Representations of symmetric groups

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Declaration of Autorship

I, Eugenio Giannelli, hereby declare that the work presented in this thesis is the result of original research carried out by myself, in collaboration with others. Where I have consulted the work of others, this is always clearly stated.

Signed:

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Abstract

In this thesis we study the ordinary and the modular representation theory of the symmetric group. In particular we focus our work on different important open questions in the area.

1. Foulkes' Conjecture

In Chapter 2 we focus our attention on the long standing open problem known as Foulkes' Conjecture. We use methods from character theory of symmetric groups to determine new information on the decomposition into irreducible characters of the Foulkes character.

2. Foulkes modules and decomposition numbers

The decomposition matrix of a finite group in prime characteristic p records the multiplicities of its p-modular irreducible representations as composition factors of the reductions modulo p of its irreducible representations in characteristic zero.

In Chapter 3 we give a combinatorial description of certain columns of the decomposition matrices of symmetric groups in odd prime characteristic. The result applies to blocks of arbitrarily high p-weight. It is obtained by studying the p-local structure of certain twists by the sign character of the Foulkes module $H^{(2^n)}$. This is joint work with Mark Wildon.

In Chapter 4 we extend the results obtained in Chapter 3 on the modular structure of $H^{(2^n)}$, to the entire class of Foulkes modules $H^{(a^n)}$ defined over any field $\mathbb{F}$ of odd prime characteristic p such that $a < p \leqslant n$. In particular we characterize the vertices of all the indecomposable summands of $H^{(a^n)}$.

3. Vertices of Specht and simple modules

In Chapter 5 we study the vertices of indecomposable Specht modules for symmetric groups. For any given indecomposable non-projective Specht module, the main theorem of the chapter describes a large p-subgroup contained in its vertex.

In Chapter 6 we consider the vertices of simple modules for the symmetric groups in prime characteristic p. The main theorem of the chapter completes the classification of the vertices of simple $\mathbb{F}S_n$-modules labelled by hook partitions. This is joint work with Susanne Danz.

Notation

Throughout p denotes a prime number and $\mathbb{F}_p$ denotes the finite field of size p. Unless otherwise stated $\mathbb{F}$ denotes an arbitrary field of prime characteristic p. Given a finite group G, we denote by $\mathbb{F}G$ the usual group algebra. All $\mathbb{F}G$-modules are intended to be finite dimensional right modules, homomorphisms between modules are composed from right to left.

Throughout n denotes a positive integer, and S_n denotes the symmetric group on the set $\{1, 2, \ldots, n\}$. We write S_A for the symmetric group on the non-empty subset $A \subseteq \{1, 2, \ldots, n\}$. If $n \in \mathbb{N}$ and p^a is the highest power of p dividing n then we write $(n)_p = p^a$ and we say that the p-part of n is p^a.

If $\sigma \in S_n$ is a permutation fixing exactly $n - r$ elements of $\{1, 2, \ldots, n\}$ then we say σ has *support* of size r and write $|\operatorname{supp} \sigma| = r$. In particular, the *support* of σ is the subset of $\{1, 2, \ldots, n\}$ consisting of the r elements that are not fixed by σ. We also use the obvious extension of this definition to subgroups of S_n.

Finally we write $\lambda \vdash n$ to indicate that λ is a partition of n, that is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$ and $\sum_{i=1}^{k} \lambda_i = n$.

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Chapter 1

Introduction and background results

1.1 Introduction and overview of main results

Let G be a finite group, $\mathbb{F}$ a field and n a natural number. A representation of G over $\mathbb{F}$ of dimension n is an homomorphism ϕ from G to the group $\mathrm{GL}_n(\mathbb{F})$ of invertible linear maps over an $\mathbb{F}$-vector space of dimension n.

The representation theory of finite groups started with the mail exchange between Frobenius and Dedekind in Spring 1896. In the following years, many mathematicians such as Burnside, Schur, Noether and others laid the foundations of the *ordinary* representation theory (i.e. $\mathbb{F} = \mathbb{C}$). Given a group G, a complex vector space V and a representation

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

we have that ρ is uniquely determined by its *ordinary character*

$$\chi : G \longrightarrow \mathbb{C}, \quad \chi(g) = \mathrm{tr}(\rho(g)).$$

We denote by $\mathrm{Irr}(G)$ the set of all the *irreducible characters* of G. An important consequence of the fundamental Maschke's theorem is that every ordinary character of G can be uniquely expressed as a linear combination of irreducible characters with coefficients in $\mathbb{N}$. The new ideas contained in this theory had a spectacular impact on classical group theory. Problems considered not achievable were solved with the new tools offered by representation and character theory. A famous example is

Burnside's $p^a q^b$ Theorem.

In 1935, Richard Brauer initiated the study of *modular* representation theory (i.e. $\mathrm{char}(\mathbb{F}) = p > 0$). The situation, in this setting, is much more delicate in respect to the ordinary case. A very important contribution of Brauer himself is the introduction of *Brauer characters.* Unfortunately two different (non irreducible) representations over the field $\mathbb{F}$ of prime characteristic p, can afford the same Brauer character. More precisely, the Brauer character of an $\mathbb{F}G$-module M determines the simple composition factors of M and their multiplicities, but does not uniquely identify the isomorphism class of M.

Symmetric groups have always played a central role in group theory. In representation theory of finite groups they provide evidence for some of the important modern *local-global conjectures* such as Alperin Weight conjecture, Donovan conjecture, McKay conjecture and Broué's abelian defect group conjecture. Moreover, the importance of the study of the representations of symmetric groups transcends group theory; in fact this topic has an important impact on the investigation of the representation theory of related algebras like the Brauer algebra, the partition algebra, the Hecke algebra of the symmetric group and the recently discovered Khovanov-Lauda-Rouquier algebras (KLR algebras). Without adding any further detail about the above listed objects, it is important to mention that these algebras are different generalizations of the group algebra of the symmetric group, have a wide range of applications in many distinct branches of mathematics (from combinatorics to quantum mechanics) and many of the ideas and the sophisticated techniques used to approach the study of them are already encoded in the classical representation theory of the symmetric group.

Despite the numerous generalisations of the algebra of the symmetric group, there are many fundamental open problems at the level of $\mathbb{F}S_n$. Motivated by some of those questions, in this thesis we focus on the study of both the ordinary and the modular representation theory of the symmetric groups.

1.1.1 Character theory

The ordinary representation theory of the symmetric group S_n was extensively studied since 1896. Remarkable results are now well known and largely understood. For example, there is a well defined bijection between simple $\mathbb{C}S_n$-modules and partitions of n. The simple modules are known as *Specht modules* and the Specht module corresponding to the partition λ of n is denoted by S^λ. We write χ^λ for the character afforded by S^λ. The *hook length formula* describes combinatorially the dimension of any simple $\mathbb{C}S_n$-module and the *Murnaghan-Nakayama formula* allows us to calculate explicitly the character table. Unfortunately (or not!) we are far from having an answer to all the possible questions. There are many interesting open problems that seem definitely out of reach at the moment.

In this work, in particular in Chapter 2, we will focus our attention on the long-standing open problem known as Foulkes' Conjecture. Stated in 1950 by H.O. Foulkes in [26] as a problem in invariant theory, significant advances towards a proof of Foulkes' Conjecture have been recently achieved via character theory of symmetric groups. Apart from the obvious appeal of a sixty years old conjecture, the study of this problem is interesting for the impact that the knowledge of the ordinary structure of the Foulkes module has on other problems in different areas such as modular representation theory and group theory.

We write $\phi^{(a^b)}$ for the ordinary character afforded by the Foulkes module of parameters a and b (the reader may wish to refer ahead to Section 1.4 for the definition of the Foulkes module). The determination of the decomposition into irreducible characters of $\phi^{(a^b)}$ is of central importance in the study of Foulkes' Conjecture. Our main contribution on these lines is Theorem 2.1.3. There we determine a wide subset of the irreducible characters of the symmetric group appearing with zero multiplicity in $\phi^{(a^b)}$. In particular we give sufficient conditions on the shape of a partition λ of ab in order to have the inner product between the characters $\phi^{(a^b)}$ and χ^λ equal to zero. Our results give a partial answer to Problem 9 of Stanley's survey article on positivity problems in algebraic combinatorics [72]. This chapter is based on the paper [28].

1.1.2 Modular representation theory

The modular representation theory of the symmetric group is the main object of this thesis. As already mentioned in the first general part of this introduction the representation theory of a finite group over a field $\mathbb{F}$ of prime characteristic p is much more complicated compared to the characteristic 0 setting. This is the case also for the symmetric group S_n. Important and natural questions that have a complete answer in the ordinary case, are obscure and apparently very hard to solve in the modular setting. For example, while the hook length formula gives us a precise combinatorial way to compute the dimension of any given simple $\mathbb{C}S_n$-module, the dimension of the simple $\mathbb{F}S_n$-modules is in general not known. In Chapters 3, 4, 5 and 6 we will study various aspects of the modular structure of important families of modules for the group algebra $\mathbb{F}S_n$.

Chapter 3 is devoted to the long standing and fundamental open problem of determining the decomposition matrix of the symmetric group. As explained in full detail in Section 1.3.5, the decomposition matrix of the symmetric group records the multiplicities of the simple $\mathbb{F}S_n$-modules as composition factors of the Specht modules S^λ defined over the field $\mathbb{F}$.

Our main contribution in the study of this problem is Theorem 3.1.1, where we completely describe a number of columns of the decomposition matrix labelled by partitions of arbitrarily high p-weight. One of the key tools used in the proof of Theorem 3.1.1 was the determination of the vertices of indecomposable summands of some twists by the sign character of the family of Foulkes modules of parameters 2 and n (Theorem 3.1.2). For a detailed account of the definition and the basic properties of vertices of indecomposable modules we refer the reader to Section 1.2.1. This chapter is based on the paper [29].

Motivated by the preliminary and specific result obtained in Theorem 3.1.2, in Chapter 4 we study the more general problem of determining the possible vertices of the family of the indecomposable summands of Foulkes modules $H^{(a^n)}$. In particular in Theorem 4.1.1 we are able to completely describe both vertices and Green correspondents of arbitrary indecomposable direct summands of $H^{(a^n)}$, whenever $a < p \leqslant n$. In this more general setting we are also able to draw corollaries on the structure of the decomposition matrix of the symmetric group. In particular in Theorem 4.1.2 we give upper bounds on some decomposition numbers in term of the multiplicities of irreducible ordinary characters in the decomposition of the ordinary Foulkes character $\phi^{(a^n)}$. This chapter is based on the paper [31].

In Chapter 5, we study the vertices of indecomposable Specht modules defined over $\mathbb{F}$. As explained in the introduction of the chapter, this is an important open problem in the representation theory of the symmetric group, largely studied by mathematicians in the last fifty years.

In Theorem 5.1.2 we give a lower bound on the vertices of indecomposable Specht modules. More precisely, for any given indecomposable Specht module S^λ we determine a large subgroup necessarily contained in a vertex of S^λ. Theorem 5.1.2 generalizes an earlier result due to Wildon in [77].

We conclude Chapter 5 by describing a family of Specht modules with maximal possible vertex. In particular, in Theorem 5.1.3 we give conditions on a partition λ in order to have the vertices of S^λ equal to the defect groups of the block where S^λ lies. The first part of the chapter is based on the paper [30].

As mentioned above, a very natural problem that lacks a satisfactory answer in modular representation theory of symmetric groups is to determine the dimensions of simple modules. As we will clarify in Section 1.2.1 below, the vertices of a simple module D are local invariants encoding interesting information about the p-part of the $\mathbb{F}$-linear dimension of D. Motivated by this open problem, in Chapter 6 we focus our attention on the study of the vertices of simple $\mathbb{F}S_n$-modules. Unfortunately a characterisation of the vertices of the full family of simple modules seems to be a very hard problem. Therefore we focus on the description of a more approachable subfamily, namely the one composed by all the simple modules labelled by hook partitions. As explained in the introduction of the chapter this family was largely studied in the last fifteen years. Given a natural number n and a hook partition $\lambda = (n-r, 1^r)$, the vertices of the simple $\mathbb{F}S_n$-module D^λ were already known except when $p \geqslant 3$, $r = p-1$ and n is congruent to p modulo p^2. In Theorem 6.1.1 we deal with this last unknown case. In particular we prove that if $r = p-1$ and n is congruent to p modulo p^2 then the vertices of D^λ are the Sylow p-subgroups of S_n. This is the last piece of information needed to completely classify vertices of simple modules labelled by hook partitions. A compact statement of this classification is given in Theorem 6.1.3. This chapter is based on the paper [14]

1.2 Background on modular representation theory

In this section we summarize without proofs some of the main results in the area of modular representation theory of finite groups. We refer the reader to [1], [2] and [62] for a complete and extensive account of the theory. Throughout this section we let $\mathbb{F}$ be a field of prime characteristic $p > 0$, G a finite group and we denote by $\mathbb{F}G$ the group algebra of G over $\mathbb{F}$.

1.2.1 Relative projectivity and vertices

An $\mathbb{F}G$-module U is a *free module* if it is isomorphic to a finite direct sum of copies of the *regular representation* $\mathbb{F}G$ naturally defined by linear extension of the action of G on itself by right multiplication. If

$$U \cong \underbrace{\mathbb{F}G \oplus \cdots \oplus \mathbb{F}G}_{r},$$

then we say that U is free of *rank* r. An $\mathbb{F}G$-module V is a *projective module* if it is a direct summand of a free module. In particular an $\mathbb{F}G$-module is indecomposable and projective if and only if it is an indecomposable direct summand of the regular module $\mathbb{F}G$. The following theorem is a fundamental characterization of projective modules (see [1, Section 5, Theorem 2]).

THEOREM 1.2.1 *Let U be an $\mathbb{F}G$-module. The following are equivalent:*

1. *U is a projective module.*

2. *If ϕ is a surjective $\mathbb{F}G$-homomorphism of the $\mathbb{F}G$-module V onto U then the kernel of ϕ is a direct summand of V.*

3. *If ϕ is a surjective $\mathbb{F}G$-homomorphism of the $\mathbb{F}G$-module V onto the $\mathbb{F}G$-module W and ψ is an $\mathbb{F}G$-homomorphism from U to W then there exists an $\mathbb{F}G$-homomorphism ζ of U to V such that $\phi\zeta = \psi$.*

Let H be a subgroup of G. An $\mathbb{F}G$-module U is called *relatively H-projective* if it is a direct summand of $\mathrm{Ind}^G(\mathrm{Res}_H\, U)$, namely the induction to G of the restriction to H of U. This is a generalization of the notion of projectivity. In fact it is not too difficult to realize that if $H = 1$ then U is relatively 1-projective if and only if it is projective. The following proposition (see [1, Section 9, Proposition 1]) characterizes relatively H-projective modules and it is an analogue of Theorem 1.2.1.

PROPOSITION 1.2.2 *Let U be an $\mathbb{F}G$-module and let H be a subgroup of G. The following are equivalent:*

1. *U is a relatively H-projective module.*

2. *If ϕ is a surjective $\mathbb{F}G$-homomorphism of the $\mathbb{F}G$-module V onto U that splits as an $\mathbb{F}H$-homomorphism then the kernel of ϕ is a direct summand of V.*

3. *If ϕ is a surjective $\mathbb{F}G$-homomorphism of the $\mathbb{F}G$-module V onto the $\mathbb{F}G$-module W and ψ is an $\mathbb{F}G$-homomorphism of U to W then there exists an $\mathbb{F}G$-homomorphism ζ from U to V such that $\phi\zeta = \psi$, provided that there is an $\mathbb{F}H$-homomorphism from U to V with this property.*

A natural question arising at this point is: given an $\mathbb{F}G$-module U, for which subgroups H of G is U relatively H-projective? The lemma below (see [1, Section 9]) is an important first step towards an answer.

LEMMA 1.2.3 *Let H be a subgroup of G and let M be a relatively H-projective $\mathbb{F}G$-module. Let P be a Sylow p-subgroup of H. Then M is relatively P-projective.*

Let now V be an indecomposable $\mathbb{F}G$-module. A subgroup Q of G that is minimal with respect to the condition that V is relatively Q-projective is called a *vertex* of V. Introduced by J. A. Green in 1959 [33], vertices of indecomposable modules over modular group algebras have proved to be important invariants linking the global and local representation theory of finite groups over fields of positive characteristic. Given a finite group G and a field $\mathbb{F}$ of characteristic $p > 0$, by Green's result, the vertices of every indecomposable $\mathbb{F}G$-module form a G-conjugacy class of p-subgroups of G. Moreover, vertices of simple $\mathbb{F}G$-modules are known to satisfy a number of very restrictive properties, most notably in consequence of Knörr's Theorem [52] below. We refer the reader to Section 1.2.2 for the definition of blocks and defect groups of a block of the group algebra $\mathbb{F}G$.

THEOREM 1.2.4 *Let B be a block of the group algebra $\mathbb{F}G$ and let S be a simple $\mathbb{F}G$-module with vertex Q lying in B. If D is a defect group of B containing Q, then $C_D(Q) \leqslant Q \leqslant D$.*

The latter, in particular, implies that vertices of simple $\mathbb{F}G$-modules lying in blocks with abelian defect groups have precisely these defect groups as their vertices. Despite this result, the precise structure of vertices of simple $\mathbb{F}G$-modules is still poorly understood, even for very concrete groups and modules.

Since an indecomposable module is projective if and only if its vertex is the trivial subgroup we have that, roughly speaking, the vertex of an indecomposable module V measures the distance of the module V from being projective.

If V has vertex Q and U is an indecomposable $\mathbb{F}Q$-module such that V is a direct summand of $\mathrm{Ind}^G(U)$, then we call U a *source* of V. The source U of V is unique up to conjugation in $N_G(Q)$. A fundamental result in the theory of vertices is the following theorem.

THEOREM 1.2.5 (GREEN CORRESPONDENCE) *There is a one-to-one correspondence between isomorphism classes of indecomposable $\mathbb{F}G$-modules with vertex Q and isomorphism classes of indecomposable $\mathbb{F}N_G(Q)$-modules with vertex Q. Moreover, if V is an indecomposable $\mathbb{F}G$-module with vertex Q then the Green correspondent U of V is the unique indecomposable summand of $\mathrm{Res}_{N_G(Q)}(V)$ having vertex Q. Equivalently V is the unique summand of $\mathrm{Ind}^G U$ having vertex Q.*

We conclude the section by stating some well known results that we will extensively use later on in the thesis.

THEOREM 1.2.6 *Let Q be a subgroup of the p-group P. If U is an absolutely indecomposable $\mathbb{F}Q$-module then $\mathrm{Ind}_Q^P(U)$ is absolutely indecomposable with vertex Q.*

Proof: See Theorem 8 in [33]. $\square$

An important consequence of theorem 1.2.6 is the following Theorem that relates the $\mathbb{F}$-linear dimension of an indecomposable module to the size of its vertex.

THEOREM 1.2.7 *Let V be an indecomposable $\mathbb{F}G$-module. Let P be a Sylow p-subgroup of G containing a vertex Q of V. Then*

$$|P : Q| \;\Big|\; \dim_{\mathbb{F}}(V).$$

Theorem 1.2.7 is a very important tool that will be frequently used in Chapters 5 and 6.

1.2.2 Block theory

In this paragraph we recall the main definitions and some of the basic properties concerning the block theory of group algebras. As usual let G be a finite group and let $\mathbb{F}$ a field of prime characteristic p such that $|G|$ is divisible by p. The group algebra $\mathbb{F}G$ can be considered as an $\mathbb{F}(G \times G)$-module via the action defined by $a(g, h) = g^{-1}ah$, for a, g and h in G. We call *p-blocks* of G the indecomposable summands $B_1, \ldots, B_k$ of the $\mathbb{F}(G \times G)$-module $\mathbb{F}G$. Equivalently

$$\mathbb{F}G = B_1 \oplus \cdots \oplus B_k,$$

is the unique decomposition of the algebra $\mathbb{F}G$ as the direct sum of indecomposable subalgebras. If V is an indecomposable $\mathbb{F}G$-module, we say that V *lies* in the p-block B_i if $VB_i = V$ and $VB_j = 0$ for all $j \neq i$.

THEOREM 1.2.8 *If B is a p-block of $\mathbb{F}G$ then B has vertex, as an $\mathbb{F}(G\times G)$-module, of the form*

$$\delta(D) = \{(d, d) \mid d \in D\},$$

where D is a p-subgroup of G.

Proof: See [1, Section 13, Theorem 4] □

It is not too difficult to notice that the vertices of a p-block B form a conjugacy class of p-subgroups of G. These are called the *defect groups* of G. If D is a defect group of a block B and has order p^d then B is said to be of *defect d*. The following theorem (see [1, Section 13, Theorem 5]) sheds light on the relation between vertices of indecomposable modules lying in B and B itself.

THEOREM 1.2.9 *Let V be an indecomposable $\mathbb{F}G$-module lying in a p-block B. If Q is a vertex of V then there exists a defect group D of B such that $Q \leqslant D$.*

In fact by [1, Section 14, Corollary 5], for every p-block B there exists an indecomposable module V lying in B having vertex equal to the defect group D of B.

The *Brauer correspondence* relates the p-blocks of $\mathbb{F}G$ to the p-blocks of subgroups of G. We define this important tool below.

Definition 1.2.10 *Let H be a subgroup of G, b a p-block of H and B a p-block of G. We say that B corresponds to b and write $B = b^G$ if b, as an $\mathbb{F}(H \times H)$-module, is a direct summand of the restriction of B to $H \times H$ and if B is the only block of G with this property.*

The following proposition underlines the connection between Brauer correspondence for blocks and Green correspondence for indecomposable modules. Again global and local information are closely related.

Proposition 1.2.11 *Let V be an indecomposable $\mathbb{F}G$-module with vertex Q lying in the p-block B of G. Let U be the Green correspondent of V, lying in the block b of $N_G(Q)$. Then $B = b^G$.*

1.2.3 The Brauer homomorphism

Let V be an $\mathbb{F}G$-module. Given a p-subgroup $Q \leqslant G$ we denote by V^Q the set

$$V^Q = \{v \in V : vg = v \text{ for all } g \in Q\}$$

of Q-fixed elements. It is easy to see that V^Q is an $\mathbb{F}N_G(Q)$-module on which Q acts trivially. For a proper subgroup R of Q, the *relative trace* map $\mathrm{Tr}_R^Q : V^R \to V^Q$ is the linear map defined by

$$\mathrm{Tr}_R^Q(v) = \sum_{g \in T} vg,$$

where the sum is over a set T of right coset representatives for R in Q. We observe that

$$\mathrm{Tr}^Q(V) := \sum_{R<Q} \mathrm{Tr}_R^Q(V^R)$$

is an $\mathbb{F}N_G(Q)$-module contained in V^Q. Moreover, for all $R < P < Q$ we have that

$$\mathrm{Tr}_R^Q(V^R) = \mathrm{Tr}_P^Q(\mathrm{Tr}_R^P(V^R)).$$

Therefore

$$\mathrm{Tr}^Q(V) = \sum_{P \in \Omega_Q} \mathrm{Tr}_P^Q(V^P),$$

where Ω_Q is the set consisting of all maximal subgroups of Q. If $P \in \Omega_Q$ then every element $g \in Q \smallsetminus P$ has the property that $\{1, g, g^2, \ldots, g^{p-1}\}$ is a set of representatives of the right cosets of P in Q; in particular, we get $\mathrm{Tr}_P^Q(v) = v + vg + \cdots + vg^{p-1}$, for $v \in V^P$.

