1}} \geq 2^{\widetilde{m}_{n-1}} \geq 2^{n-1} \geq n$.

2. Let $x = \widetilde{m}_{n-1}$, then

$$0 \leq \frac{x}{m_n^x} \leq \frac{x}{2^x}$$

by hypothesis. Passing to the limits we obtain the claim.

3. Let $x = \widetilde{m}_{n-1}$. The claim follows readily from the fact that the function $2^x/x$ tends to infinity as x does.

$\square$

The previous lemma describes the very fast growth of the function $\widetilde{m}_n : \mathbb{N} \to \mathbb{N}$. The next lemma is a standard result of calculus.

Lemma 6.2.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ two bounded real sequences. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ converges to 0, then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

We are now ready for the proof of Theorem 6.1.1. Remember that, for a sequence of integers $(m_k)_{k\in\mathbb{N}}$, we write $\widetilde{m}_1 = m_1$ and $\widetilde{m}_{n+1} = m_{n+1}^{\widetilde{m}_n}$ for $n \in \mathbb{N}$.

Proof of Theorem 6.1.1. Set $N_k = \ker(G \to \widetilde{S}_k)$ for $k \in \mathbb{N}$ and let $\mathcal{G} = \{N_k\}_{k\in\mathbb{N}}$ be the unique maximal descending chain of open normal subgroups of G. Set $\dim_H = \dim_{H,\mathcal{G}}$. By definition, $\dim_H(\{1\}) = 0$ and $\dim_H(G) = 1$. To prove the theorem it will be sufficient to build subgroups $H_n^\alpha \leq \widetilde{S}_n$ such that H_{n+1}^α projects onto H_n^α and

$$\lim_{n\to\infty} \frac{\log |H_n^\alpha|}{\log |G : N_n|} = \alpha \tag{6.2.1}$$

for every $\alpha \in (0, 1)$. Suppose that $\alpha \in (0, 1)$, then there is a sequence $(p_n/q_n)_{n\in\mathbb{N}}$ of positive rationals such that $\lim_{n\to\infty} p_n/q_n = \alpha$ and $p_n/q_n \geq \alpha$. Remember Notation 2 from Chapter 5 and define $c_1 = m_1$, $o_1 = 1$ and

$$K_2^\alpha = \prod_{i=1}^{\left\lfloor \frac{p_1}{q_1} \cdot m_1 \right\rfloor} S_2^{m_1}(i) \leq \widetilde{S}_2.$$

By definition of exponentiation, it is clear that K_2^α has $c_2 = m_2^{m_1 - \left\lfloor \frac{p_1}{q_1} \cdot c_1 \right\rfloor \cdot o_1} = m_2^{m_1 - \left\lfloor \frac{p_1}{q_1} \cdot m_1 \right\rfloor}$ orbits on $\widetilde{\boldsymbol{m}}_2$. Again an easy computation shows that each orbit of K_2^α has the same cardinality $o_2 = m_2^{\left\lfloor \frac{p_1}{q_1} \cdot c_1 \right\rfloor \cdot o_1}$. Suppose $n \geq 2$ and assume that for $j = 2, \ldots, n$ we already defined subgroups $K_j^\alpha \leq \widetilde{S}_j$ such that:

(1) for $j = 2, \ldots, n$, the subgroup K_j^α has exactly c_j pairwise disjoint orbits $\{O_j(1), \ldots, O_j(c_j)\}$ for its action on $\widetilde{\boldsymbol{m}}_j$,

(2) for $j = 2, \ldots, n$ and $i \in \boldsymbol{c_j}$, $|O_j(i)| = o_j$,

(3) for $j = 2, \ldots, n$, $\widetilde{m}_j = c_j \cdot o_j$ and

(4) for every $j = 2, \ldots, n$, the subset $O_n = \bigcup_{i=1}^{\lfloor p_n/q_n \cdot c_n \rfloor} O_n(i)$ of $\boldsymbol{m_n}^{\widetilde{m}_{n-1}}$ is K_j^α-invariant.

Define a new subgroup K_{n+1}^α of $S_{n+1}^{\widetilde{m}_n}$ by

$$K_{n+1}^\alpha = \prod_{i=1}^{\left\lfloor \frac{p_n}{q_n} \cdot c_n \right\rfloor} S_{n+1}^{\widetilde{m}_n}(O_n(i)).$$

By definition of exponentiation, the number of orbits of K_{n+1}^α on $\widetilde{\boldsymbol{m}}_{n+1}$ corresponds to the number of possible choices for the components of $\boldsymbol{m_{n+1}}^{\widetilde{m}_n}$ that are not moved by K_{n+1}^α, that is $(m_{n+1})^{\widetilde{m}_n - \left\lfloor \frac{p_n}{q_n} \cdot c_n \right\rfloor \cdot o_n}$ and the size of an orbit of K_{n+1}^α on $\widetilde{\boldsymbol{m}}_{n+1}$ will simply be

$$o_{n+1} = \frac{\widetilde{m}_{n+1}}{(m_{n+1})^{\widetilde{m}_n - \left\lfloor \frac{p_n}{q_n} \cdot c_n \right\rfloor \cdot o_n}} = (m_{n+1})^{\left\lfloor \frac{p_n}{q_n} \cdot c_n \right\rfloor \cdot o_n}.$$