The *Brauer quotient* of V with respect to Q is the $\mathbb{F}N_G(Q)$-module $V(Q)$ defined by

$$V(Q) = V^Q / \sum_{R<Q} \mathrm{Tr}_R^Q(V^R).$$

LEMMA 1.2.12 *Let U and V be two $\mathbb{F}G$-modules and let P be a p-subgroup of G, then*

$$(U \oplus V)(P) \cong U(P) \oplus V(P)$$

as $\mathbb{F}(N_G(P))$-modules.

Proof: It is easy to observe that $(U \oplus V)^P = U^P \oplus V^P$. Moreover, if R is a maximal subgroup of P and $\{1, g, g^2, \ldots, g^{p-1}\}$ is a set of representatives for the cosets of R in P, then

$$\mathrm{Tr}_R^P((U \oplus V)^R) = \{\sum_{i=0}^{p-1}(u,v)g^i \mid (u,v) \in U^R \oplus V^R\} = \mathrm{Tr}_R^P(U^R) \oplus \mathrm{Tr}_R^P(V^R).$$

Hence $(U \oplus V)(P) \cong U(P) \oplus V(P)$. $\square$

The next proposition (see [9, (1.3)]) is fundamental for many arguments used later in this thesis.

PROPOSITION 1.2.13 *If V is an indecomposable $\mathbb{F}G$-module and Q is a p-subgroup of G, then $V(Q) \neq 0$ implies that Q is contained in a vertex of V.*

In the following example we show that the converse of Proposition 1.2.13 is not true in general.

EXAMPLE 1.2.14 *Let p be a prime, let G be the cyclic group of order p^2 and let σ be a generator of G. Let J be the $\mathbb{F}_pG$-submodule of $\mathbb{F}_pG$ defined by*

$$J = \left\langle \sigma^i - 1 \mid i \in \{1, 2, \ldots, p^2 - 1\} \right\rangle_{\mathbb{F}_p},$$

where the action of $\mathbb{F}_pG$ is defined as the natural linear extension of the action of G on itself by right multiplication. Since J is (p^2-1)-dimensional, Theorem 1.2.7 implies that G is the vertex of J. It is easy to see that J^G is a 1-dimensional module linearly generated by $v = 1 + \sigma + \cdots + \sigma^{p^2-1}$. Consider now the element u defined by

$$u = \sum_{i=0}^{p-1} \sigma^{ip}.$$

Clearly $u \in J^{\langle \sigma^p \rangle}$ and we have that

$$v = Tr^G_{\langle \sigma^p \rangle}(u).$$

Therefore we conclude that $J(G) = 0$.

The following proposition will be very useful for proving Theorem 6.1.1 in Chapter 6. The proof is straightforward, and is thus left to the reader.

PROPOSITION 1.2.15 *Let G be a finite group, let V be an $\mathbb{F}G$-module with $\mathbb{F}$-basis B, and let P be a p-subgroup of G. Suppose that there is some $b_0 \in B$ satisfying the following properties:*

(i) *$b_0 \in V^P$;*

(ii) *whenever $Q <_{\max} P$, $u \in V^Q$ and $\mathrm{Tr}^P_Q(u) = \sum_{b \in B} a_b(u)b$, for $a_b(u) \in \mathbb{F}$, one has $a_{b_0}(u) = 0$.*

Then $b_0 + \mathrm{Tr}^P(V) \in V(P) \smallsetminus \{0\}$, and therefore $V(P) \neq 0$.

1.2.4 Brauer homomorphism and p-permutation modules

In this subsection we summarize the principal results from [9]. We start by defining p-permutation modules and describing some of their basic properties. In particular we focus on the behaviour of this family of modules under the Brauer map defined in Section 1.2.3. The results presented in this section will be used extensively in Chapters 3 and 4.

Let G be a finite group. An $\mathbb{F}G$-module V is said to be a *p-permutation module* if whenever P is a p-subgroup of G, there exists an $\mathbb{F}$-basis $\mathcal{B}$ of V whose elements are permuted by P. In this case we say that $\mathcal{B}$ is a *p-permutation basis* of V with respect to P, and write $V = \langle \mathcal{B} \rangle$. This definition immediately implies that the direct sum of p-permutation modules is a p-permutation module. It is also easily seen that if V has a p-permutation basis with respect to a Sylow p-subgroup P of G then V is a p-permutation module. It is enough to observe that if Q is another Sylow p-subgroup of G and g is an element of G such that $Q = P^g$, then the set $\mathcal{B}'$ defined by

$$\mathcal{B}' = \{bg \mid b \in \mathcal{B}\}$$

is a p-permutation basis for V with respect to Q.

The following proposition characterizing p-permutation modules is proved in [9, (0.4)]. Notice that if V and W are $\mathbb{F}G$-modules we write $V \mid W$ to indicate that V is isomorphic to a direct summand of W.

Proposition 1.2.16 *An indecomposable $\mathbb{F}G$-module V is a p-permutation module if and only if there exists a p-subgroup P of G such that $V \mid \mathrm{Ind}_P^G(\mathbb{F})$.*

Thus an indecomposable $\mathbb{F}G$-module is a p-permutation module if and only if it has trivial source. It follows that the restriction or induction of a p-permutation module is still p-permutation, as is any summand of a p-permutation module.

Lemma 1.2.17 *Let $\mathbb{F}$ be a field of prime characteristic p. Let G be a finite group and let K and H be subgroups of G such that $K \leqslant H \leqslant G$. Suppose that p does not divide $|H : K|$. Then $\mathrm{Ind}_H^G(\mathbb{F})$ is a direct summand of $\mathrm{Ind}_K^G(\mathbb{F})$.*

Proof: From the hypothesis we deduce that a Sylow p-subgroup of H is contained in K. Therefore the trivial $\mathbb{F}H$-module $\mathbb{F}_H$ is relatively K-projective, by Lemma 1.2.3. Hence

$$\mathbb{F}_H \mid \mathrm{Ind}_K^H(\mathrm{Res}_K^H(\mathbb{F}_H)) = \mathrm{Ind}_K^H(\mathbb{F}_K).$$

The statement now follows by inducing up to G. $\square$

The following theorem is proved in [9, 3.2(1)] and is the first and most important evidence of the nice behaviour of p-permutation modules under the Brauer homomorphism.

Theorem 1.2.18 *Let V be an indecomposable p-permutation $\mathbb{F}G$-module and let Q be a vertex of V. Let R be a p-subgroup of G. Then $V(R) \neq 0$ if and only if $R \leq Q^g$ for some $g \in G$.*

If V is an $\mathbb{F}G$-module with p-permutation basis $\mathcal{B}$ with respect to a Sylow p-subgroup P of G and $R \leq P$, then taking for each orbit of R on $\mathcal{B}$ the sum of the elements in that orbit, we obtain a basis for V^R. The sums over vectors lying in orbits of size p or more are relative traces from proper subgroups of R, and so $V(R)$ is equal to the $\mathbb{F}$-span of

$$\{v + Tr^R(V) \mid v \in \mathcal{B}^R\},$$

where $\mathcal{B}^R = \{v \in \mathcal{B} \mid vg = v \text{ for all } g \in R\}$. Notice that, to ease the notation we will sometimes equivalently denote $\mathcal{B}^R$ by $\mathcal{B}(R)$. Thus Theorem 1.2.18 has the following corollary, which we shall use throughout Chapters 3 and 4.

Corollary 1.2.19 *Let V be a p-permutation $\mathbb{F}G$-module with p-permutation basis $\mathcal{B}$ with respect to a Sylow p-subgroup P of G. Let $R \leq P$. The $\mathbb{F}N_G(R)$-module $V(R)$ is equal to $\langle b + \mathrm{Tr}^R(V) \mid b \in \mathcal{B}^R \rangle$ and V has an indecomposable summand with a vertex containing R if and only if $\mathcal{B}^R \neq \varnothing$.*

The next result [9, 3.4] explains what is now known as Broué correspondence.

THEOREM 1.2.20 *An indecomposable p-permutation module V has vertex Q if and only if $V(Q)$ is a projective $\mathbb{F}N_G(Q)/Q$-module. Furthermore*

- *The Brauer functor sending V to $V(Q)$ is a bijection between the set of indecomposable p-permutation $\mathbb{F}G$-modules with vertex Q and the set of indecomposable projective $\mathbb{F}(N_G(Q)/Q)$-modules. Regarded as an $\mathbb{F}N_G(Q)$-module, $V(Q)$ is the Green correspondent of V.*

- *Let V be a p-permutation $\mathbb{F}G$-module and E an indecomposable projective $\mathbb{F}(N_G(Q)/Q)$-module. Then E is a direct summand of $V(Q)$ if and only if its correspondent U (i.e. the $\mathbb{F}G$-module U such that $U(Q) \cong E$) is a direct summand of V.*

Some important consequences that we will use extensively in the thesis are stated below.

COROLLARY 1.2.21 *Let U be a p-permutation $\mathbb{F}G$-module and let Q be a p-subgroup of G. Then $U(Q)$ is a p-permutation $\mathbb{F}N_G(Q)$-module.*

Proof: Let R be a Sylow p-subgroup of $N_G(Q)$ and let P be a Sylow p-subgroup of G containing R. Denote by $\mathcal{B}_P$ a p-permutation basis of U with respect to P. By Corollary 1.2.19 we have that the $\mathbb{F}N_G(Q)$-module $U(Q)$ has linear basis $\mathcal{B}_P(Q)$. It is easy to observe that $\mathcal{B}_P(Q)$ is a p-permutation basis with respect to R. This completes the proof. $\square$

COROLLARY 1.2.22 *Let G and H be two finite groups and let C be a subgroup of G. Let U be an indecomposable p-permutation $\mathbb{F}G$-module, V be an indecomposable p-permutation $\mathbb{F}H$-module and $W_1, \ldots, W_k$ be indecomposable $\mathbb{F}C$-modules. Then*

- *If $\mathrm{Res}_C(U) = W_1 \oplus \cdots \oplus W_k$, then there is a vertex R of U and vertices $Q_1, \ldots, Q_k$ of $W_1, \ldots, W_k$ respectively, such that $Q_i \leqslant R$ for all $i \in \{1, 2, \ldots, k\}$.*

- *The indecomposable $\mathbb{F}(G \times H)$-module $U \boxtimes V$ has a vertex containing $Q \times P$, where Q and P are vertices of U and V respectively.*

LEMMA 1.2.23 *Let Q and R be p-subgroups of a finite group G and let U be a p-permutation $\mathbb{F}G$-module. Let $K = N_G(R)$. If R is normal in Q then $\mathrm{Res}_{N_K(Q)} U(Q)$ and $\big(U(R)\big)(Q)$ are isomorphic as $\mathbb{F}N_K(Q)$-modules.*

Proof: Let P be a Sylow p-subgroup of $N_G(R)$ containing Q and let $\mathcal{B}$ be a p-permutation basis for U with respect to P. By Corollary 1.2.19 we have $U(Q) = \langle \mathcal{B}(Q) \rangle$ as an $\mathbb{F}N_G(Q)$-module. In particular

$$\mathrm{Res}_{N_K(Q)}(U(Q)) = \langle \mathcal{B}(Q) \rangle$$

as an $\mathbb{F}N_K(Q)$-module. On the other hand $U(R) = \langle \mathcal{B}(R) \rangle$ as an $\mathbb{F}N_G(R)$-module. Now $\mathcal{B}(R)$ is a p-permutation basis for $U(R)$ with respect to P. Since P contains Q we have $\big(U(R)\big)(Q) = \langle \mathcal{B}(R) \rangle(Q) = \langle (\mathcal{B}(R))(Q) \rangle = \langle \mathcal{B}(Q) \rangle$, as $\mathbb{F}N_K(Q)$-modules, as required. $\square$

Lemma 1.2.24 *Let G and H be finite groups and let U and U' be p-permutation modules for $\mathbb{F}G$ and $\mathbb{F}H$, respectively. If $Q \leq G$ is a p-subgroup then*

$$(U \boxtimes U')(Q) = U(Q) \boxtimes U',$$

where on the left-hand side Q is regarded as a subgroup of $G \times H$ in the obvious way.

Proof: This follows easily from Corollary 1.2.19 by taking p-permutation bases for U and U'. $\square$

Proposition 1.2.25 *Let M be a p-permutation $\mathbb{F}G$-module and P be a p-subgroup of G. If $M(P)$ is an indecomposable $\mathbb{F}N_G(P)$-module then M has a unique indecomposable summand U such that P is contained in a vertex of U.*

Proof: Suppose by contradiction that there exist V_1 and V_2 indecomposable summands of M with vertices Q_1 and Q_2 respectively, such that $P \leqslant Q_1 \cap Q_2$. Then by Lemma 1.2.12 we have that

$$V_1(P) \oplus V_2(P) \mid M(P).$$

This contradicts the indecomposability of $M(P)$ since by Theorem 1.2.20 we have that $V_i(P) \neq 0$ for $i \in \{1, 2\}$. $\square$

Lemma 1.2.26 *Let M be an indecomposable p-permutation module and let $P \leqslant G$ be a vertex of M. Let Q be a subgroup of P. Suppose that M lies in the block B of $\mathbb{F}G$. If $M(Q)$, considered as an $\mathbb{F}N_G(Q)$-module, has a summand in the block b of $\mathbb{F}N_G(Q)$, then b^G is defined and $b^G = B$.*

Proof: See [77, Lemma 7.4]. $\square$

In Chapters 3 and 4 we will need the following well known Scott's lifting theorem for p-permutation modules (see for instance [2, Theorem 3.11.3]). We denote by $\mathbb{Z}_p$ the ring of p-adic integers. If M is an $\mathbb{F}_pG$-permutation module with permutation basis $\mathcal{B}$, we denote by $M_{\mathbb{Z}_p}$ the canonical lift of M defined by

$$M_{\mathbb{Z}_p} = \langle b \mid b \in \mathcal{B} \rangle_{\mathbb{Z}_p}.$$

Theorem 1.2.27 *If U is a direct summand of a permutation $\mathbb{F}_pG$-module M then there is a $\mathbb{Z}_pG$-module $U_{\mathbb{Z}_p}$, unique up to isomorphism, such that $U_{\mathbb{Z}_p}$ is a direct summand of $M_{\mathbb{Z}_p}$ and $U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong U$.*

An argument that we shall use several times is stated in the lemma below:

Lemma 1.2.28 *If P is a p-group and Q is a subgroup of P then the permutation module $\mathrm{Ind}_Q^P(\mathbb{F})$ is indecomposable, with vertex Q.* $\square$

Proof: This is a straightforward application of Proposition 1.2.6. $\square$

We conclude the section by recalling the definition and the basic properties of Scott modules. We refer the reader to [9, Section 2] for a more detailed account. Given a subgroup H of G there exists a unique indecomposable summand U of the permutation module $\mathrm{Ind}_H^G(\mathbb{F})$ such that the trivial $\mathbb{F}G$-module is a submodule of U. This can be seen by observing that

$$\dim_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}G}(\mathbb{F}, \mathrm{Ind}_H^G(\mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}H}(\mathbb{F}, \mathbb{F})) = 1,$$

therefore the multiplicity of the trivial $\mathbb{F}G$-module as a direct summand of the socle of $\mathrm{Ind}_H^G(\mathbb{F})$ is equal to 1. We say that U is the *Scott module* of G associated to H and we denote it by $\mathrm{Sc}(G, H)$. The following theorem summarizes the main properties of Scott modules (see [9, Theorems (2.1) and (3.2)]).

Theorem 1.2.29 *Let G be a finite group, H a subgroup of G and P a Sylow p-subgroup of H. Then the Scott module $\mathrm{Sc}(G, P)$ is isomorphic to $\mathrm{Sc}(G, H)$ and is uniquely determined up to isomorphism among the summands of $\mathrm{Ind}_H^G(\mathbb{F})$ by either of the following properties:*

- *The trivial $\mathbb{F}G$-module is isomorphic to a submodule of $\mathrm{Sc}(G, P)$.*
- *The trivial $\mathbb{F}G$-module is isomorphic to a quotient of $\mathrm{Sc}(G, P)$.*

Moreover, $\mathrm{Sc}(G, P)$ has vertex P and the Broué correspondent $(\mathrm{Sc}(G, P))(P)$ is isomorphic to the projective cover of the trivial $\mathbb{F}(N_G(P)/P)$-module.

1.3 Background on the representation theory of symmetric groups

In this section we collect the main results in both the ordinary and the modular representation theory of the symmetric group.

1.3.1 The combinatorics of partitions and tableaux

A composition ℓ of a natural number n is a finite sequence of non-negative integers $\ell = (\ell_1, \ldots, \ell_k)$, such that $\sum_{i=1}^{k} \ell_i = n$. The non-negative integers $\ell_1, \ell_2, \ldots, \ell_k$ are called the *parts* of the composition. For example $\ell = (2, 3, 1, 1, 0)$ is a composition of 7. A partition λ of a natural number n (denoted by $\lambda \vdash n$) is a composition of n whose parts are strictly positive and non-increasingly ordered. For example $\lambda = (3, 2, 1, 1)$ is a partition of 7 and is called the *underlying partition* of $\ell = (2, 3, 1, 1, 0)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of a natural number n. For all $i \in \{1, \ldots, \lambda_1\}$ denote by λ'_i the natural number defined by

$$\lambda'_i = |\{\lambda_j \mid \lambda_j \geqslant i\}|.$$

Let $s = \lambda_1$ and define $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ to be *conjugate partition* of λ. Given $\mu = (\mu_1, \ldots, \mu_t)$ a partition of m, we say that λ is a *subpartition* of μ, and write $\lambda \subseteq \mu$, if $k \leqslant t$ and $\lambda_j \leqslant \mu_j$, for all j such that $1 \leqslant j \leqslant k$.

There is a natural partial order on the set of partitions of a positive integer n known as the *dominance order*. Given $\lambda, \mu \vdash n$, denote by $p(\lambda)$ and $p(\mu)$ the number of parts of λ and μ respectively. We say λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i$$

for all j such that $1 \leq j \leq \min\bigl(p(\lambda), p(\mu)\bigr)$.

The *Young diagram* $[\lambda]$ of λ is an array of boxes, left aligned, having λ_j boxes in the j^{th}-row for all $j \in \{1, \ldots, k\}$.

A λ-*tableau* is an assignment of the numbers $\{1, 2, \ldots, n\}$ to the boxes of the Young diagram of λ such that no number appears twice. We will denote by $t(i, j)$ the number assigned to the box of t in row i and column j. Given a λ-tableau t, the *transposed* tableau of t is the λ'-tableau obtained by assigning to the box in row i and column j the value $t(j, i)$. We will denote by t' the transposed tableau of t.

The symmetric group S_n acts naturally on the set of λ-tableaux by permuting the entries within the boxes. We call *row-standard* any λ-tableau having the entries of each row ordered increasingly from left to right. Similarly a λ-tableau is called *column-standard* if the entries of each column are increasingly ordered from top to bottom. When a λ-tableau is both row-standard and column-standard it is called *standard*. Given a λ-tableau v we will denote by $\overline{v}$ the row-standard tableau obtained from v by sorting its rows in increasing order. We will call $\overline{v}$ the *row-straightening of* v.

For example, if $\lambda = (3, 2, 1, 1)$ then in Figure 1.1 below are represented a λ-tableau u and the row-straightening $\overline{u}$ of u.

$$u = \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 5 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} 6 & \multicolumn{2}{|c}{} \\ \cline{1-1} 7 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \qquad \overline{u} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{|c}{} \\ \cline{1-2} 6 & \multicolumn{2}{|c}{} \\ \cline{1-1} 7 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}$$

Figure 1.1: Two $(3, 2, 1, 1)$-tableaux

There is a fundamental ordering on the set of standard λ-tableaux known (again) as the *dominance order*. In order to define it we must define the dominance order on compositions. Let $\ell = (\ell_1, \dots, \ell_k)$ and $b = (b_1, \dots, b_s)$ be compositions of the natural number n. We say that ℓ dominates b, and write $\ell \trianglerighteq b$, if

$$\sum_{i=1}^{r} \ell_i \geqslant \sum_{i=1}^{r} b_i,$$

for all $r \in \mathbb{N}$. (If r is larger than the number of parts of ℓ or b then take the corresponding part to be 0). If t is a standard tableau, then we denote by $\mathrm{com}(t^{\leqslant j})$ the composition recording the number of entries $\leqslant j$ in each row of t. For example if $t = \overline{u}$, where u is as in Figure 1.1, then $\mathrm{com}(t^{\leqslant 7}) = (3, 2, 1, 1)$ and $\mathrm{com}(t^{\leqslant 4}) = (3, 1, 0, 0)$. Let λ be a partition of n. If t and v are standard λ-tableaux then we say that t dominates v if

$$\mathrm{com}(t^{\leqslant j}) \trianglerighteq \mathrm{com}(v^{\leqslant j}),$$

for all j with $1 \leqslant j \leqslant n$. Notice, for example, that the tableau $\overline{u}$ in Figure 1.1 is the most dominant $(3, 2, 1, 1)$-standard tableau. Following the usual convention, we will re-adopt the $\trianglerighteq$ symbol for dominance order on standard tableaux.

We conclude the section by defining a special subclass of partitions of a natural number n that we will consider extensively in the rest of the thesis, in particular in Chapters 2 and 6. Let k be a natural number smaller than n and let λ be the partition defined by

$$\lambda = (n - k, \underbrace{1, \dots, 1}_{k}) = (n - k, 1^k).$$

We realize that the Young diagram $[\lambda]$ has the shape of a hook. For this reason λ is called a *hook partition*.

1.3.2 Young permutation modules and Specht modules

We turn now to a brief account of the theory of Specht modules for S_n. We refer the reader to [39] for further details and examples. We start by introducing the following definition. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n we denote by S_λ the subgroup of S_n defined by

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}.$$

The subgroups S_λ are called *Young subgroups* of S_n for all λ partitions of n.