Therefore K_{n+1}^α satisfies properties (1)-(3), we will prove that K_{n+1}^α also satisfies property (4). By definition, O_{n+1} is K_{n+1}^α-invariant. Let C be the complement of O_n in $\boldsymbol{m_n}^{\widetilde{m}_{n-1}}$. By definition of K_{n+1}^α and exponentiation, for any orbit O of K_{n+1}^α on $\boldsymbol{m_{n+1}}^{\widetilde{m}_n}$ and for any $c \in C$ there exist $f_c \in \boldsymbol{m_{n+1}}$ such that

$$O = \{\underline{x} \in \boldsymbol{m_{n+1}}^{\widetilde{m}_n} \mid \underline{x}(c) = f_c \text{ for } c \in C\}.$$

It follows that any orbit of K_{n+1}^{α} on $\boldsymbol{m_{n+1}}^{\widetilde{m}_n}$ is identified by its values in the C-coordinates. By Lemma 6.2.1, property (4) yields that C is K_j^{α}-invariant for every $j = 2, \dots, n-1$ and this implies that O_{n+1} is K_j^{α}-invariant. Therefore, property (4) is satisfied by K_{n+1}^{α}.

Set $H_2^{\alpha} = K_2^{\alpha}$ and $H_{n+1}^{\alpha} = H_n^{\alpha} \cdot K_{n+1}^{\alpha}$ for $n \geq 2$. By property (4), it follows that H_n^{α} is a subgroup of $\widetilde{S}_n$ and, by definition of H_n^{α}, it is clear that H_{n+1}^{α} projects onto H_n^{α}. Set $H^{\alpha} = \varprojlim H_n^{\alpha}$.

We will now work on the quantity at the numerator of (6.2.1).

$$\log |H_n^{\alpha}| = \log \left(\prod_{k=2}^{n} |S_k|^{\left\lfloor \frac{p_{k-1}}{q_{k-1}} \cdot c_{k-1} \right\rfloor \cdot o_{k-1}} \right) = \sum_{k=2}^{n} \left\lfloor \frac{p_{k-1}}{q_{k-1}} \cdot c_{k-1} \right\rfloor \cdot o_{k-1} \cdot \log |S_k|.$$

Set

$$a_k = \left(\frac{p_{k-1}}{q_{k-1}} \widetilde{m}_{k-1} \log |S_k| \right)_{k \geq 2} \quad \text{and} \quad b_k = \left(\left\lfloor \frac{p_{k-1}}{q_{k-1}} \cdot c_{k-1} \right\rfloor \cdot o_{k-1} \cdot \log |S_k| \right)_{k \geq 2},$$

then $\sum_{k=2}^{n} b_k$ is less than $\sum_{k=2}^{n} a_k$ for every $n \geq 2$ and $(b_n/a_n)_{n \geq 2}$ tends to 1. We shall now prove that for every real constant $C > 0$ there exists $N(C) \in \mathbb{N}$ such that $Ca_{k-1} \leq a_k$ for every $k \geq N(C)$. Fix $\varepsilon > 0$, since $(p_n/q_n)_{n \in \mathbb{N}}$ is convergent, we have that there exists $N_2 \in \mathbb{N}$ such that $p_{n-1}q_n/q_{n-1}p_n \leq 1 + \varepsilon$ for every $n \geq N_2$. By Lemma 6.2.2 and the hypothesis on $|S_k|$, there exists $N_1 \in \mathbb{N}$ such that

$$\begin{aligned} Ca_{k-1} = C\frac{p_{k-2}}{q_{k-2}} \widetilde{m}_{k-2} \log |S_{k-1}| &= C \left(\frac{q_{k-1}}{p_{k-1}} \frac{p_{k-2}}{q_{k-2}} \right) \frac{p_{k-1}}{q_{k-1}} \widetilde{m}_{k-2} \log |S_{k-1}| \\ &\leq C(1+\varepsilon)A\widetilde{m}_{k-2} \frac{p_{k-1}}{q_{k-1}} \log |S_k| \leq \widetilde{m}_{k-1} \frac{p_{k-1}}{q_{k-1}} \log |S_k| = a_k \end{aligned} \tag{6.2.2}$$

for every $k \geq N(C) = \max\{N_1, N_2, M\}$. Next, we will use (6.2.2) to show by induction on n that $\sum_{k=N(2)}^{n} a_k \leq 2a_n$ for $n \geq N(2)$. It is clear that $a_{N(2)} \leq 2a_{N(2)}$. Suppose by inductive hypothesis that $\sum_{k=N(2)}^{n-1} a_k \leq 2a_{n-1}$, then $\sum_{k=N(2)}^{n} a_k \leq 2a_{n-1} + a_n \leq 2a_n$, for $n \geq N(2)$. Summing up we can now show that $\sum_{k=2}^{n} a_k \leq 3a_n$ for all n large enough. By (6.2.2), $(a_k)_{k \geq 2}$ is increasing and $N(2)a_{N(2)} \leq N(2)a_{n-1} \leq a_n$ for all $n \geq \max\{N(2), N(N(2))\}$. Therefore we have $\sum_{k=2}^{n} a_k \leq N(2)a_{N(2)} + 2a_n \leq 3a_n$ for every $n \geq \max\{N(2), N(N(2))\}$. The above discussion yields

$$0 \leq \frac{1}{a_n} \cdot \sum_{k=2}^{n-1} a_k \leq \frac{3a_{n-1}}{a_n} \leq \frac{3a_{n-1}}{a_n}$$

for every n large enough. Hence, by Lemma 6.2.2 and Lemma 6.2.3, it follows that

$$\lim_{n\to\infty} \frac{a_{n-1}}{a_n} = \lim_{n\to\infty} \frac{\widetilde{m}_{n-2} \cdot \frac{p_{n-2}}{q_{n-2}} \cdot \log|S_{n-1}|}{\widetilde{m}_{n-1} \cdot \frac{p_{n-1}}{q_{n-1}} \cdot \log|S_n|} = \lim_{n\to\infty} \frac{\widetilde{m}_{n-2}}{\widetilde{m}_{n-1}} \cdot d_n = 0$$

where $\{d_n\}_{n\in\mathbb{N}}$ is a bounded sequence by hypothesis. In particular, we have

$$0 \leq \frac{1}{b_n} \sum_{k=2}^{n-1} b_k \cdot \frac{b_n}{a_n} \leq \frac{1}{a_n} \cdot \sum_{k=2}^{n-1} a_k \leq \frac{3a_{n-1}}{a_n}$$

and, since $(b_n/a_n)_{n\geq 2}$ tends to 1 for n that tends to infinity, it follows that

$$\lim_{n\to\infty} \sum_{k=2}^{n} \frac{b_k}{b_n} = 1. \tag{6.2.3}$$

Set $\widetilde{m}_0 = 1$. It possible to reproduce exactly the above steps with the sequence $(\widetilde{m}_k \log|S_{k+1}|)_{k\geq 0}$ instead of $(a_k)_{k\geq 2}$ to show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\widetilde{m}_k \log|S_{k+1}|}{\widetilde{m}_n \log|S_{n+1}|} = 1. \tag{6.2.4}$$

Next, we will deal with the denominator of (6.2.1). We notice that

$$\frac{\left\lfloor \frac{p_n}{q_n} \cdot c_n \right\rfloor \cdot o_n}{\widetilde{m}_n} = \frac{\frac{p_n}{q_n} \cdot c_n \cdot o_n - \left\{ \frac{p_n}{q_n} \cdot c_n \right\} \cdot o_n}{\widetilde{m}_n} = \frac{p_n}{q_n} + \frac{\left\{ \frac{p_n}{q_n} \cdot c_n \right\} \cdot o_n}{\widetilde{m}_n}$$

and, by definition of c_n and Lemma 6.2.2, $c_n \geq \widetilde{m}_n^{\alpha} \geq n^{\alpha}$ which tends to infinity as n does. Moreover, remember that $o_n/\widetilde{m}_n = 1/c_n$. Therefore

$$\lim_{n\to\infty} \frac{\left\lfloor \frac{p_n}{q_n} \cdot c_n \right\rfloor \cdot o_n}{\widetilde{m}_n} = \lim_{n\to\infty} \frac{p_n}{q_n} = \alpha. \tag{6.2.5}$$

Remember that that we set $\widetilde{m}_0 = 1$. Let us now summarize everything in the computation of (6.2.1):

$$\frac{\log|H_n^\alpha|}{\log|G:N_n|} = \frac{\sum\limits_{k=2}^{n} \left\lfloor \frac{p_{k-1}}{q_{k-1}} \cdot c_{k-1} \right\rfloor \cdot o_{k-1} \cdot \log|S_k|}{\sum\limits_{k=1}^{n} \widetilde{m}_{k-1} \cdot \log|S_k|} =$$

$$= \frac{\frac{1}{b_n} \cdot \sum\limits_{k=2}^{n} b_k}{1/(\widetilde{m}_{n-1} \cdot \log|S_n|) \cdot \sum\limits_{k=1}^{n} \widetilde{m}_{k-1} \cdot \log|S_k|} \cdot \frac{\left\lfloor \frac{p_{n-1}}{q_{n-1}} \cdot c_{n-1} \right\rfloor \cdot o_{n-1} \cdot \log|S_n|}{\widetilde{m}_{n-1} \cdot \log|S_n|}.$$

By (6.2.3) and (6.2.4), the first factor of the product tends to 1 for n that tends to infinity and, by (6.2.5), the limit of the second factor is α for n that tends to infinity. Therefore $\dim_H(H^\alpha) = \alpha$, as expected. □

Appendix A

On Subgroup growth of IIEs

A.1 Introduction

The results in this section are mainly contained in the excellent book on subgroup growth [15] by Lubotzky and Segal.