We say that two λ-tableaux t and u are *row-equivalent* if the entries in each row of t are the same as the entries in the corresponding row of u (for instance the tableaux u and $\overline{u}$ in Figure 1.1 are row equivalent). It is easy to see that this defines an equivalence relation on the set of λ-tableaux. We will denote by $\{t\}$ the row-equivalence class of t and we will say that $\{t\}$ is a λ-*tabloid*. The symmetric group S_n acts naturally on the set of λ-tabloids, therefore we can define M^λ to be the S_n-permutation module generated as a vector space by the set of all λ-tabloids. The module M^λ is called a *Young permutation module*. Since S_n acts transitively on the set of λ-tabloids and since the stabilizer in S_n of a fixed λ-tabloid $\{t\}$ is conjugate to the Young subgroup S_λ we have the following important isomorphism of $\mathbb{F}S_n$-modules:

$$M^\lambda \cong \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{F}), \quad \text{for all } \lambda \vdash n.$$

Given any λ-tableau t we denote by $C(t)$ the column stabilizer of t, namely the subgroup of S_n that fixes the columns of t setwise. The λ-*polytabloid* corresponding to the λ-tableau t is the following element of M^λ:

$$e_t = \sum_{g \in C(t)} \mathrm{sgn}(g)\{t\}g.$$

The Specht module S^λ is the submodule of M^λ linearly generated by the polytabloids. When considered over the field of complex numbers, the family of Specht modules

$$\{S^\lambda \mid \lambda \vdash n\},$$

is a complete set of non-isomorphic simple $\mathbb{C}S_n$-modules. For every partition λ of n we denote by π^λ and χ^λ the ordinary characters afforded by M^λ and S^λ respectively.

Notice that for all $h \in S_n$ we have that $e_t h = e_{th}$ for any given λ-tableau t. Moreover if $g \in C(t)$ then it is easy to observe that $e_t g = \mathrm{sgn}(g) e_t$. Finally we will say that e_t is a *standard polytabloid* if t is a standard λ-tableau. One of the main theorems about the structure of Specht modules is the following Standard Basis

Theorem proved by James in [39]. Here we present it in its stronger version with the contribution made by Wildon in [77, Proposition 4.1].

THEOREM 1.3.1 *The set of standard λ-polytabloids is a $\mathbb{Z}$-basis for the Specht module S^λ defined over $\mathbb{Z}$. Moreover if v is a column-standard λ-tableau then its row-straightening $\overline{v}$ is a standard λ-tableau and*

$$e_v = e_{\overline{v}} + x,$$

where x is a $\mathbb{Z}$-linear combination of standard λ-polytabloids e_t such that $\overline{v} \rhd t$.

In particular we observe that the dimension of the Specht module S^λ equals the number of standard λ-tableaux. One of the nicest results in the representation theory of symmetric groups is the *hook length formula*. The hook length formula is a closed combinatorial formula which gives the dimensions of Specht modules. We need to introduce a bit of notation in order to state the related theorem. Let b be a box of the Young diagram of λ. We denote by H_b the *hook* on b, namely the subset of boxes of $[\lambda]$ lying either to the right or below b, including b itself. We define the *hook length* h_b to be the number of boxes in H_b.

THEOREM 1.3.2 *Let λ be a partition of n, then*

$$\dim(S^\lambda) = \frac{n!}{\prod_{b\in[\lambda]} h_b}.$$

For example the dimension of $S^{(3,2,1)}$ is $\frac{6!}{5\cdot 3^2} = 16$.

We present below some fundamental results concerning the ordinary representation theory and the character theory of symmetric groups. We refer the reader to [39] for the proofs of such well known theorems.

The theorem below gives important information concerning the decomposition of Young permutation modules into irreducible Specht modules.

THEOREM 1.3.3 *Let λ and μ be two partitions of n. If the $\mathbb{C}S_n$-module S^λ is a direct summand of M^μ then λ dominates μ.*

THEOREM 1.3.4 (BRANCHING THEOREM) *Let μ be a partition of n. Let Λ be the set of all the partitions of $n+1$ corresponding to the Young diagrams obtained by adding a box to the Young diagram of μ. Then the induced $\mathbb{C}S_{n+1}$-module $\mathrm{Ind}^{S_{n+1}}(S^\mu)$ decomposes as follows:*

$$\mathrm{Ind}^{S_{n+1}}(S^\mu) = \bigoplus_{\lambda\in\Lambda} S^\lambda.$$

The following theorems will be extensively used in Chapter 2. They are both straightforward corollaries of the Littlewood–Richardson rule, as stated in [39, Chapter 16].

THEOREM 1.3.5 *Let k be a natural number such that $k < n$ and let λ be a partition of $n-k$. If $\mathcal{L}$ is the set of all the partitions of n corresponding to the Young diagrams obtained by adding k boxes, no two in the same column, to the Young diagram of λ, then*

$$(\chi^\lambda \times 1_{S_k})\big\uparrow_{S_{n-k}\times S_k}^{S_n} = \sum_{\mu\in\mathcal{L}} \chi^\mu.$$

THEOREM 1.3.6 *Let k be a natural number such that $k < n$ and let λ be a partition of $n-k$. Denote by sgn_k the sign character of the symmetric group S_k (i.e. the character afforded by the Specht module $S^{(1^k)}$). If $\mathcal{K}$ is the set of all the partitions of n corresponding to the Young diagrams obtained by adding k boxes, no two in the same row, to the Young diagram of λ, then*

$$(\chi^\lambda \times \mathrm{sgn}_k)\big\uparrow_{S_{n-k}\times S_k}^{S_n} = \sum_{\mu\in\mathcal{K}} \chi^\mu.$$

THEOREM 1.3.7 *Let k be a natural number such that $k < n$, let λ be a partition of $n-k$, let μ be a partition of k and let ν be a partition of n. If*

$$\left\langle (\chi^\lambda \times \chi^\mu)\big\uparrow^{S_n}, \chi^\nu \right\rangle \neq 0$$

then $\lambda, \mu \subseteq \nu$, and $p(\nu) \leqslant p(\lambda) + p(\mu)$.

1.3.3 The structure of Sylow p-subgroups of symmetric groups

We pause for a moment the study of the representation theory to recall the structure of the Sylow p-subgroups of symmetric groups, as described for example in [45, Chapter 4]. Then, after fixing a convenient notation we prove a number of properties of the Sylow p-subgroups of S_n and their subgroups that we will use in Chapters 3, 4 and 6. In particular, in Section 6.3 we will also extend some of the results presented in this section.

Let P_p be the cyclic group $\langle(1,2,\ldots,p)\rangle \leqslant S_p$ of order p. Let further $P_1 := \{1\}$ and, for $d \geqslant 1$, we set

$$P_{p^{d+1}} := P_{p^d} \wr P_p := \{(\sigma_1,\ldots,\sigma_p;\pi) : \sigma_1,\ldots,\sigma_p \in P_{p^d},\ \pi \in P_p\}.$$

Recall that, for $d \geqslant 2$, the multiplication in $P_{p^d} \wr P_p$ is given by

$$(\sigma_1, \ldots, \sigma_p; \pi)(\sigma'_1, \ldots, \sigma'_p; \pi') = (\sigma_1 \sigma'_{(1)\pi}, \ldots, \sigma_p \sigma'_{(p)\pi}; \pi\pi'),$$

for $(\sigma_1, \ldots, \sigma_p; \pi)$, $(\sigma'_1, \ldots, \sigma'_p; \pi') \in P_{p^d}$.

We shall always identify P_{p^d} with a subgroup of S_{p^d} in the usual way. That is, $(\sigma_1, \ldots, \sigma_p; \pi) \in P_{p^d}$ is identified with the element $\overline{(\sigma_1, \ldots, \sigma_p; \pi)} \in S_{p^d}$ that is defined as follows: if $j \in \{1, \ldots, p^d\}$ is such that $j = b + p^{d-1}(a-1)$, for some $a \in \{1, \ldots, p\}$ and some $b \in \{1, \ldots, p^{d-1}\}$ then

$$(j)\overline{(\sigma_1, \ldots, \sigma_p; \pi)} := (b)\sigma_a + p^{d-1}((a)\pi - 1).$$

Via this identification, P_{p^d} is generated by the elements $g_1, \ldots, g_d \in S_{p^d}$, where

$$g_j := \prod_{k=1}^{p^{j-1}} (k, k + p^{j-1}, k + 2p^{j-1}, \ldots, k + (p-1)p^{j-1}) \quad (1 \leqslant j \leqslant d)\,. \qquad (1.1)$$

With this notation, we have $P_p \leqslant P_{p^2} \leqslant \cdots \leqslant P_{p^{d-1}} \leqslant P_{p^d}$, and the base group of the wreath product $P_{p^{d-1}} \wr P_p$ has the form

$$\prod_{i=0}^{p-1} g_d^{-i} \cdot P_{p^{d-1}} \cdot g_d^i.$$

In particular, we can write P_{p^d} as

$$P_{p^d} = (P_{p^{d-1}} \times P_{p^{d-1}}^{g_d} \times \cdots \times P_{p^{d-1}}^{(g_d^{p-1})}) \rtimes \langle g_d \rangle\,.$$

Since the order of a Sylow p-subgroup of S_{p^d} equals the p-part of $p^d!$, an easy inductive argument is now enough to show that P_{p^d} is a Sylow p-subgroup of S_{p^d}, for all $d \in \mathbb{N}$.

Definition 1.3.8 *Let $g_1, \ldots, g_d$ be the generators of P_{p^d} fixed in (1.1). For $j \in \{1, \ldots, d-1\}$, let*

$$g_{j,j+1} := \prod_{i=0}^{p-1} g_{j+1}^{-i} g_j g_{j+1}^i,$$

and for $l \in \{1, \ldots, d-j-1\}$, we inductively set

$$g_{j,j+1,\ldots,j+l+1} := \prod_{i=0}^{p-1} g_{j+l+1}^{-i} \cdot g_{j,j+1,\ldots,j+l} \cdot g_{j+l+1}^i\,.$$

To ease the notation we denote the element $g_{1,2,\ldots,j}$ by z_j for all $j \in \{1, 2, \ldots, d\}$. In Section 6.3 we will extensively describe these elements. Here we want to focus on the study of some properties of the element $z_d = g_{1,2,\ldots,d}$. In particular we observe that for all $j \in \{1, 2, \ldots, d-1\}$ we can express z_d as

$$z_d = \prod_{k_{j+1}=0}^{p-1} \prod_{k_{j+2}=0}^{p-1} \cdots \prod_{k_d=0}^{p-1} (g_{j+1}^{k_{j+1}} \cdots g_d^{k_d})^{-1} z_j (g_{j+1}^{k_{j+1}} \cdots g_d^{k_d}).$$

Moreover for all $j \in \{1, 2, \ldots d-1\}$ we have that g_j commutes with $y_j := z_j^{(g_{j+1}^{k_{j+1}} \cdots g_d^{k_d})}$, for all $k_{j+1}, \ldots, k_d \in \{0, 1, \ldots, p-1\}$. This follows by observing that $\mathrm{supp}(g_j) \cap \mathrm{supp}(y_j) = \emptyset$ unless $k_r = 0$ for all $r \in \{j+1, \ldots, d\}$, in which case $y_j = z_j$ and g_j commutes with z_j by construction. This implies that z_d commutes with g_j for all $j \in \{1, 2, \ldots, d\}$ and therefore we deduce that $\langle z_d \rangle$ is a central subgroup of P_{p^d}. This fact will be used later in Chapters 3 and 4.

Now let $n \in \mathbb{N}$ be arbitrary, and consider the p-adic expansion $n = \sum_{i=0}^{r} n_i p^i$ of n, where $0 \leqslant n_i \leqslant p-1$ for $i \in \{0, \ldots, r\}$, and where we may suppose that $n_r \neq 0$. By [45, 4.1.22, 4.1.24], the Sylow p-subgroups of S_n are isomorphic to the direct product $\prod_{i=0}^{r} (P_{p^i})^{n_i}$. For subsequent computations it will be useful to fix a particular Sylow p-subgroup P_n of S_n as follows: for $i \in \{t \in \mathbb{N} \mid n_t \neq 0\}$ and $1 \leqslant j_i \leqslant n_i$, let $k(j_i) := \sum_{l=0}^{i-1} n_l p^l + (j_i - 1)p^i$ and

$$P_{p^i, j_i} := (1, 1+k(j_i)) \cdots (p^i, p^i + k(j_i)) \cdot P_{p^i} \cdot (1, 1+k(j_i)) \cdots (p^i, p^i + k(j_i))\,.$$

Now set

$$P_n := P_{p,1} \times \cdots \times P_{p,n_1} \times \cdots \times P_{p^r,1} \times \cdots \times P_{p^r, n_r}\,.$$

Notice that for $i \in \{1, \ldots, r\}$ and $j_i \in \{1, \ldots, n_i\}$, the direct factor P_{p^i, j_i} of P_n is determined by i and its support $\mathrm{supp}(P_{p^i, j_i})$.

Example 1.3.9 *Suppose that $p = 3$. Then $P_3 = \langle g_1 \rangle$, $P_9 = \langle g_1, g_2 \rangle$, $P_{27} = \langle g_1, g_2, g_3 \rangle$ and $z_3 \in Z(P_{27})$, where*

$$g_1 = (1,2,3)\,,$$

$$g_2 = (1,4,7)(2,5,8)(3,6,9)\,,$$

$$g_3 = (1,10,19)(2,11,20)(3,12,21)(4,13,22) \cdots (8,17,26)(9,18,27)\,,$$

$$z_2 = g_{1,2} = g_1 g_1^{g_2} g_1^{(g_2^2)} = (1,2,3)(4,5,6)(7,8,9)\,,$$

$$z_3 = g_{1,2,3} = z_2 z_2^{g_3} z_2^{(g_3^2)} = (1,2,3)(4,5,6)(7,8,9) \cdots (22,23,24)(25,26,27)\,.$$

Moreover, $P_{51} \cong P_3 \times P_3 \times P_9 \times P_9 \times P_{27}$.

The following Lemma, proved by Danz in [12], gives an interesting picture of the lattice of subgroups of P_n.

LEMMA 1.3.10 ([12, LEMMA 2.1, REMARK 2.2]) *Let n be a natural number with p-adic expansion $n = \sum_{i=0}^{r} n_i p^i$. Let $P \leqslant P_n$ be such that P is S_n-conjugate to P_{p^i}, for some $i \in \{1, \ldots, r\}$. Then $P \leqslant P_{p^l, j_l}$, for some $l \in \{i, \ldots, r\}$ and some $1 \leqslant j_l \leqslant n_l$. Moreover, P_{p^l, j_l} has precisely p^{l-i} subgroups that are S_n-conjugate to P_{p^i}, and these are pairwise P_{p^l, j_l}-conjugate to each other.*

REMARK 1.3.11 *Let again $n \in \mathbb{N}$ with p-adic expansion $n = \sum_{i=0}^{r} n_i p^i$. Let $P \leqslant P_n$ be such that P is S_n-conjugate to P_{p^i}, for some $i \in \{1, \ldots, r\}$, so that $P \leqslant P_{p^l, j_l}$, for some $l \in \{i, \ldots, r\}$ and some $1 \leqslant j_l \leqslant n_l$, by Lemma 1.3.10. Note that the subgroups of P_{p^l, j_l} that are S_n-conjugate to P_{p^i} are uniquely determined by their supports. In particular, if $i = 1$ then P is generated by one of the p-cycles $(1, \ldots, p), \ldots, (n - n_0 - p + 1, \ldots, n - n_0) \in P_n$.*

1.3.4 Blocks of symmetric groups

In the first part of Section 1.3.2 we gave an account of *characteristic free* properties of the representations of symmetric groups. The last part of the section was focused on the ordinary character theory. We now turn our interest to the modular case.

Let $\mathbb{F}$ be a field of prime characteristic p. The p-blocks of the symmetric group algebra $\mathbb{F}S_n$ are described combinatorially by Nakayama's Conjecture, first proved by Brauer and Robinson in two connected papers [67] and [7]. In order to state this result, we must recall some definitions.

Let λ be a partition. A *p-hook* in λ is a connected part of the rim of the Young diagram of λ consisting of exactly p boxes, whose removal leaves the diagram of a partition. By repeatedly removing p-hooks from λ we obtain the *p-core* of λ (usually denoted by $\gamma(\lambda)$); the number of hooks we remove is the *p-weight* of λ (usually denoted by $w(\lambda)$). When the prime p is clearly fixed, we will sometimes talk about core and weight instead of p-core and p-weight. In general a *p-core partition* is a partition that does not have removable p-hooks. The best way to understand and perform these kind of operations on partitions is to use *James' abacus*. We refer the reader to [45, Chapter 2] for the definition and a detailed account of the combinatorial properties of the abacus.

THEOREM 1.3.12 (NAKAYAMA'S CONJECTURE) *Let p be prime. The p-blocks of S_n are labelled by pairs (γ, w), where γ is a p-core and $w \in \mathbb{N}_0$ is the associated weight,* *such that $|\gamma|+wp=n$. Thus the Specht module S^λ lies in the block labelled by (γ, w) if and only if λ has p-core γ and weight w.* $\square$

We denote the block of weight w corresponding to the p-core γ by $B(\gamma, w)$. The following description of the Brauer correspondence for blocks of symmetric groups is critical to the proofs of Proposition 3.4.1 in Chapter 3 and Proposition 4.3.1 in Chapter 4.

THEOREM 1.3.13 *Let V be an indecomposable p-permutation module lying in the block $B(\gamma, w)$ of S_n. Suppose that R is a subgroup of a vertex of V and that R moves exactly the first rp elements of $\{1, \dots, n\}$; that is* $\mathrm{supp}(R) = \{1, \dots, rp\}$. *Then $N_{S_n}(R) \cong N_{S_{rp}}(R) \times S_{n-rp}$. Moreover,*

(i) *$N_{S_{rp}}(R)$ has a unique block, b say.*

(ii) *The blocks $b \otimes B(\gamma, w-r)$ and $B(\gamma, w)$ are Brauer correspondents.*

(iii) *As an $\mathbb{F}N_{S_n}(R)$-module, $V(R)$ lies in $b \otimes B(\gamma, w-r)$.*

Proof: Part (i) is an immediate corollary of Lemma 2.6 and the following sentence of [10]. Part (ii) is stated in (2) on page 166 of [10], and then proved as a corollary of the characterisation of maximal Brauer pairs given in Proposition 2.12 of [10]. In order to prove part (iii) we let W be an indecomposable summand of $V(R)$ lying in the block c of $\mathbb{F}N_{S_n}(R)$. By Lemma 1.2.26 we deduce that the Brauer correspondent c^{S_n} is defined. Moreover, by part (i) we have that

$$c = b \otimes B(\overline{\gamma}, t),$$

where b is the unique block of $\mathbb{F}N_{S_{rp}}(R)$ and $B(\overline{\gamma}, t)$ is a p-block of $\mathbb{F}S_{n-rp}$. By [73, Exercise 27.4] $V(R)$ is a direct summand of $\mathrm{Res}_{N_{S_n}(R)}(V)$, therefore

$$0 \neq W \cdot c \mid \mathrm{Res}_{N_{S_n}(R)}(V) \cdot \mathrm{Res}_{N_{S_n}(R) \times N_{S_n}(R)}(c^{S_n}).$$

In particular we deduce that $V \cdot c^{S_n} \neq 0$ and so that V lies in the block c^{S_n}. This implies that $c^{S_n} = B(\gamma, w)$ and by part (ii) we obtain

$$B(\gamma, w) = c^{S_n} = (b \otimes B(\overline{\gamma}, t))^{S_n} = B(\overline{\gamma}, t+r).$$

Clearly we must have $\gamma = \overline{\gamma}$ and $t = w - r$, as required. $\square$

We conclude the section stating the following important result (see [45, Theorem 6.2.45]) describing the defect groups of blocks of the symmetric group.

Theorem 1.3.14 *The defect group of a block of the symmetric group of weight w is a Sylow p-subgroup of S_{wp}.*

In particular Theorem 1.3.14 implies that a p-block $B(\gamma, w)$ of $\mathbb{F}S_n$ has abelian defect group if and only if $w < p$.

In [70], Scopes studied the Morita equivalence classes of blocks of symmetric groups. In particular in [70, Theorem 1] it is proved that for any $w \in \mathbb{N}$ there are only finitely many distinct families of Morita equivalent blocks of p-weight w. It follows immediately that the Donovan conjecture holds for blocks of the symmetric group.

1.3.5 The decomposition matrices of symmetric groups

Let $\mathbb{F}$ be a field of prime characteristic p. A partition λ of a natural number n is said to be *p-regular* ($\lambda \vdash_p n$) if it has at most $p-1$ parts of any given size. The simple $\mathbb{F}S_n$-modules are labelled by the p-regular partitions of n. In particular we can find each simple $\mathbb{F}S_n$-module D^λ as the top composition factor of the Specht module $S^\lambda_{\mathbb{F}}$ defined over $\mathbb{F}$. More precisely, we have the following fundamental theorem.

Theorem 1.3.15 *Let $\mathbb{F}$ be a field of prime characteristic p. If λ is a p-regular partition of n then the $\mathbb{F}S_n$-module $S^\lambda_{\mathbb{F}}$ has a unique top composition factor, $S^\lambda_{\mathbb{F}}/\mathrm{rad}(S^\lambda_{\mathbb{F}}) = D^\lambda$. Moreover, the set $\{D^\lambda \mid \lambda \vdash_p n\}$ is a complete set of non isomorphic simple $\mathbb{F}S_n$-modules.*

A central open problem in the representation theory of symmetric groups is to determine for all $\lambda \vdash n$ and for all $\mu \vdash_p n$ the multiplicity $d_{\lambda\mu}$ of the simple $\mathbb{F}S_n$-module D^μ as a composition factor of the Specht module $S^\lambda_{\mathbb{F}}$. Such non-negative integers $d_{\lambda\mu}$ (equivalently denoted by $[S^\lambda : D^\mu]$) are called *decomposition numbers* of S_n. Decomposition numbers are usually recorded in a matrix $D_n(p)$ known as the *decomposition matrix* of the symmetric group S_n in prime characteristic p.

We collect below some of the main properties of decomposition matrices.

Theorem 1.3.16 *Let λ be a p-regular partition of n and let μ be a partition of n that is not dominated by λ. Then $d_{\lambda\lambda} = 1$ and $d_{\mu\lambda} = 0$.*

Proof: See [39, Theorem 12.1]. □

Another easy and very important fact is implied by Theorem 1.3.12.

Theorem 1.3.17 *Let λ be a p-regular partition of n and let μ be a partition of n. Suppose that the p-core $\gamma(\lambda)$ of λ is not equal to $\gamma(\mu)$. Then $d_{\mu\lambda} = 0$.*

Proof: By Theorem 1.3.12 we deduce that $S^\mu_{\mathbb{F}}$ and D^λ lie in different p-blocks of $\mathbb{F}S_n$. Therefore D^λ can not be a composition factor of $S^\mu_{\mathbb{F}}$. $\square$

We refer the reader to Chapter 3 for further properties of the decomposition matrix and for a detailed account of the state of the art concerning the determination of decomposition numbers.

1.4 Young modules and Foulkes modules

1.4.1 Motivation

An interesting problem in the representation theory of finite groups is the study of the properties shared by indecomposable summands of permutation modules, namely indecomposable p-permutation modules. In the case of the symmetric group the first major achievement in this sense is the description of the indecomposable summands of the family of Young permutation modules $\{M^\lambda \mid \lambda \vdash n\}$. Young permutation modules were deeply studied by James in [40], Klyachko in [49] and Grabmeier in [32]. In their work, they completely parametrized the indecomposable summands (known as *Young modules*) of such modules and developed a Green correspondence for those summands. Their original description of the modular structure of Young modules was based on Schur algebras. More recently Erdmann in [18] described completely the Young modules using only the representation theory of the symmetric groups. (The proof of Lemma 3 of [18] contains some errors. A correction to that proof was later given by Erdmann and Schroll in [19]). We would like to state here Erdmann's result. In order to do this we need to introduce the following definition. Let λ be a partition of a natural number n. We say that λ is a *p-restricted* partition if the conjugate partition λ' is p-regular. Moreover notice that if a partition λ is not p-restricted, then there exist a unique natural number k and unique p-restricted partitions $\lambda(0), \lambda(1), \dots, \lambda(k) \neq \emptyset$, such that

$$\lambda = \sum_{m=0}^{k} \lambda(m) p^m.$$

The above expression is called the *p-adic expansion* of λ. We will denote by r_m the degree $|\lambda(m)|$ of $\lambda(m)$ for each $m \in \{0, 1, \dots, k\}$.

THEOREM 1.4.1 (THEOREMS 1 AND 2 OF [18]) *Let n be a natural number and $\mathbb{F}$ a field of prime characteristic p. There is a set of indecomposable $\mathbb{F}S_n$-modules Y^μ, one for each partition μ of n, such that the following holds for all partitions λ of n.*

1. *M^λ is isomorphic to a direct sum of Young modules Y^μ with $\mu \trianglerighteq \lambda$ and with Y^λ appearing exactly once.*

2. *$Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$.*

3. *The Young module Y^λ is projective if and only if λ is p-restricted.*

4. *Let λ be not p-restricted and suppose that $\lambda = \sum_{m=0}^{k} \lambda(m)p^m$ is the p-adic expansion of λ. Consider ρ to be the partition of n which has r_m parts equal to p^m for every $m \in \{0, 1, \ldots, k\}$, where $r_m = |\lambda(m)|$. Then Y^λ has vertex a Sylow p-subgroup of S_ρ.*

In this thesis we will begin a systematic study of a new family of permutation modules, the Foulkes modules. In this section we will give the definition and we will describe the main properties of Foulkes modules. Motivated by the problem known as the *Foulkes Conjecture* (see 2.1.1 for the statement), in Chapter 2 we will analyse the ordinary character afforded by Foulkes modules. In Chapters 3 and 4 we will study the modular structure of Foulkes modules. In particular in Chapter 3 we will use some properties of this family of permutation modules to determine parts of the decomposition matrix of S_n. On the other hand, in Chapter 4 we will focus on the indecomposable summands of these permutation modules. More precisely we will give a precise description of their vertices and Green correspondents. We will also compare the obtained results with the known structure of Young modules; this will allow us to disprove a modular version of Foulkes Conjecture, in Proposition 4.2.11.

1.4.2 Definitions and preliminary results

The purpose of this paragraph is to introduce the definition and some of the basic properties of Foulkes modules. We will also introduce the main pieces of notation that we will frequently use in Chapters 2, 3 and 4.

Let $\mathbb{F}$ be a field (of arbitrary characteristic) and a, n two non-zero natural numbers. Let $\Omega^{(a^n)}$ be the collection of all set partitions of $\{1, 2, \ldots, an\}$ into n sets of size a. We will denote an arbitrary element $\omega \in \Omega^{(a^n)}$ by

$$\omega = \{\omega_1, \omega_2, \cdots, \omega_n\},$$

where $\omega_j \subseteq \{1, 2, \cdots, an\}$, $|\omega_j| = a$ and $\omega_i \cap \omega_j = \emptyset$ for all $1 \leqslant i < j \leqslant n$. We will call ω_j a *set of* ω. For example the set partition $\{\{1, 2, 3\}, \{4, 5, 6\}\}$ is an element of $\Omega^{(3^2)}$. Observe that given two natural numbers a and n we have that

$$|\Omega^{(a^n)}| = \frac{(an)!}{(a!)^n(n!)}.$$

The symmetric group S_{an} acts transitively in a natural way on $\Omega^{(a^n)}$ by permuting the numbers in each set of every set partition. Let $H^{(a^n)}$ be the $\mathbb{F}S_{an}$-permutation module generated as a $\mathbb{F}$-vector space by the elements of $\Omega^{(a^n)}$, with the action of S_{an} defined as the natural linear extension of the action on $\Omega^{(a^n)}$. The module $H^{(a^n)}$ is called the *Foulkes module of parameters a and n*. Since the action of S_{an} is transitive on $\Omega^{(a^n)}$ and the stabilizer of any set partition $\omega \in \Omega^{(a^n)}$ is isomorphic to $S_a \wr S_n$ it is finally easy to deduce that

$$H^{(a^n)} \cong \mathrm{Ind}_{S_a \wr S_n}^{S_{an}}(\mathbb{F}).$$

An easy application of Lemma 1.2.17 clarifies the relation between the Foulkes module $H^{(a^n)}$ and the Young permutation module M^μ where μ is the partition of an having n parts of size a.

PROPOSITION 1.4.2 *If the field $\mathbb{F}$ has characteristic 0 or it has prime characteristic p strictly greater than n then the Foulkes module $H^{(a^n)}$ is a direct summand of the Young permutation module M^μ.*

Proof: If the characteristic of the field $\mathbb{F}$ is $p > 0$ then let K be the Young subgroup of S_{an} defined by

$$K = \underbrace{S_a \times \cdots \times S_a}_{n}.$$

By definition M^μ is the module obtained by inducing the trivial $\mathbb{F}K$-module to the full symmetric group S_{an}. Moreover p does not divide $n! = |S_a \wr S_n : K|$, therefore by Lemma 1.2.17 we deduce the statement. We leave the characteristic 0 case to the reader. $\square$

In particular we deduce that when the underlying field $\mathbb{F}$ has prime characteristic p strictly greater than n, then every indecomposable summand of $H^{(a^n)}$ is a Young module. In Chapter 4 we will show that the situation is completely different when $a < p \leqslant n$. More precisely, in Corollary 4.2.10 we will prove that no non-projective indecomposable summand of $H^{(a^n)}$ is a Young module.

Chapter 2

Foulkes' Conjecture

2.1 Introduction and outline

Foulkes' Conjecture is a long standing open problem in the areas of ordinary representation theory of the symmetric group, algebraic combinatorics and invariant theory. It was stated by H.O.Foulkes in 1950 as an invariant theoretic problem. We need to introduce some notation and definitions to state an equivalent character theoretical version of the conjecture.

As alredy mentioned in Chapter 1 (Section 1.4), for a and b natural numbers, we denote by $\Omega^{(a^b)}$ the collection of all set partitions of $\{1, 2, \ldots, ab\}$ into b sets each of size a. Throughout this chapter $H^{(a^b)}$ denotes the corresponding $\mathbb{C}S_{ab}$-Foulkes module. Let $\phi^{(a^b)}$ be the permutation character of S_{ab} afforded by $H^{(a^b)}$.

At the end of Section 1 of [26], Foulkes made a conjecture which can be stated as follows.

Conjecture 2.1.1 (Foulkes' Conjecture) *Let a and b be natural numbers such that $a \geqslant b$. Then the $\mathbb{C}S_{ab}$-module $H^{(a^b)}$ is a direct summand of the $\mathbb{C}S_{ab}$-module $H^{(b^a)}$.*

The conjecture has been only proved to be true when $b = 2$ by Thrall (see [74]), when $b = 3$ by Dent (see [15, Main Theorem]), when $b = 4$ by McKay (see [57, Theorem 1.2]) and when b is very large compared to a by Brion (see [8, Corollary 1.3]).

It is also possible to restate Foulkes' Conjecture as an inequality between multiplicities, namely that, for all a and b natural numbers such that $a \geqslant b$ and for all partitions λ of ab,

$$\left\langle \phi^{(a^b)}, \chi^\lambda \right\rangle \leqslant \left\langle \phi^{(b^a)}, \chi^\lambda \right\rangle,$$

where χ^λ is the irreducible character of S_{ab} canonically labelled by λ. From this point of view the decomposition of the Foulkes module as a direct sum of simple modules becomes central. Except in the case when $a = 2$ or $b = 2$ (see [74, Chapter 2] and [69]) and when $b = 3$ (see [15, Theorem 4.1]), little is known about the multiplicities of simple modules in this decomposition. In [45, Theorem 5.4.34] an explicit fomula is given for the specific case of simple modules labelled by two-row partitions: in this case Foulkes' Conjecture holds with equality. In [64] Paget and Wildon gave a combinatorial description of the minimal partitions that label simple modules appearing as summands of Foulkes modules.

In this chapter, we will prove a number of new results on when these multiplicities vanish. We start in Section 2.2 below, by proving some properties of the Foulkes character. In particular we will describe its restriction to the subgroups $S_r \times S_{ab-r}$ of S_{ab}. In Section 2.3 we prove by using only the character theory of symmetric groups the following result which shows that no Specht module labelled by a hook partition $(ab - r, 1^r)$ is a direct summand of the Foulkes module $H^{(a^b)}$. The result was already proved, with the language of Schur functions by Langley and Remmel in [54]. We call r the *leg length* of the hook partition.

Theorem 2.1.2 *If a, b and r are natural numbers such that $1 \leqslant r < ab$, then*

$$\left\langle \phi^{(a^b)}, \chi^{(ab-r,1^r)} \right\rangle = 0.$$

In Section 2.4 we extend this result, by giving a sufficient condition on a partition λ of ab for $\langle \phi^{(a^b)}, \chi^\lambda \rangle$ to equal zero.

We need the following notation: let $\alpha = (\alpha_1, \ldots, \alpha_t)$ be a partition of $m \in \mathbb{N}$, let $k \in \mathbb{N}$ be such that $k \geqslant t$ and $ab - k - m \geqslant \alpha_1 + 1$. Denote by $[k : \alpha]$ the partition of ab defined by

$$[k : \alpha] = (ab - k - m, \alpha_1 + 1, \ldots, \alpha_t + 1, 1^{k-t}).$$

Notice that the value of ab will be always clear from the context. It is obvious that every partition of ab can be expressed uniquely in the form $[k : \alpha]$. We will call α the *inside-partition* of $[k : \alpha]$ (see figure 2.1).

The main result of the chapter is as follows.

Theorem 2.1.3 *Let a, b and k be natural numbers and let $[k : \alpha]$ be a partition of ab with $\alpha = (\alpha_1, \ldots, \alpha_t)$ and $t \leqslant k$. Let $n := \sum_{j=2}^{t} \alpha_j$. Suppose that $k > n$ and*

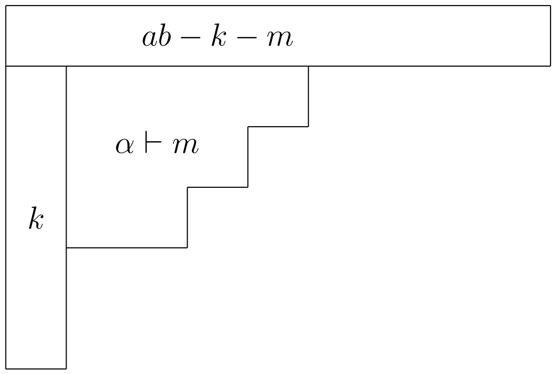

Figure 2.1: The shape of $[k:\alpha]$

$\alpha_1 < \frac{1}{2}(k-n)(k-n+1)$. *Then*

$$\left\langle \phi^{(a^b)}, \chi^{[k:\alpha]} \right\rangle = 0.$$

Notice that for every simple $\mathbb{C}S_{ab}$-module labelled by λ, a partition of ab satisfying the hypothesis of Theorem 2.1.3, Foulkes' Conjecture holds with equality. Indeed for all $a \geqslant b$ we have

$$\left\langle \phi^{(a^b)}, \chi^{\lambda} \right\rangle = 0 = \left\langle \phi^{(b^a)}, \chi^{\lambda} \right\rangle,$$

since there is not any restriction on a and b in the statement of the theorem.

By Proposition 2.2.1 below, if $\langle \phi^{(a^b)}, \chi^{\lambda} \rangle \neq 0$ then λ has at most b parts. When we consider only characters labelled by such partitions, it occurs that a significant proportion of the characters appearing with zero multiplicity in $\phi^{(a^b)}$ satisfy the hypotheses of Theorem 2.1.3. For example, computations using the computer algebra package MAGMA [5] show that there are 1909 partitions λ of 30 with at most 10 parts such that $\langle \phi^{(3^{10})}, \chi^{\lambda} \rangle = 0$; of these 492 satisfy the hypotheses of Theorem 2.1.3.

For an important subclass of partitions to which Theorem 2.1.3 applies we refer the reader to Corollary 2.4.4.

2.2 More on the ordinary Foulkes character

Here we present some properties of the Foulkes $\mathbb{C}S_{ab}$-module $H^{(a^b)}$ that will be needed to prove the two main theorems.

Proposition 2.2.1 *Let λ be a partition of ab such that $p(\lambda) > b$. Then*

$$\left\langle \phi^{(a^b)}, \chi^\lambda \right\rangle = 0.$$

Proof: By the characteristic 0 part of Proposition 1.4.2 we have that $H^{(a^b)}$ is a direct summand of $M^{(a^b)}$. Moreover by Theorem 1.3.3 we deduce that no Specht module labelled by a partition with more than b parts can possibly be a summand of $M^{(a^b)}$. This concludes the proof. $\square$

Definition 2.2.2 *Let r, a and b be natural numbers. We define $P(r)_a^b$ to be the set of all partitions of r with at most b parts and first part of size at most a.*

An element of $\Omega^{(a^b)}$ can be denoted by $\{A_1, \ldots, A_b\}$, where for each $1 \leqslant j \leqslant b$, A_j is a subset of $\{1, 2, \ldots ab\}$ of size a and for all i, j such that $1 \leqslant i < j \leqslant b$ it holds that $A_i \cap A_j = \emptyset$.

Definition 2.2.3 *Let r be a natural number such that $r < ab$ and let λ be in $P(r)_a^b$. We will say that an element*

$$\{A_1, \ldots, A_b\} \in \Omega^{(a^b)}$$

is linked *to λ if the composition of r whose parts are*

$$\big|\{1, 2, \ldots, r\} \cap A_i\big| \quad \textit{for } 1 \leqslant i \leqslant b$$

has underlying partition λ.

Definition 2.2.4 *Let r be a natural number, such that $r < ab$ and let λ be in $P(r)_a^b$. We denote by $\mathcal{O}(\lambda)$ the set of all the set partitions in $\Omega^{(a^b)}$ linked to λ and by V^λ the transitive permutation module for $\mathbb{C}\big(S_r \times S_{ab-r}\big)$ linearly spanned by the elements of $\mathcal{O}(\lambda)$.*

In the following proposition we show how the restriction

$$\mathrm{Res}_{S_r \times S_{ab-r}}(H^{(a^b)})$$

of the Foulkes module decomposes into a direct sum of transitive permutation modules. Such decompositions will be used in all the proofs of the main theorems of this chapter.

PROPOSITION 2.2.5 *Let r be a natural number such that $r < ab$. Then*

$$\operatorname{Res}_{S_r \times S_{ab-r}}(H^{(a^b)}) = \bigoplus_{\lambda \in P(r)_a^b} V^\lambda.$$

Proof: Let $G = S_r \times S_{ab-r}$. The restriction of $H^{(a^b)}$ to G decomposes as a direct sum of transitive permutation modules, one for each orbit of G on $\Omega^{(a^b)}$. Observe that two set partitions $\mathcal{P}$, $\mathcal{Q} \in \Omega^{(a^b)}$ are in the same orbit of G on $\Omega^{(a^b)}$ if and only if $\mathcal{P}$ and $\mathcal{Q}$ are linked to the same partition $\lambda \in P(r)_a^b$. The result follows. $\square$

An immediate corollary of Proposition 2.2.5 is the following result about the multiplicity of characters labelled by two-row partitions, proved by a different argument in [45, Theorem 5.4.34].

COROLLARY 2.2.6 *Let r, a and b be natural numbers. Then*

1. $\left\langle \phi^{(a^b)}, \pi^{(ab-r,r)} \right\rangle = \left\langle \phi^{(b^a)}, \pi^{(ab-r,r)} \right\rangle = \left| P(r)_a^b \right|$
2. $\left\langle \phi^{(a^b)}, \chi^{(ab-r,r)} \right\rangle = \left\langle \phi^{(b^a)}, \chi^{(ab-r,r)} \right\rangle = \left| P(r)_a^b \right| - \left| P(r-1)_a^b \right|$

Proof: By definition of Young permutation module (see Section 1.3.2) we have that

$$\pi^{(ab-r,r)} = 1_{S_{ab-r} \times S_r} \big\uparrow^{S_{ab}}.$$

Therefore, by Frobenius reciprocity and Proposition 2.2.5 we have

$$\begin{aligned} \left\langle \phi^{(a^b)}, \pi^{(ab-r,r)} \right\rangle &= \left\langle \phi^{(a^b)} \big\downarrow_{S_{ab-r} \times S_r}, 1_{S_{ab-r} \times S_r} \right\rangle \\ &= \sum_{\lambda \in P(r)_a^b} \left\langle \chi_{V^\lambda}, 1_{S_{ab-r} \times S_r} \right\rangle \\ &= \sum_{\lambda \in P(r)_a^b} 1 = \left| P(r)_a^b \right|. \end{aligned}$$

To complete the proof of part (i), it suffices to observe that the conjugation of partitions induces a bijective map between $P(r)_a^b$ and $P(r)_b^a$ for all r, a and b natural numbers.

Part (ii) follows from (i) since $\chi^{(ab-r,r)} = \pi^{(ab-r,r)} - \pi^{(ab-(r-1),r-1)}$. $\square$

We conclude this section with the definition and a description of a generalized Foulkes module that will be used in the proof of Theorem 2.1.3.

DEFINITION 2.2.7 *Let $\eta = (a_1^{b_1}, \ldots, a_r^{b_r})$ be a partition of n, where $a_1 > a_2 > \ldots >$*

$a_r > 0$, *and let* $G = S_{a_1 b_1} \times \cdots \times S_{a_r b_r} \leqslant S_n$. *We define*

$$H^{\eta} = \mathrm{Ind}_G^{S_n} \left(H^{(a_1^{b_1})} \boxtimes H^{(a_2^{b_2})} \boxtimes \cdots \boxtimes H^{(a_r^{b_r})} \right)$$

We denote by ψ^{η} *the character of the generalized Foulkes module* H^{η}.

DEFINITION 2.2.8 *Let* $\eta = (\eta_1, \ldots, \eta_r)$ *be a partition of* n. *Define* Ω^{η} *to be the collection of all the set partitions of* $\{1, 2, \ldots, n\}$ *into* r *sets of sizes* $\eta_1, \eta_2, \ldots, \eta_r$.

The symmetric group S_n acts naturally on Ω^{η} by permuting the numbers in each set of any set partition. The proof of the following proposition is left to the reader.

PROPOSITION 2.2.9 *Let* $\eta = (\eta_1, \ldots, \eta_r)$ *be a partition of* n. *Then* $H^{\eta} \cong \mathbb{C}\Omega^{\eta}$, *as* $\mathbb{C}S_n$*-modules.*

2.3 The multiplicities of hook characters are zero

In this section we will prove that no Specht module labelled by a hook partition $(ab - r, 1^r)$ appears in the Foulkes module $H^{(a^b)}$.

DEFINITION 2.3.1 *Let* U *be a* $\mathbb{C}S_n$*-module affording the character* χ. *For all* $k \in \mathbb{N}$ *we denote by* $\bigwedge^k U$ *the* k^{th} *exterior power of* U, *and by* $\bigwedge^k \chi$ *the corresponding character.*

Let sgn_k be the sign character of the symmetric group S_k for any natural number k. We leave to the reader the proof of the following well known result.

LEMMA 2.3.2 *Let* n *and* k *be natural numbers, then*

$$\bigwedge^k \pi^{(n-1,1)} = (\mathrm{sgn}_k \times 1_{n-k}) \big\uparrow^{S_n}.$$

LEMMA 2.3.3 *Let* n *and* k *be natural numbers such that* $1 \leqslant k \leqslant n$. *Then*

$$\chi^{(n-k,1^k)} = \bigwedge^k \chi^{(n-1,1)} \quad \textit{and} \quad \bigwedge^k \pi^{(n-1,1)} = \chi^{(n-k,1^k)} + \chi^{(n-(k-1),1^{k-1})}.$$

Proof: The first equality follows from [59, Proposition 2.3]. The second equality is a straightforward application of Theorem 1.3.5. $\square$

In the following proposition we will calculate the inner product between the Foulkes character $\phi^{(a^b)}$ and the character $\bigwedge^k \pi^{(n-1,1)}$. This is a fundamental step in the proof of Theorem 2.1.2.

PROPOSITION 2.3.4 *Let a, b and k be natural numbers and let $\psi := \pi^{(ab-1,1)}$. Then*

$$\left\langle \phi^{(a^b)}, \bigwedge^k \psi \right\rangle = \begin{cases} 0 & \text{if } k \geqslant 2 \\ 1 & \text{if } k = 0, 1. \end{cases}$$

Proof: Firstly consider the case $k \geqslant 2$. Let $K = S_{\{1,2,\dots,k\}} \times S_{\{k+1,\dots,ab\}} \cong S_k \times S_{ab-k} \leqslant S_{ab}$. By Lemma 2.3.2

$$\left\langle \phi^{(a^b)}, \bigwedge^k \psi \right\rangle = \left\langle \phi^{(a^b)}, (\mathrm{sgn}_k \times 1_{ab-k}) \uparrow^{S_{ab}} \right\rangle = \left\langle \phi^{(a^b)} \downarrow_K, \mathrm{sgn}_k \times 1_{ab-k} \right\rangle.$$

The final inner product is not equal to zero only if there exist $\mathbb{C}K$-submodules of $\mathrm{Res}_K(H^{(a^b)})$ whose associated character is $\mathrm{sgn}_k \times 1_{S_{n-k}}$. By Proposition 2.2.5 it suffices to show that if $\lambda \in P(k)_a^b$ then V^λ has no submodule affording the character $\mathrm{sgn}_k \times 1_{ab-k}$. Suppose that $u \in V^\lambda$ spans such a submodule. Let $u = \sum_{\mathcal{P}} c_{\mathcal{P}} \mathcal{P}$, where the sum is over all set partitions $\mathcal{P} \in \mathcal{O}(\lambda)$. Choose $\mathcal{Q}$ such that $c_{\mathcal{Q}} \neq 0$.