Definition A.1.1. *Let G be a profinite group and $n \in \mathbb{N}$. Define the number*

$$s_n(G) = |\{H \leq_o G \mid |G : H| \leq n\}|.$$

*Let $f : \mathbb{N} \to \mathbb{R}$ be a function. We say that the profinite group G has **subgroup growth type** f if there exist $a, b \in \mathbb{R}$ such that $a, b > 0$ and*

$$\begin{cases} s_n(G) \leq f(n)^a & \text{for every } n \\ s_n(G) \geq f(n)^b & \text{for infinitely many } n \end{cases}.$$

The subgroup growth type of a profinite group G is the "best possible upper bound" for the function $s_n(G) : \mathbb{N} \to \mathbb{N}$.

We have already seen in Lemma 2.3.8 that the subgroup growth type of a finitely generated profinite group G is at most $n \cdot (n!)^{\mathrm{d}(G)-1}$. This bound is best possible as shown in [15, Chapter 2], in fact, the free profinite group on d generators has subgroup growth type $n \cdot (n!)^{\mathrm{d}(G)-1}$.

Is it possible to classify the profinite groups that have specific a type of subgroup growth function? This question led to a beautiful theory, developed by many mathematicians over the past thirty years. One of the main results of this theory is the characterization of the profinite groups with *polynomial subgroup growth*. We say that a profinite group G has **polynomial subgroup growth** if there exists $c \in \mathbb{N}$ such that $s_n(G) \leq n^c$ for every $n \in \mathbb{N}$. The **rank** of a profinite group G is the integer defined by:

$$\mathrm{rk}(G) = \sup\{\mathrm{d}(H) \mid H \leq_o G\}.$$

Theorem A.1.2. ([15, Theorem 5.1]) *Let G be a finitely generated profinite group. Then G has polynomial subgroup growth if and only if G is virtually soluble of finite rank.*

Another question that we might ask is whether all the positive and increasing functions from $\mathbb{N}$ to $\mathbb{R}$ are the subgroup growth type of a certain profinite group. This question is known as the "Subgroup Growth Gap Problem".

The answer is substantially "yes" and we are going to cite from [15, Chapter 13.3] some definitions and lemmas that will be used in Chapter 5.

Definition A.1.3. *Let r, t be positive constants. Let $(S_k)_{k\in\mathbb{N}}$ be a sequence of finite groups satisfying the following conditions for all k:*

*N.*1 $|S_k| \geq |S_{k-1}|$;

*N.*2 $\mathrm{rk}(S_k) \leq r$;

*N.*3 *S_k contains an elementary abelian subgroup E_k such that $|S_k| \leq |E_k|^t$;*

*N.*4 *if μ_k is the minimal index of any proper subgroup in S_k, then $\mu_k \geq \mu_{k-1}$ and $|S_k| \leq \mu_k^t$.*

*N.*5 *and* $\lim_{n\to\infty} \mu_k = \infty$.

*A sequence $\mathcal{S}$ satisfying conditions N.1-N.5 above is said **nice** with constants r and t.*

Suppose that each S_k has a transitive and faithful action on m_k points, so we can embed S_k as a transitive subgroup of $\mathrm{Sym}(m_k)$. Put $\widehat{m}_1 = m_1$ and $\widehat{S}_1 = S_1$, $\widehat{m}_{k+1} = m_{k+1} \cdot \widehat{m}_k$ and $\widehat{S}_{k+1} = S_{k+1} \wr \widehat{S}_k \leq \mathrm{Sym}(\widehat{m}_{k+1})$ for $k \in \mathbb{N}$. Consider the infinitely iterated permutational wreath product $\varprojlim \widehat{S}_k$.

Lemma A.1.4. ([15, Lemma 13.3.1]) *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a nice sequence of permutation groups with constants r and t. If $n < \mu_{k+1}$ then $s_n(\varprojlim \widehat{S}_k) \leq n^{2tr\widehat{m}_k}$.*

Lemma A.1.5. ([15, Lemma 13.3.2]) *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a nice sequence of permutation groups with constants r and t. For $n = |S_{k+1}|^{2\widehat{m}_k}$ we have $s_n(\varprojlim \widehat{S}_k) \geq n^{\widehat{m}_k/8t}$.*

Therefore, for a nice sequence $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ of permutation groups with constants r and t, we have that the subgroup growth type of $\varprojlim \widehat{S}_k$ is $n^{\widehat{m}_{l(n)}}$ where $l(n) = \min\{k \in \mathbb{N} \mid n < \mu_{k+1}\}$. Notice that, by condition N.5, $l(n)$ exists for every natural number n.