If $\lambda_1 > 1$ then there exist $x, y \leqslant k$ such that x and y appear in the same set in $\mathcal{Q}$. Hence $\mathcal{Q}(x\ y) = \mathcal{Q}$ and $u(x\ y) = -u$, therefore $c_{\mathcal{Q}} = -c_{\mathcal{Q}} = 0$ which is a contradiction. Therefore $\lambda = (1^k)$.

If $\lambda = (1^k)$ then

$$\mathcal{Q} = \big\{\{1, x_2^1, \dots, x_a^1\}, \{2, x_2^2, \dots, x_a^2\}, \dots, \{k, x_2^k, \dots, x_a^k\}, \dots\big\}$$

for some $x_j^i \in \{k+1, \dots, ab\}$. Taking $\tau = (1\ 2)(x_1^1\ x_1^2) \cdots (x_a^1\ x_a^2)$, we obtain a contradiction again, since $\mathcal{Q}\tau = \mathcal{Q}$ but $u\tau = -u$. Hence there are no $\mathbb{C}K$-submodules of $\mathrm{Res}_K(H^{(a^b)})$ having character $\mathrm{sgn}_k \times 1_{S_{ab-k}}$. The two cases $k = 0$ and $k = 1$ are easy and are left to the reader. $\square$

We are now ready to prove Theorem 2.1.2. This theorem follows at once from Proposition 2.3.4, since, from Lemma 2.3.3 we have that

$$\chi^{(ab-r,1^r)} = \bigwedge^r \chi^{(ab-1,1)} = (-1)^r \sum_{k=0}^r (-1)^k \bigwedge^k \psi.$$

We end this section with a corollary of Theorem 2.1.2 that will be needed in the proof of Theorem 2.1.3. Recall that ψ^η is the character of the generalized Foulkes module H^η, as defined in Definition 2.2.7.

COROLLARY 2.3.5 *Let $\eta = (a_1^{b_1}, \dots, a_t^{b_t})$ be a partition of n, where $a_1 > \dots > a_t$. If*

$r \geq t$ *then*

$$\left\langle \psi^{\eta}, \chi^{(n-r,1^r)} \right\rangle = 0.$$

Proof: From the definition of generalized Foulkes module, we can write ψ^{η} as a character induced from

$$\phi^{(a_1^{b_1})} \times \cdots \times \phi^{(a_t^{b_t})}.$$

It follows from Theorems 1.3.7 and 2.1.2 that in order to obtain $\chi^{(n-r,1^r)}$ as an irreducible constituent of the induced character, we have to take the trivial character in each factor. Therefore

$$\left\langle \psi^{\eta}, \chi^{(n-r,1^r)} \right\rangle = \left\langle \left(1_{S_{a_1 b_1}} \times \cdots \times 1_{S_{a_t b_t}}\right)\big\uparrow^{S_n}, \chi^{(n-r,1^r)} \right\rangle.$$

Observe that the right-hand side is the multiplicity of $\chi^{(n-r,1^r)}$ in the Young permutation character $\pi^{(a_1 b_1, \dots, a_t b_t)}$. By Theorem 1.3.3, the constituents of $\pi^{(a_1 b_1, \dots, a_t b_t)}$ are labelled by partitions with at most t parts, so we need $t \geq r+1$ to get a non-zero multiplicity. $\square$

2.4 A sufficient condition for zero multiplicity

In this section we will prove Theorem 2.1.3 by an inductive argument. Part of the section will be devoted to the proof of the base step of such induction.

Firstly we need to state two technical lemmas. Let $\beta < ab$ be a natural number. Denote by K the subgroup $S_{\{1,2,\dots,\beta\}} \times S_{\{\beta+1,\dots,ab\}} \cong S_\beta \times S_{ab-\beta}$. Let λ be in $P(\beta)_a^b$ and let V^λ and $\mathcal{O}(\lambda)$ be as in Definition 2.2.4. Then by a standard result on orbit sums we have the following lemma.

LEMMA 2.4.1 *The largest* $\mathbb{C}K$*-submodule of* V^λ *on which* S_β *acts trivially is*

$$U := \Big\langle \sum_{\sigma \in S_\beta} \mathcal{P}\sigma \mid \mathcal{P} \in \mathcal{O}(\lambda) \Big\rangle_{\mathbb{C}}.$$

With the next lemma we will understand precisely the structure of this particular module U.

LEMMA 2.4.2 *Let* $\lambda \in P(\beta)_a^b$ *and denote by* $\mathbb{C}_{S_\beta}$ *the trivial* $\mathbb{C}S_\beta$*-module. If* $\lambda = (\lambda_1, \dots, \lambda_r)$ *then*

$$U \cong \mathbb{C}_{S_\beta} \boxtimes H^{\eta}$$

where $\eta = (a^{(b-r)}, a-\lambda_r, \dots, a-\lambda_2, a-\lambda_1)$ *and* H^η *is a generalized* $\mathbb{C}(S_{ab-\beta})$*-Foulkes* *module.*

Proof: By Proposition 2.2.9 it suffices to show that the set

$$\mathcal{W} := \big\{ \sum_{\sigma \in S_\beta} \mathcal{P}\sigma \ \mid \ \mathcal{P} \in \mathcal{O}(\lambda) \big\}$$

is isomorphic as a $S_{ab-\beta}$-set to the set Ω^η of all η-partitions of $\{\beta+1, \dots, ab\}$.

Let $X = \{\beta+1, \beta+2, \dots, ab\}$. We define a map $f_\lambda : \mathcal{O}(\lambda) \longrightarrow \Omega^\eta$ by

$$\mathcal{P}f_\lambda = \{A_1 \cap X, A_2 \cap X, \dots, A_b \cap X\}$$

where $\mathcal{P} = \{A_1, \dots, A_b\}$.

It is easy to see that f_λ is well defined since $\mathcal{O}(\lambda)f_\lambda \subseteq \Omega^\eta$ by definition of $\mathcal{O}(\lambda)$. The map f_λ is surjective, and for all $\mathcal{P}$ and $\mathcal{Q}$ in $\mathcal{O}(\lambda)$ we have that $\mathcal{P}f_\lambda = \mathcal{Q}f_\lambda$ if and only if $\mathcal{P}$ and $\mathcal{Q}$ are in the same S_β-orbit of $\mathcal{O}(\lambda)$. It is easy to see that f_λ is an $S_{ab-\beta}$-map and that for all $\tau \in S_{\{1,\dots,\beta\}}$ we have that $(\mathcal{P}\tau)f_\lambda = \mathcal{P}f_\lambda$, since τ fixes the numbers greater than β.

To conclude the proof we define

$$\tilde{f}_\lambda : \mathcal{W} \longrightarrow \Omega^\eta$$

by

$$\big(\sum_{\sigma \in S_\beta} \mathcal{P}\sigma \big) \tilde{f}_\lambda = \mathcal{P}f_\lambda$$

for all $\mathcal{P} \in \mathcal{O}(\lambda)$. The map $\tilde{f}_\lambda$ is well defined and the surjectivity of $\tilde{f}_\lambda$ follows directly from the surjectivity of f_λ. The map $\tilde{f}_\lambda$ is also injective since

$$\big(\sum_{\sigma \in S_\beta} \mathcal{P}\sigma \big) \tilde{f}_\lambda = \big(\sum_{\sigma \in S_\beta} \mathcal{Q}\sigma \big) \tilde{f}_\lambda \Longleftrightarrow \mathcal{P}f_\lambda = \mathcal{Q}f_\lambda \Longleftrightarrow \mathcal{P} = \mathcal{Q}\tau \Longleftrightarrow \sum_{\sigma \in S_\beta} \mathcal{P}\sigma = \sum_{\sigma \in S_\beta} \mathcal{Q}\sigma$$

for some $\tau \in S_\beta$.

Finally $\tilde{f}_\lambda$ is an $S_{ab-\beta}$-map since f_λ is an $S_{ab-\beta}$-map and $\sigma\tau = \tau\sigma$ for all $\sigma \in S_\beta$ and $\tau \in S_{ab-\beta}$. Therefore $\tilde{f}_\lambda$ is the desired isomorphism. $\square$

In the following proposition we use the notation $[k : \alpha]$ as defined in the introduction of the chapter. In particular we consider partitions $[k : \alpha]$ of ab with trivial inside-partition α (one row). The proposition is actually the base step of the inductive proof of Theorem 2.1.3.

Proposition 2.4.3 *Let a, b and k be natural numbers. For all $\beta < \frac{1}{2}k(k+1)$ we have*

$$\left\langle \phi^{(a^b)}, \chi^{[k:(\beta)]} \right\rangle = 0.$$

Proof: By Theorem 1.3.5 and Frobenius reciprocity, we have

$$\begin{aligned} \left\langle \phi^{(a^b)}, \chi^{[k:(\beta)]} \right\rangle &\leqslant \left\langle \phi^{(a^b)}, \left(1_{S_\beta} \times \chi^{(ab-(k+\beta),1^k)}\right)\big\uparrow_{S_\beta \times S_{ab-\beta}}^{S_{ab}} \right\rangle \\ &= \left\langle \phi^{(a^b)}\big\downarrow_{S_\beta \times S_{ab-\beta}}, 1_{S_\beta} \times \chi^{(ab-(k+\beta),1^k)} \right\rangle. \end{aligned}$$

Let $K := S_\beta \times S_{ab-\beta}$. By Proposition 2.2.5 we have:

$$\mathrm{Res}_K(H^{(a^b)}) = \bigoplus_{\lambda \in P(\beta)_a^b} V^\lambda.$$

Fix $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_s^{m_s}) \in P(\beta)_a^b$. Let $r := \sum_{i=1}^s m_i$ be the number of parts of λ. We are now interested in submodules $U \subseteq V^\lambda$ such that $S_{\{1,2,\ldots,\beta\}} \cong S_\beta$ acts trivially on U. By Lemmas 2.4.1 and 2.4.2, the largest submodule U of V^λ is isomorphic to $H^\eta \boxtimes \mathbb{C}_{S_\beta}$, where $\eta = (a^{(b-r)}, (a-\lambda_s)^{m_s}, \ldots, (a-\lambda_1)^{m_1})$. From now on we will denote $\zeta = ((a-\lambda_s)^{m_s}, \ldots, (a-\lambda_1)^{m_1})$. Note that

$$U \cong H^\eta \boxtimes \mathbb{C}_{S_\beta} \cong \mathrm{Ind}^{S_{ab-\beta}}\left(H^{(a^{b-r})} \boxtimes H^\zeta\right) \boxtimes \mathbb{C}_{S_\beta}.$$

Hence

$$\begin{aligned} \left\langle \chi_{V^\lambda}, \chi^{(ab-(k+\beta),1^k)} \times 1_{S_\beta} \right\rangle &= \left\langle \chi_U, \chi^{(ab-(k+\beta),1^k)} \times 1_{S_\beta} \right\rangle \\ &= \left\langle \left(\phi^{(a^{(b-r)})} \times \psi^\zeta\right)\big\uparrow^{S_{ab-\beta}} \times 1_{S_\beta}, \chi^{(ab-(k+\beta),1^k)} \times 1_{S_\beta} \right\rangle \\ &= \left\langle \left(\phi^{(a^{(b-r)})} \times \psi^\zeta\right)\big\uparrow^{S_{ab-\beta}}, \chi^{(ab-(k+\beta),1^k)} \right\rangle \\ &= \sum_{\nu,\mu} d_\nu^\mu \left\langle (\chi^\nu \times \chi^\mu)\big\uparrow^{S_{ab-\beta}}, \chi^{(ab-(k+\beta),1^k)} \right\rangle \end{aligned}$$

where χ^ν is an irreducible character of $S_{a(b-r)}$ with non zero multiplicity in $\phi^{(a^{(b-r)})}$, χ^μ is an irreducible character of $S_{ar-\beta}$ having non zero multiplicity in ψ^ζ, and d_ν^μ is the multiplicity of their tensor product in the decomposition of H^η. Notice that the last sum is not equal to zero if and only if there exist ν and μ such that $(\chi^\nu \times \chi^\mu)\big\uparrow^{S_{ab-\beta}}$ contains a hook character of $S_{ab-\beta}$ having leg length equal to k in its decomposition. By Theorem 1.3.7, we have that both ν and μ must be subpartitions of $(ab-(k+\beta), 1^k)$. This means that ν and μ are hooks or trivial partitions. In particular we deduce from Theorem 2.1.2 that $\nu = (a(b-r))$. Hence, again from

Theorem 1.3.7, we need μ to be a hook with leg length at least $k-1$ to have

$$\left\langle (\chi^{\nu} \times \chi^{\mu})\big\uparrow^{S_{ab-\beta}}, \chi^{(ab-(k+\beta),1^k)} \right\rangle \neq 0.$$

On the other hand

$$\psi^{\zeta} = \left(\phi^{((a-\lambda_1)^{m_1})} \times \cdots \times \phi^{((a-\lambda_s)^{m_s})}\right)\big\uparrow^{S_{ar-\beta}}$$

So by Corollary 2.3.5 we have that the hooks that have non-zero multiplicity in the decomposition of ψ^{ζ} have at most s parts, where s is the number of different parts of λ.

We observe that the smallest number $\tilde{\beta}$ having a partition λ with k different parts is $\frac{k(k+1)}{2}$, with $\lambda = (k, k-1, \ldots, 2, 1)$. So under our hypothesis $\beta < \frac{k(k+1)}{2}$ we obtain that χ^{μ} cannot be a hook character with leg length at least $k-1$. Hence for all $\lambda \in P(\beta)_a^b$ we have that

$$\left\langle \chi_{V^{\lambda}}, \chi^{(ab-(k+\beta),1^k)} \times 1_{S_{\beta}} \right\rangle = 0.$$

This completes the proof. $\square$

We are now ready to prove Theorem 2.1.3.

Proof: [Theorem 2.1.3] We proceed by induction on t, the number of parts of the inside-partition α. If $t = 1$ then

$$\left\langle \phi^{(a^b)}, \chi^{[k:(\alpha_1)]} \right\rangle = 0$$

by Proposition 2.4.3.

Suppose now that $t > 1$ and the theorem holds when the inside-partition has less then t parts. Denote by ν the partition defined by

$$\nu = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{t-1}).$$

By Theorem 1.3.5, Lemma 2.4.2 and Frobenius reciprocity we have that

$$\begin{aligned}
\left\langle \phi^{(a^b)}, \chi^{[k:\alpha]} \right\rangle &\leqslant \left\langle \phi^{(a^b)}, \left(\chi^{[k:\nu]} \times 1_{S_{\alpha_t}}\right)\big\uparrow^{S_{ab}} \right\rangle \\
&= \left\langle \phi^{(a^b)}\big\downarrow_{S_{ab-\alpha_t}\times S_{\alpha_t}}, \chi^{[k:\nu]} \times 1_{S_{\alpha_t}} \right\rangle \\
&= \sum_{\lambda \in P(\alpha_t)^b_a} \left\langle \chi_{V^\lambda}, \chi^{[k:\nu]} \times 1_{S_{\alpha_t}} \right\rangle \\
&= \sum_{\lambda \in P(\alpha_t)^b_a} \left\langle \chi_{U^\lambda}, \chi^{[k:\nu]} \times 1_{S_{\alpha_t}} \right\rangle \\
&= \sum_{\lambda \in P(\alpha_t)^b_a} \left\langle \left(\phi^{(a^{(b-p(\lambda))})} \times \psi^{(a-\lambda_{p(\lambda)},\dots,a-\lambda_1)}\right)\big\uparrow^{S_{ab-\alpha_t}}, \chi^{[k:\nu]} \right\rangle \\
&= \sum_{\lambda \in P(\alpha_t)^b_a} \Big(\sum_{\zeta,\mu} d^\lambda_{\zeta\mu} \left\langle \left(\chi^\zeta \times \chi^\mu\right)\big\uparrow^{S_{ab-\alpha_t}}, \chi^{[k:\nu]} \right\rangle \Big),
\end{aligned}$$

where, for each $\lambda \in P(\alpha_t)^b_a$, U^λ is the largest $\mathbb{C}(S_{ab-\alpha_t} \times S_{\alpha_t})$ submodule of V^λ on which S_{α_t} acts trivially and $\sum_{\zeta,\mu} d^\lambda_{\zeta\mu}(\chi^\zeta \times \chi^\mu)$ is the decomposition into irreducible characters of the character $\phi^{(a^{(b-p(\lambda))})} \times \psi^{(a-\lambda_{p(\lambda)},\dots,a-\lambda_1)}$.

For every $\lambda \in P(\alpha_t)^b_a$, observe that every simple summand S^μ of $H^{(a-\lambda_{p(\lambda)},\dots,a-\lambda_1)}$ is a simple summand of the Young permutation module $M^{(a-\lambda_{p(\lambda)},\dots,a-\lambda_1)}$. Hence by Theorem 1.3.3 we have that the partition μ has at most $p(\lambda)$ parts; in particular it has at most α_t parts. It follows that, by Theorem 1.3.7, we need ζ to have at least $k+1-\alpha_t$ parts, and to be a subpartition of $[k:\nu]$ in order to have

$$\left\langle \left(\chi^\zeta \times \chi^\mu\right)\big\uparrow^{S_{ab-\alpha_t}}, \chi^{[k:\nu]} \right\rangle \neq 0.$$

Therefore ζ must be of the form

$$[k_\zeta : \beta] \vdash a(b-p(\lambda)).$$

By Theorem 1.3.7 we must have that

- $\beta = (\beta_1, \dots, \beta_s) \subseteq \nu$, and
- $k_\zeta \geqslant k - \alpha_t$.

We conclude proving that such a ζ cannot label any irreducible summand of the Foulkes character $\phi^{(a^{(b-p(\lambda))})}$.

Define $n_\zeta := \sum_{j=2}^s \beta_j$; if $s=1$ then let $n_\zeta = 0$. We observe that such a partition ζ has inside-partition β having $s \leqslant t-1$ parts and it satisfies the initial hypothesis, since

- $k_\zeta \geqslant k - \alpha_t > n - \alpha_t \geqslant n_\zeta$, and
- $\beta_1 \leqslant \alpha_1 < \frac{(k-n)(k-n+1)}{2} \leqslant \frac{(k_\zeta - \sum_{j=2}^{t-1}\alpha_j)(k_\zeta - \sum_{j=2}^{t-1}\alpha_j+1)}{2} \leqslant \frac{(k_\zeta - n_\zeta)(k_\zeta - n_\zeta+1)}{2}$.

Hence χ^ζ has zero multiplicity in $\phi^{(a^{(b-p(\lambda))})}$ by induction. Therefore

$$\left\langle \phi^{(a^b)}, \chi^{[k:\alpha]} \right\rangle \leqslant \sum_{\lambda \in P(\alpha_t)_a^b} \Big(\sum_{\zeta,\mu} d_{\zeta\mu}^{\lambda} \left\langle (\chi^\zeta \times \chi^\mu)\big\uparrow^{S_{ab-\alpha_t}}, \chi^{[k:\nu]} \right\rangle \Big) = 0.$$

The theorem is then proved. $\square$

As mentioned in the introduction, and as we will prove in the following corollary, a consequence of our main theorem is that every Specht module labelled by a partition having leg length equal to k and at most k boxes inside the hook has zero multiplicity, except when the k boxes are *column-shaped* (i.e. the inside-partition is (1^k)). In that particular case we are able to prove that the multiplicity equals 1, for all the values of $k < b$. The proof is similar to that of Proposition 2.4.3 and is omitted. In [15, Lemma 3.3] Dent proved the same result in the specific case $k = b - 1$.

COROLLARY 2.4.4 *Let a, b, k and m be natural numbers. Let $m \leqslant k$ and α be a partition of m not equal to (1^k). Then*

$$\left\langle \phi^{(a^b)}, \chi^{[k:\alpha]} \right\rangle = 0$$

Proof: Let α be an arbitrary partition of m not equal to (1^k). Then

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t).$$

Write $n := \sum_{j=2}^{t} \alpha_j$. We will show that $\chi^{[k:\alpha]}$ satisfies the hypothesis of Theorem 2.1.3 that:

- $k > n$, and
- $\alpha_1 < \frac{1}{2}(k-n)(k-n+1)$.

The first condition is trivial since

$$k \geqslant m = \alpha_1 + n.$$

To prove the second condition proceed by contradiction: suppose that

$$\alpha_1 \geqslant \frac{(k-n)(k-n+1)}{2}.$$

Then

$$k - n \geqslant \frac{(k-n)(k-n+1)}{2}.$$

This implies $k - n = 0$ or $k - n = 1$. The first situation is impossible because $0 = k - n \geqslant \alpha_1 > 0$. The second is also impossible because $0 < \alpha_1 \leqslant k - n = 1$ implies $\alpha_1 = 1$ and $\alpha_1 + n = k$ with $\alpha = (1^k)$. $\square$

Chapter 3

The decomposition matrix of the symmetric group

This chapter is based on the paper [29]. The results were obtained in collaboration with my Ph.D. supervisor Dr Mark Wildon. We equally contributed to prove all the main theorems of the chapter, except for Corollary 3.5.6 which was a very good idea of Dr Mark Wildon alone.

3.1 Introduction and outline

A central open problem in the representation theory of finite groups is to find the decomposition matrices of symmetric groups. The main result of this chapter gives a combinatorial description of certain columns of these matrices in odd prime characteristic. This result applies to certain blocks of arbitrarily high weight. Another notable feature is that it is obtained almost entirely by using the methods of local representation theory.

As already explained in Section 1.3.5, we recall that the rows of the decomposition matrix of the symmetric group S_n of degree n in prime characteristic p are labelled by the partitions of n, and the columns by the p-regular partitions of n, that is, partitions of n with at most $p-1$ parts of any given size. The entry $d_{\mu\nu}$ of the decomposition matrix records the number of composition factors of the Specht module S^μ, defined over a field of characteristic p, that are isomorphic to the simple module D^ν, first defined by James in [41] as the unique top composition factor of S^ν.

Given an odd number p, a p-core γ and $k \in \mathbb{N}_0$, let $w_k(\gamma)$ denote the minimum number of p-hooks that when added to γ give a partition with exactly k odd parts.

Let $\mathcal{E}_k(\gamma)$ denote the set of partitions with exactly k odd parts that can be obtained from γ by adding $w_k(\gamma)$ disjoint p-hooks. Our main theorem is as follows.

THEOREM 3.1.1 *Let p be an odd prime. Let γ be a p-core and let $k \in \mathbb{N}_0$. Let $n = |\gamma| + w_k(\gamma)p$. If $k \geqslant p$ suppose that*

$$w_{k-p}(\gamma) \neq w_k(\gamma) - 1.$$

Then $\mathcal{E}_k(\gamma)$ is equal to the disjoint union of subsets $\mathcal{X}_1, \ldots, \mathcal{X}_c$ such that each $\mathcal{X}_j$ has a unique maximal partition ν_j in the dominance order. Each ν_j is p-regular and the column of the decomposition matrix of S_n in characteristic p labelled by ν_j has 1s in the rows labelled by partitions in $\mathcal{X}_j$, and 0s in all other rows.