In [15, Chapter 13.3] the authors actually prove something more. We will say that a function $f : \mathbb{N} \to \mathbb{R}$ **grows gently** if f is unbounded, non-decreasing, positive and there exist a positive constants $A, N \in \mathbb{R}$ such that

$$f(x^{\log x}) \leq Af(x)$$

for all $x \geq N$.

Theorem A.1.6. ([15, Theorem 13.3.4]) *Let $f : \mathbb{N} \to \mathbb{R}$ be a gently growing function. Then there exists a a nice sequence of permutation groups $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ such that the infinitely iterated permutational wreath product $\varprojlim \widehat{S}_k$ has subgroup grow type $n^{f(n)}$.*

[15, Theorem 13.3.4] solves the "Subgroup Growth Gap Problem" for gently growing functions. It is possible to show, via a very different construction, that there are no gaps also for non-gently growing functions (see [15, Chapter 13.2]).

A.2 Subgroup growth of IIEs obtained from nice sequences

In this section we concentrate on the subgroup growth of infinitely iterated exponentiations of type $\mathcal{S}$, where $\mathcal{S}$ is a nice sequence of finite non-abelian simple permutation groups. The content of this section is an easy application of the methods of [15, Chapter 13.3]. The next result is a basic tool that will be used in the following proofs.

Proposition A.1. ([15, Proposition 1.9.1]) *Let c, r be two constants such that $(c-1)r \geq 2$. Suppose that $1 = G_k \lhd G_{k-1} \lhd \ldots \lhd G_0 = G$. Put $Q_i = G_{i-1}/G_i$ and suppose that $\mu(Q_i)^c \geq |Q_i|$ and $\mathrm{rk}(Q_i) \leq r$ for all i.*

Then, for every n,

$$s_n(G) \leq n^{kcr}.$$

Remember that for a sequence $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ of permutation groups we set $\widetilde{m}_0 = 1$, $\widetilde{m}_1 = m_1$, $\widetilde{S}_1 = S_1$, $\widetilde{m}_{k+1} = m_{k+1}^{\widetilde{m}_k}$ and $\widetilde{S}_{k+1} = S_{k+1} \circledR \widetilde{S}_k \leq \mathrm{Sym}(\widetilde{m}_{k+1})$ for $k \in \mathbb{N}$.

Lemma A.1. *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a nice sequence of permutation groups with constants r and t. If $n < \mu_{k+1}$ then $s_n(\varprojlim \widetilde{S}_k) \leq n^{3tr\widetilde{m}_k}$.*

Proof. The proof follows the steps of [15, Lemma 13.3.1]. Using a method very similar to the one used in the proof of Theorem 6.1.1 it is possible to show that

$$\sum_{i=0}^{k} \widetilde{m}_i < 3\widetilde{m}_k$$

for every k large enough. If $i > k$ then, by hypothesis, $\mu_i \geq \mu_{k+1} > n$ so $S_i^{\widetilde{m}_{i-1}}$ has no proper subgroup of index n or less. Therefore every subgroup of index less to n contains $S_i^{\widetilde{m}_{i-1}}$, and hence

$$s_n(\widetilde{S}_{i+1}) = s_n(\widetilde{S}_i) = \ldots = s_n(\widetilde{S}_{k+1}).$$

Now, $\widetilde{S}_{k+1}$ has a subnormal series of length $\sum_{i=0}^{k} \widetilde{m}_i < 3\widetilde{m}_k$, whose factors S_j satisfy conditions N.2, N.4 and N.5 of a nice sequence of groups. Applying [15, Proposition 1.9.1], we deduce that

$$s_n(\widetilde{S}_{k+1}) \leq n^{3tr\widetilde{m}_k}.$$

$\square$

Lemma A.2. *Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a nice sequence of permutation groups with constants r and t. For $n = |S_{k+1}|^{2\widetilde{m}_k}$ we have $s_n(\varprojlim \widetilde{S}_k) \geq n^{\widetilde{m}_k/8t}$.*

Proof. Again we follow the proof of [15, Lemma 13.3.2]. By definition of $\widetilde{S}_{k+1}$ and by Lemma 6.2.2, it is clear that

$$|\widetilde{S}_{k+1}| = \prod_{i=0}^{k} |S|_{i+1}^{\widetilde{m}_i} \leq |S_{k+1}|^{\sum_{i=0}^{k} \widetilde{m}_i} < |S_{k+1}|^{2\widetilde{m}_k} = n.$$