We leave it as a simple exercise to show that $w_k(\gamma)$ is well-defined. It may clarify the main hypothesis in Theorem 3.1.1 to remark that since $w_k(\gamma) \leq w_{k-p}(\gamma) + 1$, we have $w_{k-p}(\gamma) \neq w_k(\gamma) - 1$ if and only if $w_{k-p}(\gamma) > w_k(\gamma) - 1$.

In particular Theorem 3.1.1 implies that if λ is a maximal partition in $\mathcal{E}_k(\gamma)$ under the dominance order, then the only non-zero entries of the column of the decomposition matrix labelled by λ are 1s in rows labelled by partitions in $\mathcal{E}_k(\gamma)$. We give some examples of Theorem 3.1.1 in Example 3.5.2.

Much of the existing work on decomposition matrices of symmetric groups has concentrated on giving complete information about blocks of small weight. In contrast, Theorem 3.1.1 gives partial information about blocks of arbitrary weight. In Proposition 3.5.4 we show that there are blocks of every weight in which Theorem 3.1.1 completely determines some columns of the decomposition matrix.

We prove Theorem 3.1.1 by studying certain twists by the sign character of the Foulkes module $H^{(2^m)}$. For m, $k \in \mathbb{N}_0$, let

$$H^{(2^m;k)} = \mathrm{Ind}_{S_{2m} \times S_k}^{S_{2m+k}} \big(H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}\big).$$

Thus when $k = 0$ we have $H^{(2^m;k)} = H^{(2^m)}$, and when $m = 0$ we have $H^{(2^m;k)} = \mathrm{sgn}_{S_k}$; if $k = m = 0$ then $H^{(2^m;k)}$ should be regarded as the trivial module for the trivial group S_0. We call $H^{(2^m;k)}$ a *twisted Foulkes module*. It is known that the ordinary characters of these modules are multiplicity-free (see Lemma 3.2.1), but as one might expect, when $\mathbb{F}$ has prime characteristic, their structure can be quite intricate. Our main contribution is Theorem 3.1.2 below, which characterizes the vertices of indecomposable summands of $H^{(2^m;k)}$ when $\mathbb{F}$ has odd characteristic. The outline of the proof of Theorem 3.1.1 given at the end of this introduction shows how the local information given by Theorem 3.1.2 is translated into our result on decomposition matrices. This step, from local to global, is the key to the argument.

Theorem 3.1.2 *Let $m \in \mathbb{N}$ and let $k \in \mathbb{N}_0$. If U is an indecomposable non-projective summand of $H^{(2^m;k)}$, defined over a field $\mathbb{F}$ of odd characteristic p, then U has as a vertex a Sylow p-subgroup Q of $(S_2 \wr S_{tp}) \times S_{(r-2t)p}$ for some $t \in \mathbb{N}_0$ and $r \in \mathbb{N}$ with $tp \leq m$, $2t \leq r$ and $(r-2t)p \leq k$. Moreover the Green correspondent of U admits a tensor factorization $V \boxtimes W$ as a module for $\mathbb{F}\big((N_{S_{rp}}(Q)/Q) \times S_{2m+k-rp}\big)$, where V and W are projective, and W is an indecomposable summand of the twisted Foulkes module $H^{(2^{m-tp};k-(r-2t)p)}$.*

Theorem 3.1.2 is a significant result in its own right. For odd primes p, it gives the first infinite family of indecomposable p-permutation modules for the symmetric group (apart from Scott modules, which always lie in principal blocks) whose vertices are not Sylow p-subgroups of Young subgroups of symmetric groups.

An important motivation for the proof of Theorem 3.1.2 is [18], in which Erdmann uses similar methods to determine the p-local structure of Young permutation modules and to establish their decomposition into Young modules. Also relevant is [63], in which Paget shows that $H^{(2^m)}$ has a Specht filtration for any field $\mathbb{F}$. Using Theorem 11 of [75], it follows that $H^{(2^m;k)}$ has a Specht filtration for every $k \in \mathbb{N}_0$. The local behaviour of $H^{(2^m)}$ in characteristic 2, which as one would expect is very different to the case of odd characteristic, was analysed in [11]; the projective summands of $H^{(2^m;k)}$ in characteristic 2 are identified in [61, Corollary 9].

Background on decomposition numbers

The problem of finding decomposition numbers for symmetric groups in prime characteristic has motivated many deep results relating the representation theory of symmetric groups to other groups and algebras. Given the depth of the subject we give only a brief survey, concentrating on results that apply to Specht modules in blocks of arbitrarily high weight.

Fix an infinite field $\mathbb{F}$ of prime characteristic p. In [48] James proved that the decomposition matrix for S_n modulo p appears, up to a column reordering, as a submatrix of the decomposition matrix for polynomial representations of $\mathrm{GL}_d(\mathbb{F})$ of degree n, for any $d \geqslant n$. In [34, 6.6g] Green gave an alternative proof of this using the Schur functor from representations of the Schur algebra to representations of symmetric groups. James later established a similar connection with representations of the finite groups $\mathrm{GL}_d(\mathbb{F}_q)$, and the Hecke algebras $\mathcal{H}_{\mathbb{F},q}(S_n)$, in the case when p divides $q-1$ (see [46]). In [17] Erdmann proved, conversely, that every decomposition number for $\mathrm{GL}_d(\mathbb{F})$ appears as an explicitly determined decomposition number for some symmetric group.

In [41] James proved that if D^ν is a composition factor of S^μ then ν dominates μ, and that if μ is p-regular then $d_{\mu\mu} = 1$. This establishes the characteristic 'wedge' shape of the decomposition matrix of S_n with 1s on its diagonal, shown in the diagram in [39, Corollary 12.3]. In [65] Peel proved that the hook Specht modules $(n-r, 1^r)$ are irreducible when p does not divide n, and described their composition factors for odd primes p when p divides n. The p-regular partitions labelling these composition factors can be determined by James' method of p-regularization [44], which gives for each partition μ of n a p-regular partition ν such that ν dominates μ and $d_{\mu\nu} = 1$. In [43] and [42], James determined the decomposition numbers $d_{\mu\nu}$ for μ of the form $(n-r, r)$ and, when $p = 2$, of the form $(n-r-1, r, 1)$. These results were extended by Williams in [79]. In [47, 5.47] James and Mathas, generalizing a conjecture of Carter, conjectured a necessary and sufficient condition on a partition μ for the Specht module S^μ, defined for a Hecke algebra $\mathcal{H}_{\mathbb{F},q}(S_n)$ over a field $\mathbb{F}$, to be irreducible. The necessity of this condition was proved by Fayers [22] for symmetric groups (the case $q = 1$), building on earlier work of Lyle [55]; later Fayers [21] proved that the condition was sufficient for symmetric groups, and also for Hecke algebras whenever $\mathbb{F}$ has characteristic zero. In [51, Theorem 1.10], Kleshchev determined the decomposition numbers $d_{\lambda\mu}$ when μ is a p-regular partition whose Young diagram is obtained from the Young diagram of λ by moving a single box. In [78], Wildon proved that in odd characteristic the rows of any decomposition matrix of a symmetric group are distinct, and so a Specht module is determined, up to isomorphism, by its multiset of composition factors; in characteristic 2 the isomorphism $(S^\mu)^\star = S^{\mu'}$, where μ' is the conjugate partition to μ, accounts for all pairs of equal rows in the decomposition matrix.

In [24] Fayers proved that the decomposition numbers in blocks of weight 3 of abelian defect are either 0 or 1. This chapter includes a valuable summary of the many techniques for computing decomposition numbers and references to earlier results on blocks of weights 1 and 2. For results on weight 3 blocks of non-abelian defect, and blocks of weight 4, the reader is referred to [25] and [23]. For further general results, including branching rules and row and column removals theorems, see [56, Chapter 6, Section 4].

Outline

The main tool used to analyse the structure of twisted Foulkes modules over fields of odd characteristic is the Brauer correspondence for p-permutation modules, as described in Section 1.2.4.

In Section 3.2 below, we collect the general results we need on twisted Foulkes modules. In particular, Lemma 3.2.1 gives their ordinary characters. The twisted Foulkes modules $H^{(2^m;k)}$ are p-permutation modules, but not permutation modules (except when $k \leq 1$), and so some care is needed when applying the Brauer correspondence. Our approach is to use Lemma 3.2.3 to construct explicit p-permutation bases: for more theoretical results on monomial modules for finite groups the reader is referred to [4].

The main part of the proof begins in Section 3.3 where we prove Theorem 3.1.2. In Section 3.4, we prove Theorem 3.1.1, by filling in the details in the following sketch. The hypotheses of Theorem 3.1.1, together with Lemma 3.2.1 on the ordinary character of $H^{(2^m;k)}$, imply that $H^{(2^m;k)}$ has a summand in the block of S_{2m+k} with p-core γ. If this summand is non-projective, then it follows from Theorem 3.1.2, using Theorem 1.3.13 on the Brauer correspondence between blocks of symmetric groups, that either $H^{(2^m;k-p)}$ has a summand in the block of S_{2m+k-p} with p-core γ, or one of $H^{(2^{m-p};k)}$ and $H^{(2^m;k-2p)}$ has a summand in the block of $S_{2m+k-2p}$ with p-core γ. All of these are shown to be ruled out by the hypotheses of Theorem 3.1.1. Hence the summand is projective. A short argument using Lemma 3.2.1, Brauer reciprocity and Scott's lifting theorem then gives Theorem 3.1.1. We also obtain the proposition below, which identifies a particular projective summand of $H^{(2^m;k)}$ in the block of S_{2m+k} with p-core γ.

PROPOSITION 3.1.3 *Let p be an odd prime, let γ be a p-core and let $k \in \mathbb{N}_0$. If $k \geqslant p$ suppose that $w_{k-p}(\gamma) \neq w_k(\gamma) - 1$. Let $2m + k = |\gamma| + w_k(\gamma)p$. If λ is a maximal partition in the dominance order on $\mathcal{E}_k(\gamma)$ then λ is p-regular and the projective cover of the simple module D^λ is a direct summand of $H^{(2^m;k)}$, where both modules are defined over a field of characteristic p.*

In Section 3.5, we give some further examples and corollaries of Theorem 3.1.1 and Proposition 3.1.3. In Lemma 3.5.3 we show that given any odd prime p, any $k \in \mathbb{N}_0$, and any $w \in \mathbb{N}$, there is a p-core γ such that $w_k(\gamma) = w$. We use these p-cores to show that the lower bound $c_{\lambda\lambda} \geqslant w+1$ on the diagonal Cartan numbers in a block of weight w, proved independently by Richards [66, Theorem 2.8] and Bessenrodt and Uno [3, Proposition 4.6(i)], is attained for every odd prime p in p-blocks of every weight. Since the endomorphism algebra of each $H^{(2^m;k)}$ is commutative (in any characteristic), it also follows that for any odd prime p and any $w \in \mathbb{N}$, there is a projective module for a symmetric group lying in a p-block of weight w whose endomorphism algebra is commutative.

3.2 Twisted Foulkes modules

Throughout this section let $\mathbb{F}$ be a field and let $m \in \mathbb{N}$, $k \in \mathbb{N}_0$. Denote, as usual, by $\Omega^{(2^m)}$ the collection of all set partitions of $\{1, \ldots, 2m\}$ into m sets each of size two. We have already defined the Foulkes module $H^{(2^m)}$ to be the permutation module with $\mathbb{F}$-basis $\Omega^{(2^m)}$, and the twisted Foulkes module $H^{(2^m;k)}$ to be $\mathrm{Ind}_{S_{2m} \times S_k}^{S_{2m+k}} \big(H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}\big)$.

Let χ^λ denote the irreducible character of S_n corresponding to the partition λ of n. When $\mathbb{F}$ has characteristic zero the ordinary character of $H^{(2^m)}$ was found by Thrall [74, Theorem III] to be $\sum_\mu \chi^{2\mu}$ where the sum is over all partitions μ of m and 2μ is the partition obtained from μ by doubling each part.

Lemma 3.2.1 *The ordinary character of $H^{(2^m;k)}$ is $\sum_\lambda \chi^\lambda$, where the sum is over all partitions λ of $2m + k$ with exactly k odd parts.*

Proof: This follows from Theorem 1.3.6 applied to the ordinary character of $H^{(2^m)}$. $\square$

We remark that an alternative proof of Lemma 3.2.1 with minimal pre-requisites can be found in [37]; the main result of [37] uses the characters of twisted Foulkes modules to construct a 'model' character for each symmetric group containing each irreducible character exactly once.

In the remainder of this section we suppose that $\mathbb{F}$ has odd characteristic p and define a module isomorphic to $H^{(2^m;k)}$ that will be used in the calculations in Section 3.2. Let S_X denote the symmetric group on the set X. Let $\Delta^{(2^m;k)}$ be the set of all elements of the form

$$\big\{\{i_1, i'_1\}, \ldots, \{i_m, i'_m\}, (j_1, \ldots, j_k)\big\}$$

where $\{i_1, i'_1, \ldots, i_m, i'_m, j_1, \ldots, j_k\} = \{1, \ldots, 2m+k\}$. Given $\delta \in \Delta^{(2^m;k)}$ of the form above, we define

$$\mathcal{S}(\delta) = \big\{\{i_1, i'_1\}, \ldots, \{i_m, i'_m\}\big\},$$
$$\mathcal{T}(\delta) = \{j_1, \ldots, j_k\}.$$

The symmetric group S_{2m+k} acts transitively on $\Delta^{(2^m;k)}$ by

$$\delta g = \big\{\{i_1 g, i'_1 g\}, \ldots, \{i_m g, i'_m g\}, (j_1 g, \ldots, j_k g)\big\}$$

for $g \in S_{2m+k}$. Let $\mathbb{F}\Delta^{(2^m;k)}$ be the permutation module for $\mathbb{F}S_{2m+k}$ with $\mathbb{F}$-basis

$\Delta^{(2^m;k)}$. Let $K^{(2^m;k)}$ be the subspace of $\mathbb{F}\Delta^{(2^m;k)}$ spanned by

$$\big\{\delta - \operatorname{sgn}(g)\delta g : \delta \in \Delta^{(2^m;k)}, g \in S_{\mathcal{T}(\delta)}\big\}.$$

Since this set is permuted by S_{2m+k}, it is clear that $K^{(2^m;k)}$ is an $\mathbb{F}S_{2m+k}$-submodule of $\mathbb{F}\Delta^{(2^m;k)}$. For $\delta \in \Delta^{(2^m;k)}$, let $\overline{\delta} \in \mathbb{F}(\Delta^{(2^m;k)}/K^{(2^m;k)})$ denote the image $\delta + K^{(2^m;k)}$ of δ under the quotient map. Let $\Omega^{(2^m;k)}$ be the subset of $\Delta^{(2^m;k)}$ consisting of those elements of the form above such that $j_1 < \ldots < j_k$. In the next lemma we use $\Omega^{(2^m;k)}$ to identify $\mathbb{F}(\Delta^{(2^m;k)}/K^{(2^m;k)})$ with $H^{(2^m;k)}$.

Lemma 3.2.2

(i) *For each $\delta \in \Delta^{(2^m;k)}$ there exists a unique $\omega \in \Omega^{(2^m;k)}$ such that $\overline{\delta} \in \{\overline{\omega}, -\overline{\omega}\}$. Moreover, for this ω we have $\mathcal{S}(\delta) = \mathcal{S}(\omega)$ and $\mathcal{T}(\delta) = \mathcal{T}(\omega)$ and there exists a unique $h \in S_{\mathcal{T}(\delta)}$ such that $\delta h = \omega$.*

(ii) *The set $\{\overline{\omega} : \omega \in \Omega^{(2^m;k)}\}$ is an $\mathbb{F}$-basis for $\mathbb{F}(\Delta^{(2^m;k)}/K^{(2^m;k)})$.*

(iii) *The $\mathbb{F}S_{2m+k}$-modules $H^{(2^m;k)}$ and $\mathbb{F}(\Delta^{(2^m;k)}/K^{(2^m;k)})$ are isomorphic.*

Proof: For brevity we write K for $K^{(2^m;k)}$. Let

$$\delta = \big\{\{i_1, i_1'\}, \ldots, \{i_m, i_m'\}, (j_1, \ldots, j_k)\big\} \in \Delta^{(2^m;k)}.$$

The unique $h \in S_{\mathcal{T}(\delta)}$ such that $\delta h \in \Omega^{(2^m;k)}$ is the permutation such that

$$j_1 h < j_2 h < \cdots < j_k h.$$

Set $\omega = \delta h$. Since $S(\omega) = S(\delta)$ and $T(\omega) = T(\delta)$ we have the existence part of (i). Since $\delta - \operatorname{sgn}(h)\delta h \in K$, it follows that $\overline{\delta} = \operatorname{sgn}(h)\overline{\omega}$, and that $\mathbb{F}(\Delta^{(2^m;k)}/K)$ is spanned by $\{\overline{\omega} : \omega \in \Omega^{(2^m;k)}\}$. Set $x = h^{-1}$. If $g \in S_{\mathcal{T}(\delta)}$ then

$$\delta - \operatorname{sgn}(g)\delta g = -\operatorname{sgn}(x)\big(\omega - \operatorname{sgn}(x)\omega x\big) + \operatorname{sgn}(x)\big(\omega - \operatorname{sgn}(xg)\omega xg\big).$$

Since $\mathcal{T}(\delta) = \mathcal{T}(\omega)$ we have x, $xg \in S_{\mathcal{T}(\omega)}$. It follows that K is spanned by

$$\{\omega - \operatorname{sgn}(y)\omega y : \omega \in \Omega^{(2^m;k)}, y \in S_{\mathcal{T}(\omega)}\}.$$

Hence $\dim(\mathbb{F}(\Delta^{(2^m;k)}/K)) \leq |\Omega^{(2^m;k)}|$ and $\dim K \leq |\Omega^{(2^m;k)}|(k! - 1)$. Since

$$\dim(\mathbb{F}\Delta^{(2^m;k)}) = |\Omega^{(2^m;k)}|\,(k!)\ ,$$

we have equality in both cases. This proves part (ii). Moreover, if $\overline{\omega} = \pm\overline{\omega'}$ for $\omega, \omega' \in \Omega^{(2^m;k)}$ then $S(\omega) = S(\omega')$ and $T(\omega) = T(\omega')$, and so $\omega = \omega'$. This proves the uniqueness in (i).

For (iii), let

$$\omega = \big\{\{1,2\}, \ldots, \{2m-1, 2m\}, (2m+1, \ldots, 2m+k)\big\} \in \Omega^{(2^m;k)}.$$

Write $S_{2m} \times S_k$ for $S_{\{1,\ldots,2m\}} \times S_{\{2m+1,\ldots,2m+k\}}$, thought of as a subgroup of S_{2m+k}. Given $h \in S_{\{1,\ldots,2m\}}$ and $x \in S_{\{2m+1,\ldots,2m+k\}}$ we have $\overline{\omega} hx = \mathrm{sgn}(x)\overline{\omega h}$. By (ii) the set $\{\overline{\omega h} : h \in S_{2m}\}$ is linearly independent. Hence the $\mathbb{F}(S_{2m} \times S_k)$-submodule of $\mathbb{F}(\Delta^{(2^m;k)}/K)$ generated by $\overline{\omega}$ is isomorphic to $H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}$. Since

$$\dim(\mathbb{F}(\Delta^{(2^m;k)}/K)) = |\Omega^{(2^m;k)}| = \binom{2m+k}{k} \dim H^{(2^m)}$$

and the index of $S_{2m} \times S_k$ in S_{2m+k} is $\binom{2m+k}{k}$, it follows that

$$\mathbb{F}(\Delta^{(2^m;k)}/K) \cong \mathrm{Ind}_{S_{2m}\times S_k}^{S_{2m+k}} \big(H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}\big)$$

as required. $\square$

Since p is odd, Lemma 1.2.17 implies that

$$\mathrm{Ind}_{(S_2 \wr S_m)\times A_k}^{(S_2\wr S_m)\times S_k}(\mathbb{F}) = \mathbb{F}_{(S_2\wr S_m)\times S_k} \oplus \big(\mathbb{F}_{S_2 \wr S_m} \boxtimes \mathrm{sgn}_{S_k}\big)$$

where A_k denotes the alternating group on $\{2m+1, \ldots, 2m+k\}$. Therefore $H^{(2^m;k)}$ is a direct summand of the module induced from the trivial $\mathbb{F}\big((S_2 \wr S_m) \times A_k\big)$-module, and so, by Proposition 1.2.16, $H^{(2^m;k)}$ is a p-permutation module.

In the following lemma we construct a p-permutation basis for $H^{(2^m;k)}$.

Lemma 3.2.3 *Let P be a p-subgroup of S_{2m+k}.*

(i) *There is a choice of signs $s_\omega \in \{+1, -1\}$ for $\omega \in \Omega^{(2^m;k)}$ such that*

$$\big\{s_\omega \overline{\omega} : \omega \in \Omega^{(2^m;k)}\big\}$$

is a p-permutation basis for $H^{(2^m;k)}$ with respect to P.

(ii) *Let $\omega = \big\{\{i_1, i'_1\}, \ldots, \{i_m, i'_m\}, (j_1, \ldots, j_k)\big\} \in \Omega^{(2^m;k)}$ and let $g \in P$. Then $\overline{\omega}$ is fixed by g if and only if $\big\{\{i_1, i'_1\}, \ldots, \{i_m, i'_m\}\big\}$ is fixed by g.*

Proof: For $\omega \in \Omega^{(2^m;k)}$ let $\gamma(\omega) = \big(\mathcal{S}(\omega), \mathcal{T}(\omega)\big)$. Let

$$\Gamma^{(2^m;k)} = \big\{\gamma(\omega) : \omega \in \Omega^{(2^m;k)}\big\}.$$

Notice that the map

$$\gamma : \Omega^{(2^m;k)} \longrightarrow \Gamma^{(2^m;k)}$$

associating to each $\omega \in \Omega^{(2^m;k)}$ the element $\gamma(\omega) \in \Gamma^{(2^m;k)}$ is a bijection. The set $\Gamma^{(2^m;k)}$ is acted on by S_{2m+k} in the obvious way. Let $\gamma_1, \ldots, \gamma_c \in \Gamma^{(2^m;k)}$ be representatives for the orbits of P on $\Gamma^{(2^m;k)}$. For each $b \in \{1, \ldots, c\}$, let $\omega_b \in \Omega^{(2^m;k)}$ be the unique element such that $\gamma_b = \gamma(\omega_b)$. Given any $\omega \in \Omega^{(2^m;k)}$ there exists a unique b such that $\gamma(\omega)$ is in the orbit of P on $\Gamma^{(2^m;k)}$ containing γ_b. Choose $g \in P$ such that $\gamma(\omega) = \gamma_b g$. Then ω and $\omega_b g$ are equal up to the order of the numbers in their k-tuples, and so there exists $h \in S_{\mathcal{T}(\omega)}$ such that $\omega_b g h = \omega$. By Lemma 3.2.2(i) we have

$$\overline{\omega_b} g = s_\omega \overline{\omega}$$

for some $s_\omega \in \{+1, -1\}$. If $\tilde{g} \in P$ is another permutation such that $\gamma(\omega) = \gamma_b \tilde{g}$ then $\overline{\omega_b} g \tilde{g}^{-1} = \pm \overline{\omega_b}$. Hence the $\mathbb{F}$-span of $\overline{\omega_b}$ is a 1-dimensional representation of the cyclic p-group generated by $g\tilde{g}^{-1}$. The unique such representation is the trivial one, so $\overline{\omega_b} g = \overline{\omega_b}\tilde{g}$. The sign s_ω is therefore well-defined. Now suppose that ω, $\omega' \in \Omega^{(2^m;k)}$ and $h \in P$ are such that $s_\omega \overline{\omega} h = \pm s_{\omega'} \overline{\omega'}$. By construction of the basis there exists $\omega_b \in \Omega^{(2^m;k)}$ and g, $g' \in P$ such that $s_\omega \overline{\omega} = \overline{\omega_b} g$ and $s_{\omega'} \overline{\omega'} = \overline{\omega_b} g'$. Therefore

$$\overline{\omega_b} g h = s_\omega \overline{\omega} h = \pm s_{\omega'} \overline{\omega'} = \pm \overline{\omega_b} g'$$

and so $\overline{\omega_b} g h {g'}^{-1} = \pm \overline{\omega_b}$. As before, the plus sign must be correct. This proves (i).