On the other hand, by property N.3 of a nice sequence, $\widetilde{S}_{k+1}$ contains $S_{k+1}^{\widetilde{m}_k}$ which in turn contains the elementary abelian subgroup $E_k^{\widetilde{m}_k}$. Suppose that $E_k = C_p^e$, then, again by property N.3 of a nice sequence, it follows that $p^{et} \geq |S_{k+1}|$. Moreover, $E_k^{\widetilde{m}_k}$ has at least $p^{e^2\widetilde{m}_k^2/4}$ subgroups. Since $|\widetilde{S}_{k+1}| < n$, we have that

$$s_n(\varprojlim \widehat{S}_k) \geq s_n(\widetilde{S}_{k+1}) = s_{|\widetilde{S}_{k+1}|}(\widetilde{S}_{k+1}) \geq p^{e^2\widetilde{m}_k^2/4} \geq |S_{k+1}|^{m^2/4t} = n^{\widetilde{m}_k/8t}.$$

$\square$

Let $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k\in\mathbb{N}}$ be a nice sequence of finite non-abelian simple permutation groups. By Lemma A.1 and Lemma A.2, the infinitely iterated exponentiation G of type $\mathcal{S}$ has subgroup growth type $n^{\widetilde{m}_{l(n)}}$ where $l(n) = \min\{t \mid n < \mu_{t+1}\}$.

Next we will show that it is possible to tune the parameters of the sequence $\mathcal{S}$ to archive a wide range of subgroup growth types of IIEs. Remember that we say that a function $f : \mathbb{N} \to \mathbb{R}$ grows gently if f is unbounded, non-decreasing, positive and there exist a positive constants $A, N \in \mathbb{R}$ such that $f(x^{\log x}) \leq Af(x)$ for all $x \geq N$.

Lemma A.3. ([15, Lemma 13.3.3]) *Let $f : \mathbb{N} \to \mathbb{R}$ be a gently growing function. Then there exist constants $B, C \in \mathbb{R}$ with $B, C > 0$ such that for every integer $m \geq C$ there exists a prime $p > m$ with*

$$f(p) \geq 2m \geq Bf(p^m).$$

Let $f : \mathbb{N} \to \mathbb{R}$ be a gently growing function and let B and C be the constants defined in the previous lemma. We will now choose a sequence of primes $\{p_k\}_{k\in\mathbb{N}}$ as in [15, Chapter 13.3]. Let $p_1 \geq \max\{12, C\}$ be a prime such that $f(p_1) \geq 12$. Suppose we chose the primes $p_1, \ldots, p_{k-1}$, set $m_i = 1 + p_i$ for $i \in \boldsymbol{k-1}$ and let $m = 6\widetilde{m}_{k-1}$. Then take $p_k = p$ where $p > m$ is the prime whose existence is guaranteed by [15, Lemma 13.3.3]. Thus

$$f(p_k) \geq 12\widetilde{m}_{k-1} \geq Bf(p_k^{6\widetilde{m}_{k-1}})$$

for every $k \in \mathbb{N}$. Let $S_k = \mathrm{PSL}_2(\mathbb{F}_{p_k})$, then $\mathrm{rk}(S_k) = 2$, $\mu_k = 1 + p_k = m_k$ and

$$p_k^2 < |S_k| = \frac{p_k(p_k^2 - 1)}{2} < |E_k|^3 < p_k^3,$$

where E_k is a subgroup of order p_k. Hence $\mathcal{P} = \{P_k = \mathrm{PSL}_2(\mathbb{F}_{p_k}) \leq \mathrm{Sym}(p_k + 1)\}_{k\in\mathbb{N}}$ is a nice sequence with constants $r = 2$ and $t = 3$.

Proposition A.2. *Let $f : \mathbb{N} \to \mathbb{R}$ be a gently growing function and let $\mathcal{P} = \{P_k = \mathrm{PSL}_2(\mathbb{F}_{p_k}) \leq \mathrm{Sym}(p_k + 1)\}_{k\in\mathbb{N}}$ as above. Then the infinitely iterated exponentiation $\varprojlim \widetilde{P}_k$ has subgroup grow type $n^{f(n)}$.*

This result is very similar to [15, Theorem 13.3.4], where it is shown that the same conclusion holds for the infinitely iterated permutational wreath product of the sequence $\mathcal{P}$.

On the other hand, condition N.5 is not always satisfied by a sequence $\mathcal{S}$, i.e. by the constant sequence $\mathcal{A} = (\mathrm{Alt}(5) \leq \mathrm{Sym}(5))_{k\in\mathbb{N}}$. The study of the subgroup growth type of the IIE of type $\mathcal{C}$ is closely related to the study of the behaviour of the function $s_n(\mathrm{Alt}(5)^k)$ for k a fixed integer. The only estimates of this quantity are as follows.

Corollary A.1. ([15, Corollary 1.9.2]) *Let S be a finite non-abelian simple group, let k be an integer and set $G = S^k$. Then there exists a constant $c > 0$ such that*

$$s_n(G) \leq n^{c \cdot \mathrm{rk}(G)^2}$$

for every $n \in \mathbb{N}$.