For (ii), suppose that $\overline{\omega} g = \overline{\omega}$. Setting $\delta = \omega g$, and noting that $\overline{\delta} = \overline{\omega}$, it follows from Lemma 3.2.2(i) that $\mathcal{S}(\omega g) = \mathcal{S}(\delta) = \mathcal{S}(\omega)$. Hence the condition in (ii) is necessary. Conversely, if $\big\{\{i_1, i_1'\}, \ldots, \{i_m, i_m'\}\big\}$ is fixed by g then g permutes $\{j_1, \ldots, j_k\}$ and so $\overline{\omega} g \in \{\overline{\omega}, -\overline{\omega}\}$. Since $g \in P$, it now follows from (i) that $\overline{\omega} g = \overline{\omega}$, as required. $\square$

In applications of Lemma 3.2.3(ii) it will be useful to note that there is an isomorphism of S_{2m}-sets between $\Omega^{(2^m)}$ and the set of fixed-point free involutions in S_{2m}, where the symmetric group acts by conjugacy. Given $\omega \in \Omega^{(2^m;k)}$ with $\mathcal{S}(\omega) = \big\{\{i_1, i_1'\}, \ldots, \{i_m, i_m'\}\big\}$, we define

$$\mathcal{I}(\omega) = (i_1, i_1') \cdots (i_m, i_m') \in S_{2m+k}.$$

By Lemma 3.2.3(ii), if $g \in S_{2m+k}$ is a p-element then g fixes $\overline{\omega}$ if and only if g commutes with $\mathcal{I}(\omega)$. Corollary 1.2.19 and Lemma 3.2.3 therefore implies the following proposition, which we shall use repeatedly in the next section.

PROPOSITION 3.2.4 *Let R be a p-subgroup of S_{2m+k} and let P be a Sylow p-subgroup of S_{2m+k} containing a Sylow p-subgroup of $N_G(R)$. There is a choice of signs $s_\omega \in \{+1, -1\}$ for $\omega \in \Omega^{(2^m;k)}$ such that*

$$\left\{s_\omega\overline{\omega} : \omega \in \Omega^{(2^m;k)},\ \mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R)\right\}.$$

is a p-permutation basis for the Brauer correspondent $H^{(2^m;k)}(R)$ with respect to $P \cap N_G(R)$.

3.3 The local structure of $H^{(2^m;k)}$

In this section we prove Theorem 3.1.2. Throughout we let $\mathbb{F}$ be a field of odd characteristic p and fix $m \in \mathbb{N}$, $k \in \mathbb{N}_0$. Any vertex of an indecomposable non-projective summand of $H^{(2^m;k)}$ must contain, up to conjugacy, one of the subgroups

$$R_r = \langle z_1 z_2 \cdots z_r \rangle$$

where z_j is the p-cycle $(p(j-1)+1, \ldots, pj)$ and $rp \le 2m+k$, so we begin by calculating $H^{(2^m;k)}(R_r)$. In the second step we show that, for any $t \in \mathbb{N}$ such that $2t \le r$, the Brauer correspondent $H^{(2^{tp};(r-2t)p)}(R_r)$ is indecomposable as an $\mathbb{F}N_{S_{rp}}(R_r)$-module and determine its vertex; in the third step we combine these results to complete the proof.

First step: Brauer correspondent with respect to R_r

Let $r \in \mathbb{N}$ be such that $rp \le 2m+k$. We define

$$T_r = \{t \in \mathbb{N}_0 : tp \leqslant m,\ 2t \le r,\ (r-2t)p \le k\}.$$

For $t \in T_r$ let

$$\mathcal{A}_{2t} = \left\{\overline{\omega}\ :\ \begin{array}{l}\omega \in \Omega^{(2^m;k)}, \mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R_r) \\ \operatorname{supp}\mathcal{I}(\omega) \text{ contains exactly } 2t \text{ orbits of } R_r \text{ of length } p\end{array}\right\}.$$

Lemma 3.3.1 *There is a direct sum decomposition of $\mathbb{F}N_{S_{2m+k}}(R_r)$-modules*

$$H^{(2^m;k)}(R_r) \cong \bigoplus_{t\in T_r} \langle \mathcal{A}_{2t}\rangle.$$

Proof: By Proposition 3.2.4 the $\mathbb{F}N_{S_{2m+k}}(R_r)$-module $H^{(2^m;k)}(R_r)$ has as a basis

$$\mathcal{A} = \bigl\{\overline{\omega} : \omega \in \Omega^{(2^m;k)},\ \mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R_r)\bigr\}.$$

Let $\omega \in \Omega^{(2^m;k)}$ be such that $\mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R_r)$. Then $\mathcal{I}(\omega)$ permutes, as blocks for its action, the orbits of R_r. It follows that the number of orbits of R_r of length p contained in $\operatorname{supp}\mathcal{I}(\omega)$ is even. Suppose this number is $2t$. Clearly $2t \le r$ and $2tp \le 2m$. The remaining $r-2t$ orbits of length p are contained in $\mathcal{T}(\omega)$. Thus $(r-2t)p \le k$, and so $t \in T_r$ and $\overline{\omega} \in \mathcal{A}_{2t}$.

Let the p-cycles corresponding to the $2t$ orbits of R_r that are contained in $\operatorname{supp}\mathcal{I}(\omega)$ be $z_{j_1}, \ldots, z_{j_{2t}}$. Let $g \in N_{S_{2m+k}}(R_r)$. Let $\omega^\star \in \Omega^{(2^m;k)}$ be such that $\overline{\omega^\star} = \pm\,\overline{\omega g}$. The p-cycles $z_{j_1}^g, \ldots, z_{j_{2t}}^g$ correspond precisely to the orbits of R_r contained in $\operatorname{supp}\mathcal{I}(\omega^\star)$. Hence $\overline{\omega^\star} \in \mathcal{A}_{2t}$, and so the vector space $\langle\mathcal{A}_{2t}\rangle$ is invariant under g. Since $\mathcal{A} = \bigcup_{t\in T_r}\mathcal{A}_{2t}$ the lemma follows. $\square$

There is an obvious factorization $N_{S_{2m+k}}(R_r) = N_{S_{rp}}(R_r) \times S_{\{rp+1,\ldots,2m+k\}}$. The next proposition establishes a corresponding tensor factorization of the $N_{S_{2m+k}}(R_r)$-module $\langle\mathcal{A}_{2t}\rangle$. The shift required to make the second factor $H^{(2^{m-tp};k-(r-2t)p)}$ a module for $\mathbb{F}S_{\{rp+1,\ldots,2m+k\}}$ is made explicit in the proof.

Proposition 3.3.2 *If $t \in T_r$ then there is an isomorphism*

$$\langle\mathcal{A}_{2t}\rangle \cong H^{(2^{tp};(r-2t)p)}(R_r) \boxtimes H^{(2^{m-tp};k-(r-2t)p)}$$

of $\mathbb{F}(N_{S_{rp}}(R_r) \times S_{\{rp+1,\ldots,2m+k\}})$-modules.

Proof: In order to simplify the notation we shall write K for the $\mathbb{F}S_{2m+k}$-submodule $K^{(2^m;k)}$ of $\mathbb{F}\Delta^{(2^m;k)}$ defined just before Lemma 3.2.2. Recall that if $\omega \in \Omega^{(2^m;k)}$ then, by definition, $\overline{\omega} = \omega + K$. Let $J = K^{(2^{tp};(r-2t)p)}$. It follows from Proposition 3.2.4, in the same way as in Lemma 3.3.1, that $H^{(2^{tp};(r-2t)p)}(R_r)$ has as a basis

$$\{\omega + J : \omega \in \Omega^{(2^{tp};(r-2t)p)},\ \mathcal{I}(\omega) \in \mathrm{C}_{S_{rp}}(R_r)\}.$$

Define Δ^+ by shifting the entries in each of the elements of $\Delta^{(2^{m-tp},k-(r-2t)p)}$ by rp, so that $\mathbb{F}\Delta^+$ is an $\mathbb{F}S_{\{rp+1,\ldots,2m+k\}}$-module, and similarly define $\Omega^+ \subseteq \Delta^+$ by shifting

$\Omega^{(2^{m-tp},k-(r-2t)p)}$ and $J^+ \subseteq \mathbb{F}\Delta^+$ by shifting the basis elements of $K^{(2^{m-tp},k-(r-2t)p)}$. Then, by Lemma 3.2.2, $H^+ = \mathbb{F}\Delta^+/J^+$ is an $\mathbb{F}S_{\{rp+1,\ldots,2m+k\}}$-module with basis

$$\{\omega^+ + J^+ : \omega^+ \in \Omega^+\}.$$

We shall define a linear map $f : \langle \mathcal{A}_{2t} \rangle \to H^{(2^{tp};(r-2t)p)} \boxtimes H^+$. Given $\omega + K \in \mathcal{A}_{2t}$ where

$$\omega = \big\{\{i_1, i_1'\}, \ldots, \{i_m, i_m'\}, (j_1, \ldots, j_k)\big\} \in \Omega^{(2^m;k)}$$

and the notation is chosen so that

$$\{i_1, i_1', \ldots, i_{tp}, i_{tp}', j_1, \ldots, j_{(r-2t)p}\} = \{1, \ldots, rp\},$$

we define $(\omega + K)f = (\alpha + J) \otimes (\alpha^+ + J^+)$ where

$$\begin{aligned} \alpha &= \big\{\{i_1, i_1'\}, \ldots, \{i_{tp}, i_{tp}'\}, (j_1, \ldots, j_{(r-2t)p})\big\} \\ \alpha^+ &= \big\{\{i_{tp+1}, i_{tp+1}'\}, \ldots, \{i_{mp}, i_{mp}'\}, (j_{(r-2t)p+1}, \ldots, j_k)\big\}. \end{aligned}$$

This defines a bijection between $\mathcal{A}_{2t}$ and the basis for $H^{(2^{tp};(r-2t)p)}(R_r) \boxtimes H^+$ afforded by the bases for $H^{(2^{tp};(r-2t)p)}(R_r)$ and H^+ just defined. The map f is therefore a well-defined linear isomorphism.

Suppose that $\omega \in \Omega^{(2^m;k)}$ is as above and let $g \in N_{S_{2m+k}}(R_r)$. Let $h \in S_{\mathcal{T}(\omega g)}$ be the unique permutation such that $(j_1 gh, \ldots, j_k gh)$ is increasing. Let $\omega^\star = \omega gh$, so $\omega^\star \in \Omega^{(2^m;k)}$ and $\overline{\omega} g = \operatorname{sgn}(h)\overline{\omega^\star}$. Since g permutes $\{1, \ldots, rp\}$ we may factorize h as $h = xx^+$ where $x \in S_{\mathcal{T}(\alpha g)}$ and $x^+ \in S_{\mathcal{T}(\alpha^+ g)}$. By definition of f we have

$$(\omega^\star + K)f = (\alpha g x + J) \otimes (\alpha^+ g x^+ + J^+).$$

Hence

$$\begin{aligned} (\omega + K)gf &= \operatorname{sgn}(h)(\omega^\star + K)f \\ &= \operatorname{sgn}(h)\operatorname{sgn}(x)\operatorname{sgn}(x^+)(\alpha g + J) \otimes (\alpha^+ g + J^+) \\ &= (\omega + K)fg. \end{aligned}$$

The map f is therefore a homomorphism of $\mathbb{F}N_{S_{2m+k}}(R_r)$-modules. Since f is a linear isomorphism, the proposition follows. $\square$

Second step: the vertex of $H^{(2^{tp};(r-2t)p)}(R_r)$

Fix $r \in \mathbb{N}$ and $t \in \mathbb{N}_0$ such that $2t \leq r$. In the second step we show that the $\mathbb{F}N_{S_{rp}}(R_r)$-module $H^{(2^{tp};(r-2t)p)}(R_r)$ is indecomposable and that it has the subgroup Q_t defined below as a vertex.

To simplify the notation, we denote $H^{(2^{tp};(r-2t)p)}(R_r)$ by M. Let C and E_t be the elementary abelian p-subgroups of $N_{S_{rp}}(R_r)$ defined by

$$C = \langle z_1 \rangle \times \langle z_2 \rangle \times \cdots \times \langle z_r \rangle,$$

$$E_t = \langle z_1 z_{t+1} \rangle \times \cdots \times \langle z_t z_{2t} \rangle \times \langle z_{2t+1} \rangle \times \cdots \times \langle z_r \rangle,$$

where the z_j are the p-cycles defined at the start of this section. For $i \in \{1, \ldots, tp\}$, let $i' = i + tp$, and for $g \in S_{\{1,\ldots,tp\}}$, let $g' \in S_{\{tp+1,\ldots,2tp\}}$ be the permutation defined by $i'g' = (ig)'$. Note that if $1 \leq j \leq t$ then $z_j' = z_{j+t}$. Let L be the group consisting of all permutations gg' where g lies in a Sylow p-subgroup of $S_{\{1,\ldots,tp\}}$ containing the group $\langle z_1, \ldots, z_t \rangle$, chosen so that $z_1 \cdots z_t$ is in its centre. Let L^+ be a Sylow p-subgroup of $S_{\{2tp+1,\ldots,rp\}}$ containing the group $\langle z_{2t+1}, \ldots, z_r \rangle$, chosen so that $z_{2t+1} \cdots z_r$ is in its centre. (The existence of such Sylow p-subgroups follows from the construction of Sylow p-subgroups of symmetric groups described in Section 1.3.3. In particular we have that $z_1 \cdots z_t$ is a product of the central elements identified after Definition 1.3.8. Moreover we observe that the non-zero powers of $z_{2t+1}, \ldots, z_r$ are the unique p-cycles in L^+, by Remark 1.3.11.) Let

$$Q_t = L \times L^+.$$

Observe that Q_t normalizes C and so $\langle C, Q_t \rangle$ is a p-group contained in $\mathrm{C}_{S_{rp}}(R_r)$. Let P be a Sylow p-subgroup of $\mathrm{C}_{S_{rp}}(R_r)$ containing $\langle Q_t, C \rangle$. Since there is a Sylow p-subgroup of S_{rp} containing R_r in its centre, P is also a Sylow p-subgroup of S_{rp}. Clearly $E_t \leq C$ and

$$R_r \leq E_t \leq Q_t \leq P \leq \mathrm{C}_{S_{rp}}(R_r).$$

If $t = 0$ then M is the sign representation of $N_{S_{rp}}(R_r)$, with p-permutation basis $\mathcal{B} = \{\omega\}$ where ω is the unique element of $\Omega^{(2^0;rp)}$. It is then clear that M has the Sylow p-subgroup Q_0 of $\mathrm{C}_{S_{rp}}(R_r)$ as a vertex. We may therefore assume that $t \in \mathbb{N}$ for the rest of this step.

By Proposition 3.2.4 there is a choice of signs $s_\omega \in \{+1, -1\}$ for $\omega \in \Omega^{(2^{tp};(r-2t)p)}$ such that

$$\mathcal{B} = \{s_\omega \overline{\omega} : \omega \in \Omega^{(2^{tp};(r-2t)p)},\ \mathcal{I}(\omega) \in \mathrm{C}_{S_{rp}}(R_r)\}$$

is a p-permutation basis for M with respect to P. Let

$$\mathcal{O}_j = \{(j-1)p+1, \ldots, jp\}$$

be the orbit of z_j on $\{1, \ldots, rp\}$ of length p. If $\mathcal{I}(\omega) \in \mathrm{C}_{S_{rp}}(R_r)$ then $\mathcal{I}(\omega)$ permutes these orbits as blocks for its action; let

$$\mathcal{I}_{\mathcal{O}}(\omega) \in S_{\{\mathcal{O}_1,\ldots,\mathcal{O}_r\}}$$

be the involution induced by the action of $\mathcal{I}(\omega)$ on the set of orbits.

Proposition 3.3.3 *The $\mathbb{F}N_{S_{rp}}(R_r)$-module M is indecomposable and has a vertex containing E_t.*

Proof: For each involution $h \in S_{\{\mathcal{O}_1,\ldots,\mathcal{O}_r\}}$ that fixes exactly $r-2t$ of the orbits $\mathcal{O}_j$, and so moves the other $2t$, define

$$\mathcal{B}(h) = \{s_\omega \overline{\omega} \in \mathcal{B} : \mathcal{I}_{\mathcal{O}}(\omega) = h\}.$$

Clearly there is a vector space decomposition

$$M = \bigoplus_h \langle \mathcal{B}(h) \rangle.$$

If $g \in C$ then $\mathcal{I}_{\mathcal{O}}(\omega g) = \mathcal{I}_{\mathcal{O}}(\omega)$ since g acts trivially on the set of orbits $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$. Therefore C permutes the elements of each $\mathcal{B}(h)$.

Let

$$h^\star = (\mathcal{O}_1, \mathcal{O}_{t+1}) \cdots (\mathcal{O}_t, \mathcal{O}_{2t}) \in S_{\{\mathcal{O}_1,\ldots,\mathcal{O}_r\}}$$

and let $s_{\omega^\star}\overline{\omega^\star} \in \mathcal{B}(h^\star)$ be the unique basis element such that

$$\mathcal{I}(\omega^\star) = (1, tp+1)(2, tp+2) \cdots (tp, 2tp)$$

(Equivalently, $\mathcal{I}(\omega^\star)$ is the unique involution in S_{2tp} that preserves the relative orders of the elements in $\mathcal{O}_j$ for $1 \le j \le 2t$ and satisfies $\mathcal{I}_{\mathcal{O}}(\omega^\star) = h^\star$.) By Lemma 3.2.3(ii) we see that the stabiliser of $\overline{\omega^\star}$ in C is the subgroup E_t. Let $s_\delta\overline{\delta} \in B(h^\star)$, then $\mathcal{I}(\delta) = \mathcal{I}(\omega^\star) = h^\star$. Without loss of generality we have that

$$\mathcal{I}(\delta) = (1, i_1)(2, i_2) \cdots (tp, i_{tp}),$$

where $\{i_{(j-1)p+1}, i_{(j-1)p+2}, \ldots, i_{jp}\} = \mathcal{O}_{t+j}$ for all $j \in \{1, \ldots, t\}$. Hence, there exist $k_1, k_2, \ldots, k_t \in \{0, 1, \ldots, p-1\}$ and a permutation

$$g = z_{t+1}^{k_1} z_{t+2}^{k_2} \cdots z_{2t}^{k_t},$$

such that $(tp+(j-1)p+1)g = i_{(j-1)p+1}$ for all $j \in \{1, \ldots, t\}$. Since $s_\delta \overline{\delta}$ is fixed by R_r, it follows that $s_{\omega^\star} \overline{\omega^\star} g = s_\delta \overline{\delta}$. Therefore any basis element in $\mathcal{B}(h^\star)$ can be obtained from $\omega^\star$ by permuting the members of $\mathcal{O}_{t+1}$, ..., $\mathcal{O}_{2t}$ by an element of C. It follows that $\mathcal{B}(h^\star)$ has size p^t and is equal to the orbit of $s_{\omega^\star} \overline{\omega^\star}$ under C. Therefore there is an isomorphism of $\mathbb{F}C$-modules $\langle \mathcal{B}(h^\star) \rangle \cong \mathrm{Ind}_{E_t}^{C}(\mathbb{F})$. By Lemma 1.2.28, $\langle \mathcal{B}(h^\star) \rangle$ is an indecomposable $\mathbb{F}C$-module with vertex E_t.

For each involution $h \in S_{\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}}$, the $\mathbb{F}C$-submodule $\langle \mathcal{B}(h) \rangle$ of M is sent to $\langle \mathcal{B}(h^\star) \rangle$ by an element of $N_{S_{rp}}(R_r)$ normalizing C. It follows that if U is any summand of M, now considered as an $\mathbb{F}N_{S_{rp}}(R_r)$-module, then the restriction of U to C is isomorphic to a direct sum of indecomposable p-permutation $\mathbb{F}C$-modules with vertices conjugate in $N_{S_{rp}}(R_r)$ to E_t. Applying Theorem 1.2.18 to these summands, we see that there exists $g \in N_{S_{rp}}(R_r)$ such that $U(E_t^g) \neq 0$. Now by Theorem 1.2.18, this time applied to the $\mathbb{F}N_{S_{rp}}(R_r)$-module U, we see that U has a vertex containing E_t^g. Hence every indecomposable summand of M has a vertex containing E_t.

We now calculate the Brauer correspondent $M(E_t)$. Let $s_\omega \overline{\omega} \in \mathcal{B}$. It follows from Lemma 3.2.3(ii) that $\overline{\omega}$ is fixed by E_t if and only if $\mathcal{I}_{\mathcal{O}}(\omega)$ is the involution $h^\star$. Hence, by Corollary 1.2.19 and Lemma 3.2.3, we have $M(E_t) = \langle \mathcal{B}(h^\star) \rangle$. We have already seen that $\langle \mathcal{B}(h^\star) \rangle$ is indecomposable as an $\mathbb{F}C$-module. Since C normalizes E_t and centralizes R_r, it follows that $M(E_t)$ is indecomposable as a module for the normalizer of E_t in $N_{S_{rp}}(R_r)$. We already know that every indecomposable summand of M has a vertex containing E_t, so it follows from Corollary 1.2.25 that M is indecomposable. $\square$

Note that if $\omega^\star$ is as defined in the proof of Proposition 3.3.3, then Q_t is a Sylow p-subgroup of $\mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega^\star)\big) \cong (S_2 \wr S_{tp}) \times S_{(r-2t)p}$. Using this observation and the p-permutation basis $\mathcal{B}$ for M it is now straightforward to prove the following proposition.