The previous corollary gives an upper bound but this is not sharp and it does not provide a lower bound. We strongly suspect that the subgroup growth type of the direct power of a finite non-abelian simple group is "very fast" but we did not carry out all the computation to confirm this.

Appendix B

Open problems

We list here a series of problems and conjectures that could lead the author's future research.

Problem 1. Let $\mathcal{S} = \{S_k\}_{k\in\boldsymbol{n}}$ be a sequence of non-abelian simple transitive subgroups of $\mathrm{Sym}(m_k)$ such that for every $k \in \boldsymbol{n}$ and all $i, j \in \boldsymbol{m}_k$ $\mathrm{St}_{S_k}(i) \neq \mathrm{St}_{S_k}(j)$. By Theorem 2.2.8, $\mathrm{d}(\varprojlim \widetilde{S}_n) = 2$.

Find explicitly two generators for the IIE of type $\mathcal{S}$.

Problem 2. Does there exist a sequence $\mathcal{S} = \{S_k\}_{k\in\boldsymbol{n}}$ of 2-generated perfect transitive subgroups of $\mathrm{Sym}(m_k)$ such that the infinitely iterated exponentiation of type $\mathcal{S}$ is not 2-generated? By Corollary 3.2.8, this would imply that the IIE of type $\mathcal{S}$ is 3-generated.

Problem 3. Let d be an integer. Let $\mathcal{S} = \{S_k\}_{k\in\boldsymbol{n}}$ be a sequence of d-generated perfect transitive subgroups of $\mathrm{Sym}(m_k)$. Define inductively the "reversed iterated exponentiation": $\mathring{m}_1 = m_1$ and $\mathring{S}_1 = S_1 \leq \mathrm{Sym}(\mathring{m}_1)$, $\mathring{m}_{n+1} = \mathring{m}_n^{m_{n+1}}$ and $\mathring{S}_{n+1} = \mathring{S}_n \textcircled{\wr} S_{n+1} \leq \mathrm{Sym}(\mathring{m}_{n+1})$ for $n \in \mathbb{N}$.

Is $\mathrm{d}(\mathring{S}_n)$ bounded above by a constant (depending on d) for every $n \in \mathbb{N}$? If this were the case any infinitely iterated mixed wreath product of type $(\mathcal{S}, (k_n)_{n\in\mathbb{N}})$ would be finitely generated.

Problem 4. Does the infinitely iterated exponentiation of the sequence $\mathcal{A} = \{\mathrm{Alt}(5) \leq \mathrm{Sym}(5)\}_{k\in\mathbb{N}}$ have infinite lower rank?

Problem 5. (Conjecture 1) Find a generalised Wilson group of lower rank r for every $r \in \mathbb{N} \cup \{\infty\}$.

Problem 6. (Conjecture 2) Prove that every generalised Wilson group is not finitely presentable.

Problem 7. Let $\mathcal{X}$ be a set of finite simple groups. Does there exist an infinitely iterated exponentiation E with set of composition factors $\mathcal{X}$ such that every finitely generated profinite group with composition factors in $\mathcal{X}$ can be embedded in E?

Problem 8. Let $\mathcal{S} = \{S_k \leq \mathrm{Sym}(m_k)\}_{k \in \mathbb{N}}$ be a sequence of finite non-abelian simple non-trivial permutation groups. Suppose that there exist a fixed natural number N such that for any $k \geq N$, there are infinitely many $j \in \mathbb{N}$ such that $S_k \leq \mathrm{Sym}(m_k)$ is a sub-permutation group of $S_j \leq \mathrm{Sym}(m_j)$. Let G be the infinitely iterated exponentiation of type $\mathcal{S}$. By Theorem 5.1.5, there exists a proper closed subgroup H of G isomorphic to G.

- Determine the possible Hausdorff dimensions of proper closed subgroups of G isomorphic to G.
- Is it possible to find a proper *open* subgroup of G isomorphic to G?

Problem 9. Suppose that there exists a constant $A > 0$ and a natural number M such that $|S_k| \leq |S_{k+1}|^A$ for all $k \geq M$ and let G be the infinitely iterated exponentiation of type $\mathcal{S} = (S_k \leq \mathrm{Sym}(m_k))_{k \in \mathbb{N}}$. Set $N_k = \ker(G \to \widetilde{S}_k)$ for $k \in \mathbb{N}$ and $\mathcal{G} = \{N_k\}_{k \in \mathbb{N}}$. By Theorem 6.1.1, for every $\alpha \in [0,1]$ there is a closed subgroup H^α of G such that $\dim_{H,\mathcal{G}}(H^\alpha) = \alpha$.

Is it possible to choose H^α finitely generated for every $\alpha \in [0,1]$?

Problem 10. Calculate the subgroup growth type of the infinitely iterated exponentiation of type $\mathcal{A} = \{\mathrm{Alt}(5) \leq \mathrm{Sym}(5)\}_{k \in \mathbb{N}}$.

Bibliography

[1] Y. Barnea and M. Larsen. Random generation in semisimple algebraic groups over local fields. *J. Algebra*, 271(1):1–10, 2004.

[2] Yiftach Barnea and Aner Shalev. Hausdorff dimension, pro-p groups, and Kac-Moody algebras. *Trans. Amer. Math. Soc.*, 349(12):5073–5091, 1997.

[3] Meenaxi Bhattacharjee. The probability of generating certain profinite groups by two elements. *Israel J. Math.*, 86(1-3):311–329, 1994.

[4] Ievgen V. Bondarenko. Finite generation of iterated wreath products. *Arch. Math. (Basel)*, 95(4):301–308, 2010.

[5] Rachel Camina. The Nottingham group. In *New horizons in pro-p groups*, volume 184 of *Progr. Math.*, pages 205–221. Birkhäuser Boston, Boston, MA, 2000.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups.* Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[7] John D. Dixon and Brian Mortimer. *Permutation groups*, volume 163 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1996.

[8] Paul Erdös. A Theorem of Sylvester and Schur. *J. London Math. Soc.*, S1-9(4):282, 1934.

[9] Mikhail Ershov and Andrei Jaikin-Zapirain. Groups of positive weighted deficiency and their applications. *J. Reine Angew. Math.*, 677:71–134, 2013.

[10] Eugenio Giannelli. Gruppi profiniti e dimensione di hausdorff. Master's thesis, Università degli Studi di Firenze, Italy, 2011.

[11] P. Hall. Eulerian functions of a group. *Quart. J. Math.*, 7:134–151, 1936.

[12] P. Hall. Wreath powers and characteristically simple groups. *Proc. Cambridge Philos. Soc.*, 58:170–184, 1962.

[13] L. A. Kalužnin, M. H. Klin, and V. Ī. Suščans′kiĭ. Exponentiation of permutation groups. I. *Izv. Vyssh. Uchebn. Zaved. Mat.*, pages 26–33, 1979.

[14] Alexander Lubotzky and Avinoam Mann. Powerful p-groups. II. p-adic analytic groups. *J. Algebra*, 105(2):506–515, 1987.

[15] Alexander Lubotzky and Dan Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.

[16] Andrea Lucchini and Federico Menegazzo. Generators for finite groups with a unique minimal normal subgroup. *Rend. Sem. Mat. Univ. Padova*, 98:173–191, 1997.

[17] Avinoam Mann. Positively finitely generated groups. *Forum Math.*, 8(4):429–459, 1996.

[18] G. A. Miller. Possible orders of two generators of the alternating and of the symmetric group. *Trans. Amer. Math. Soc.*, 30(1):24–32, 1928.

[19] Peter M. Neumann. Some questions of Edjvet and Pride about infinite groups. *Illinois J. Math.*, 30(2):301–316, 1986.

[20] C. Wayne Patty. *Foundations of topology*. Jones and Bartlett Publishers, Boston, MA, second edition, 2009.

[21] Martyn Quick. Probabilistic generation of wreath products of non-abelian finite simple groups. II. *Internat. J. Algebra Comput.*, 16(3):493–503, 2006.

[22] E. W. Read. On the Schur multiplier of a wreath product. *Illinois J. Math.*, 20(3):456–466, 1976.

[23] Colin D. Reid. Inverse system characterizations of the (hereditarily) just infinite property in profinite groups. *Bull. Lond. Math. Soc.*, 44(3):413–425, 2012.

[24] Dan Segal. The finite images of finitely generated groups. *Proc. London Math. Soc. (3)*, 82(3):597–613, 2001.

[25] Aner Shalev. Lie methods in the theory of pro-p groups. In *New horizons in pro-p groups*, volume 184 of *Progr. Math.*, pages 1–54. Birkhäuser Boston, Boston, MA, 2000.

[26] John G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. *Bull. Amer. Math. Soc.*, 74:383–437, 1968.

[27] John G. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. II – VI. *Pacific J. Math.*, 33:451–536, 1970; ibid. 39:483–534, 1971; ibid. 48:511–592, 1973; ibid. 50:215–297, 1974; ibid. 51:573–630, 1974.

[28] Matteo Vannacci. Finite generation of iterated wreath products in product action. *to appear on Arch. der Math.*

[29] James Wiegold. Growth sequences of finite groups. III. *J. Austral. Math. Soc. Ser. A*, 25(2):142–144, 1978.

[30] John S. Wilson. Groups with every proper quotient finite. *Proc. Cambridge Philos. Soc.*, 69:373–391, 1971.

[31] John S. Wilson. *Profinite groups*, volume 19 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1998.

[32] John S. Wilson. Large hereditarily just infinite groups. *J. Algebra*, 324(2):248–255, 2010.