Proposition 3.3.4 *The indecomposable $\mathbb{F}N_{S_{rp}}(R_r)$-module M has Q_t as a vertex.*

Proof: By Corollary 1.2.19, if Q is a subgroup of P maximal subject to $\mathcal{B}^Q \neq \varnothing$ then Q is a vertex of M. By Lemma 3.2.3(ii), a basis element $s_\omega \overline{\omega} \in B$ is fixed by a p-subgroup Q of P if and only if $Q \leq \mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega)\big)$. Taking $\omega = \omega^\star$ we see that there is a vertex of M containing Q_t. On the other hand, $\mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega)\big)$ is conjugate in S_{rp} to $\mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega^\star)\big)$, and so if $Q \leq \mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega)\big)$ then $|Q| \leq |Q_t|$. It follows that Q_t is a vertex of M. $\square$

Third step: proof of Theorem 3.1.2

For the remainder of the proof we shall regard $S_{(r-2t)p}$ as acting on $\{2tp+1, \ldots, rp\}$. We denote by D_t the p-group $C \cap N_{S_{rp}}(Q_t)$. Notice that $\langle D_t, Q_t \rangle$ is a p-group since it is a subgroup of $\langle C, Q_t \rangle \leqslant P$. We shall need the following lemma to work with modules for $N_{S_{rp}}(Q_t)$.

Lemma 3.3.5 *The unique Sylow p-subgroup of $N_{S_{rp}}(Q_t)$ is the subgroup $\langle D_t, Q_t \rangle$ of P.*

Proof: Let $x \in N_{S_{rp}}(Q_t)$. If $2t+1 \leq j \leq r$ then the conjugate z_j^x of the p-cycle $z_j \in E_t$ is a p-cycle in Q_t. Since Q_t normalizes E_t, it permutes the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_r$ of E_t as blocks for its action. No p-cycle can act non-trivially on these blocks, so $z_j^x \in \langle z_{2t+1}, \ldots, z_r \rangle$. Hence if $1 \leq j \leq t$ then $(z_j z_{j+t})^x \in \langle z_1 z_{t+1}, \ldots, z_t z_{2t} \rangle$. It follows that $N_{S_{rp}}(Q_t)$ factorizes as

$$N_{S_{rp}}(Q_t) = N_{S_{2tp}}(L) \times N_{S_{(r-2t)p}}(L^+)$$

where L and L^+ are as defined at the start of the second step. Moreover, we see that $N_{S_{rp}}(Q_t)$ permutes, as blocks for its action, the sets $\mathcal{O}_1 \cup \mathcal{O}_{t+1}, \ldots, \mathcal{O}_t \cup \mathcal{O}_{2t}$ and $\mathcal{O}_{2t+1}, \ldots, \mathcal{O}_r$.

Let $h \in N_{S_{rp}}(Q_t)$ be a p-element. We may factorize h as gg^+ where $g \in N_{S_{2tp}}(L)$ and $g^+ \in N_{S_{(r-2t)p}}(L^+)$ are p-elements. Since $\langle L^+, g^+ \rangle$ is a p-group and L^+ is a Sylow p-subgroup of $S_{(r-2t)p}$, we have $g^+ \in L^+$. Let

$$X = \{\mathcal{O}_1 \cup \mathcal{O}_{t+1}, \ldots, \mathcal{O}_t \cup \mathcal{O}_{2t}\}.$$

The group $\langle L, g \rangle$ permutes the sets in X as blocks for its action. Let

$$\pi : \langle L, g \rangle \to S_X$$

be the corresponding group homomorphism. By construction L acts on the sets $\mathcal{O}_1, \ldots, \mathcal{O}_t$ as a Sylow p-subgroup of $S_{\{\mathcal{O}_1, \ldots, \mathcal{O}_t\}}$; hence $L\pi$ is a Sylow p-subgroup of S_X. Since $\langle L, g \rangle$ is a p-group, there exists $\tilde{g} \in L$ such that $g\pi = \tilde{g}\pi$. Let $y = g\tilde{g}^{-1}$. Since y acts trivially on X, we may write

$$y = g_1 \ldots g_t$$

where $g_j \in S_{\mathcal{O}_j \cup \mathcal{O}_{j+t}}$ for each j. The p-group $\langle L, y\rangle$ has as a subgroup $\langle z_j z_{j+t}, y\rangle$. The permutation group induced by this subgroup on $\mathcal{O}_j \cup \mathcal{O}_{j+t}$, namely $\langle z_j z_{j+t}, g_j\rangle$, is a p-group acting on a set of size $2p$. Since p is odd, the unique Sylow p-subgroup of $S_{\mathcal{O}_j \cup \mathcal{O}_{j+t}}$ containing $z_j z_{j+t}$ is $\langle z_j, z_{j+t}\rangle$. Hence $g_j \in \langle z_j, z_{j+t}\rangle$ for each j. Therefore $y \in \langle z_1, \ldots, z_t, z_{t+1}, \ldots, z_{2t}\rangle \leq C$. We also know that $y \in \langle L, g\rangle \leqslant N_{S_{2tp}}(L) \leqslant N_{S_{rp}}(Q_t)$. Therefore $y \in D_t$, and since $\tilde{g} \in Q_t$, it follows that $g \in \langle D_t, Q_t\rangle$. Hence $h = gg^+ \in \langle D_t, Q_t\rangle \leqslant \langle C, Q_t\rangle \leqslant P$.

Conversely, the subgroup $\langle D_t, Q_t\rangle$ is contained in $N_{S_{rp}}(Q_t)$ because both D_t and Q_t are. It follows that $\langle D_t, Q_t\rangle$ is the unique Sylow p-subgroup of $N_{S_{rp}}(Q_t)$. $\square$

We are now ready to prove Theorem 3.1.2. We repeat the statement below for the reader's convenience.

THEOREM 3.1.2 *Let $m \in \mathbb{N}$ and let $k \in \mathbb{N}_0$. If U is an indecomposable non-projective summand of $H^{(2^m;k)}$, defined over a field $\mathbb{F}$ of odd characteristic p, then U has as a vertex a Sylow p-subgroup Q of $(S_2 \wr S_{tp}) \times S_{(r-2t)p}$ for some $t \in \mathbb{N}_0$ and $r \in \mathbb{N}$ with $tp \leq m$, $2t \leq r$ and $(r-2t)p \leq k$. Moreover the Green correspondent of U admits a tensor factorization $V \boxtimes W$ as a module for $\mathbb{F}\big((N_{S_{rp}}(Q)/Q) \times S_{2m+k-rp}\big)$, where V and W are projective, and W is an indecomposable summand of the twisted Foulkes module $H^{(2^{m-tp};k-(r-2t)p)}$.*

Proof: Let $r \in \mathbb{N}$ be maximal such that the subgroup R_r is contained in a vertex of U. Let $K = N_{S_{rp}}(R_r)$. By Lemma 3.3.1 and Proposition 3.3.2 there is an isomorphism of $N_{S_{2m+k}}(R_r)$-modules

$$H^{(2^m;k)}(R_r) \cong \bigoplus_{t \in T_r} \big(H^{(2^{tp};(r-2t)p)}(R_r) \boxtimes H^{(2^{m-tp};k-(r-2t)p)}\big)$$

compatible with the factorization $N_{S_{2m+k}}(R_r) = K \times S_{2m+k-rp}$, where we regard $S_{2m+k-rp}$ as acting on $\{rp+1, \ldots, 2m+k\}$ and shift each module $H^{(2^{m-tp};k-(r-2t)p)}$ appropriately.

For $t \in T_r$, let $M_t = H^{(2^{tp};(r-2t)p)}(R_r)$. By Proposition 3.3.4, each M_t is indecomposable as an $\mathbb{F}K$-module. Hence, by the Krull–Schmidt Theorem, there is a subset $T' \subset T_r$, and for each $t \in T'$, a non-zero summand W_t of $H^{(2^{m-tp};k-(r-2t)p)}$, such that

$$U(R_r) \cong \bigoplus_{t \in T'} M_t \boxtimes W_t$$

as $\mathbb{F}(K \times S_{2m+k-rp})$-modules. By Proposition 3.3.4, M_t has Q_t as a vertex for each non-zero $t \in T'$. It is clear that $M_0 = \mathrm{sgn}_{S_{rp}}(R_r)$ has vertex Q_0 as an $\mathbb{F}N_{S_{rp}}(R_r)$-module. Let ℓ be the least element of T'. If $t > \ell$ then Q_t does not contain a conjugate of the subgroup E_ℓ of Q_ℓ. Hence, by Theorem 1.2.18, we have $M_t(Q_\ell) = 0$. It now follows from Lemmas 1.2.23 and 1.2.24 that there is an isomorphism of $\mathbb{F}(N_K(Q_\ell) \times S_{2m+k-rp})$-modules

$$U(Q_\ell) \cong \big(U(R_r)\big)(Q_\ell) \cong M_\ell(Q_\ell) \boxtimes W_\ell.$$

Since M_ℓ has Q_ℓ as a vertex, we have $M_\ell(Q_\ell) \neq 0$. It follows that U has a vertex Q containing Q_ℓ.

Let $\mathcal{B}$ be the p-permutation basis for M_ℓ defined in the second step. Since $\mathcal{B}$ is permuted by the Sylow p-subgroup P of K, it follows from Corollary 1.2.19 and Lemma 3.3.5 that $\mathcal{C} = \mathcal{B}^{Q_\ell}$ is a p-permutation basis for the $\mathbb{F}N_K(Q_\ell)$-module $M_\ell(Q_\ell)$ with respect to the Sylow p-subgroup $\langle D_\ell, Q_\ell \rangle$ of $N_K(Q_\ell)$. Since W_ℓ is isomorphic to a direct summand of the p-permutation module $H^{(2^{m-\ell p};k-(r-2\ell)p)}$ it has a p-permutation basis $\mathcal{C}^+$ with respect to a Sylow p-subgroup P^+ of $S_{\{rp+1,\ldots,2m+k\}}$. Therefore

$$\mathcal{C} \times \mathcal{C}^+ = \{v \otimes v^+ : v \in \mathcal{C}, v^+ \in \mathcal{C}^+\}$$

is a p-permutation basis for $M_\ell(Q_\ell) \boxtimes W_\ell$ with respect to the Sylow subgroup $\langle D_\ell, Q_\ell \rangle \times P^+$ of $N_K(Q_\ell) \times S_{2m+k-rp}$.

Suppose, for a contradiction, that Q strictly contains Q_ℓ. Since Q is a p-group there exists a p-element $g \in N_Q(Q_\ell) \leqslant N_{S_{2m+k}}(Q_\ell)$ such that $g \notin Q_\ell$. Now Q_ℓ has orbits of length at least p on $\{1, \ldots, rp\}$ and fixes $\{rp+1, \ldots, 2m+k\}$. Since g permutes these orbits as blocks for its action, we may factorize g as $g = hh^+$ where $h \in N_{S_{rp}}(Q_\ell)$ and $h^+ \in S_{2m+k-rp}$. By Lemma 3.3.5 we have that $\langle Q_\ell, h \rangle \leq N_K(Q_\ell)$.

Corollary 1.2.19 now implies that $(\mathcal{C} \times \mathcal{C}^+)^{\langle Q_\ell, g \rangle} \neq \varnothing$. Let $v \otimes v^+ \in \mathcal{C} \times \mathcal{C}^+$ be such that $(v \otimes v^+)g = v \otimes v^+$. Then $v \in \mathcal{B}^{\langle Q_\ell, h \rangle}$. But Q_ℓ is a vertex of M_ℓ, so it follows from Corollary 1.2.19 that $h \in Q_\ell$. Hence h^+ is a non-identity element of Q. By taking an appropriate power of h^+ we find that Q contains a product of one or more p-cycles with support contained in $\{rp+1, \ldots, 2m+k\}$. This contradicts our assumption that r was maximal such that R_r is contained in a vertex of U.

Therefore U has vertex Q_ℓ. We saw above that there is an isomorphism $U(Q_\ell) \cong M_\ell(Q_\ell) \boxtimes W_\ell$ of $\mathbb{F}(N_K(Q_\ell) \times S_{2m+k-rp})$-modules. This identifies $U(Q_\ell)$ as a vector space on which $N_{S_{2m+k}}(Q_\ell) = N_{S_{rp}}(Q_\ell) \times S_{2m+k-rp}$ acts. It is clear from the isomorphism in Proposition 3.3.2 that $N_{S_{rp}}(Q_\ell)$ acts on the first tensor factor and $S_{2m+k-rp}$ acts on the second. Hence the action of $N_K(Q_\ell)$ on $M_\ell(Q_\ell)$ extends to an action of $N_{S_{rp}}(Q_\ell)$ on $M_\ell(Q_\ell)$ and we obtain a tensor factorization $V \boxtimes W_\ell$ of $U(Q_\ell)$ as a $N_{S_{rp}}(Q_\ell) \times S_{2m+k-rp}$-module. An outer tensor product of modules is projective if and only if both factors are projective, so by Theorem 1.2.20, V is a projective

$\mathbb{F}(N_{S_{rp}}(Q_\ell)/Q_\ell)$-module, W_ℓ is a projective $\mathbb{F}S_{2m+k-rp}$-module, and $U(Q_\ell)$ is the Green correspondent of U. $\square$

3.4 Proofs of Theorem 3.1.1 and Proposition 3.1.3

In this section we prove Proposition 3.1.3, and hence Theorem 3.1.1. It will be convenient to assume that $H^{(2^m;k)}$ is defined over $\mathbb{F}_p$. Proposition 3.1.3 then follows for an arbitrary field of characteristic p by change of scalars. We assume the common hypotheses for these results, so γ is a p-core such that $2m+k = |\gamma| + w_k(\gamma)p$ and if $k \geqslant p$ then

$$w_{k-p}(\gamma) \neq w_k(\gamma) - 1.$$

Let λ be a maximal element of $\mathcal{E}_k(\gamma)$ under the dominance order.

Write $H_{\mathbb{Q}}^{(2^m;k)}$ for the twisted Foulkes module defined over the rational field. This module has an ordinary character given by Lemma 3.2.1. In particular it has χ^λ as a constituent, and so the rational Specht module $S_{\mathbb{Q}}^\lambda$ is a direct summand of $H_{\mathbb{Q}}^{(2^m;k)}$. Therefore, by reduction modulo p, each composition factor of S^λ (now defined over $\mathbb{F}_p$) appears in $H^{(2^m;k)}$. In particular $H^{(2^m;k)}$ has non-zero block component for the block $B(\gamma, w_k(\gamma))$ with p-core γ and weight $w_k(\gamma)$. We now use Theorem 3.1.2 to show that any such block component is projective.

Proposition 3.4.1 *The block component of $H^{(2^m;k)}$ for the block $B(\gamma, w_k(\gamma))$ of S_{2m+k} is projective.*

Proof: Suppose, for a contradiction, that $H^{(2^m;k)}$ has a non-projective indecomposable summand U in $B(\gamma, w_k(\gamma))$. By Theorem 3.1.2, the vertex of U is a Sylow subgroup Q_t of $(S_2 \wr S_{tp}) \times S_{(r-2t)p}$ for some $r \in \mathbb{N}$ and $t \in \mathbb{N}_0$ such that $tp \leq m$, $2t \leq r$ and $(r-2t)p \leq k$.

Suppose first of all that $2t < r$. In this case there is a p-cycle $g \in Q_t$. Replacing Q_t with a conjugate, we may assume that $g = (1, \dots, p)$ and so $\langle g \rangle = R_1$ where R_1 is as defined at the start of the first step in Section 3.3. By Lemma 3.3.1 and Proposition 3.3.2, we have that $k \geqslant p$ and $U(R_1)$ is a direct summand of

$$H^{(2^m;k)}(R_1) = \mathrm{sgn}_{S_p}(\langle g \rangle) \boxtimes H^{(2^m;k-p)}.$$

Hence there exists an indecomposable summand W of $H^{(2^m;k-p)}$ such that

$$\mathrm{sgn}_{S_p}(\langle g \rangle) \boxtimes W \mid U(R_1).$$

By Theorem 1.3.13, W lies in the block $B(\gamma, w_k(\gamma) - 1)$ of S_{2m+k-p}. In particular, this implies that $H^{(2^m;k-p)}$ has a composition factor in this block. Therefore there is a constituent χ^μ of the ordinary character of $H^{(2^m;k-p)}$ such that S^μ lies in $B(\gamma, w_k(\gamma)-1)$. But then, by Lemma 3.2.1, μ is a partition with p-core γ having exactly $k-p$ odd parts and weight $w_k(\gamma)-1$. Adding a single vertical p-hook to μ gives a partition of weight $w_k(\gamma)$ with exactly k odd parts. Hence $w_{k-p}(\gamma) = w_k(\gamma) - 1$, contrary to the hypothesis on $w_{k-p}(\gamma)$.

Now suppose that $2t = r$. Let $g = (1, \ldots, p)(p+1, \ldots, 2p)$. Then $g \in Q_t$ by definition and $\langle g \rangle = R_2$. By Lemma 3.3.1 and Proposition 3.3.2 we have that $U(R_2)$ is a direct summand of

$$H^{(2^m;k)}(R_2) = \Big(H^{(2^p)}(\langle g \rangle) \boxtimes H^{(2^{m-p};k)}\Big) \bigoplus \Big(\mathrm{sgn}_{S_{2p}}(\langle g \rangle) \boxtimes H^{(2^m;k-2p)}\Big)$$

where the second summand should be disregarded if $k < 2p$. It follows that either there is an indecomposable summand V of $H^{(2^{m-p};k)}$ such that

$$H^{(2^p)}(\langle g \rangle) \boxtimes V \mid U(R_2),$$

or $k \geqslant 2p$ and there is an indecomposable summand W of $H^{(2^m;k-2p)}$ such that

$$\mathrm{sgn}_{S_{2p}}(\langle g \rangle) \boxtimes W \mid U(R_2).$$

Again we use Theorem 1.3.13. In the first case the theorem implies that V lies in the block $B(\gamma, w_k(\gamma) - 2)$ of $S_{2m+k-2p}$. Hence there is a constituent χ^μ of the ordinary character of $H^{(2^{m-p};k)}$ such that μ is a partition with p-core γ and weight $w_k(\gamma) - 2$ having exactly k odd parts. This contradicts the minimality of $w_k(\gamma)$. In the second case W also lies in the block $B(\gamma, w_k(\gamma) - 2)$ of $S_{2m+k-2p}$ and there is a constituent χ^μ of the ordinary character of $H^{(2^m;k-2p)}$ such that μ is a partition with p-core γ and weight $w_k(\gamma) - 2$ having exactly $k - 2p$ odd parts. But then by adding a single vertical p-hook to μ we reach a partition with weight $w_k(\gamma) - 1$ having exactly $k - p$ odd parts. Once again this contradicts the hypothesis that $w_{k-p}(\gamma) \neq w_k(\gamma) - 1$. $\square$

For ν a p-regular partition, let P^ν denote the projective cover of the simple module D^ν. To finish the proof of Proposition 3.1.3 we must show that if λ is a maximal element of $\mathcal{E}_k(\gamma)$ then P^λ is one of the projective summands of $H^{(2^m;k)}$ in the block $B(\gamma, w_k(\gamma))$. For this we need a lifting result for summands of the monomial module $H^{(2^m;k)}$, which we prove using the analogous result for permutation modules, as stated in Theorem 1.2.27. Let $\mathbb{Z}_p$ denote the ring of p-adic integers and let $H_{\mathbb{Z}_p}^{(2^m;k)}$

denote the twisted Foulkes module defined over $\mathbb{Z}_p$.

LEMMA 3.4.2 *If U is a direct summand of $H^{(2^m;k)}$ then there is a $\mathbb{Z}_pS_{2m+k}$-module $U_{\mathbb{Z}_p}$, unique up to isomorphism, such that $U_{\mathbb{Z}_p}$ is a direct summand of $H^{(2^m;k)}_{\mathbb{Z}_p}$ and $U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong U$.*

Proof: Let A_k denote the alternating group on $\{2m+1, \ldots, 2m+k\}$. Let $M = \mathrm{Ind}_{S_2 \wr S_m \times A_k}^{S_{2m+k}}(\mathbb{F}_p)$ be the permutation module of S_{2m+k} acting on the cosets of $S_2 \wr S_m \times A_k$ and let $M_{\mathbb{Z}_p} = \mathrm{Ind}_{S_2 \wr S_m \times A_k}^{S_{2m+k}}(\mathbb{Z}_p)$ be the corresponding permutation module defined over $\mathbb{Z}_p$. Since p is odd, the trivial $\mathbb{Z}_p(S_2 \wr S_m \times S_k)$ module is a direct summand of $\mathrm{Ind}_{S_2 \wr S_m \times A_k}^{S_2 \wr S_m \times S_k}(\mathbb{Z}_p)$. Hence, inducing up to S_{2m+k} (as in the remark after Lemma 3.2.2), we see that $M_{\mathbb{Z}_p} = H^{(2^m;k)}_{\mathbb{Z}_p} \oplus M'_{\mathbb{Z}_p}$ where $M'_{\mathbb{Z}_p}$ is a complementary $\mathbb{Z}_pS_{2m+k}$-module, and $M = H^{(2^m;k)} \oplus M'$ where M' is the reduction modulo p of $M'_{\mathbb{Z}_p}$.

By Scott's lifting theorem (see Theorem 1.2.27), reduction modulo p is a bijection between the summands of $M_{\mathbb{Z}_p}$ and the summands of M. By the same result, this bijection restricts to a bijection between the summands of the permutation module $M'_{\mathbb{Z}_p}$ and the summands of M'. Since U is a direct summand of M there is a summand $U_{\mathbb{Z}_p}$ of $M_{\mathbb{Z}_p}$, unique up to isomorphism, such that $U_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong U$. By the remarks just made, $U_{\mathbb{Z}_p}$ is isomorphic to a summand of $H^{(2^m;k)}_{\mathbb{Z}_p}$. $\square$

Let $P^\nu_{\mathbb{Z}_p}$ be the $\mathbb{Z}_p$-free $\mathbb{Z}_pS_{2m+k}$-module whose reduction modulo p is P^ν. By Brauer reciprocity (see for instance [71, §15.4]), the ordinary character of $P^\nu_{\mathbb{Z}_p}$ is

$$\psi^\nu = \sum_\mu d_{\mu\nu}\chi^\mu. \qquad (\star)$$

By Theorem 1.3.16, we have that if $d_{\mu\nu} \neq 0$ then ν dominates μ. Therefore the sum above may be taken over those partitions μ dominated by ν.

Proof: [Proposition 3.1.3] We have seen that each summand of $H^{(2^m;k)}$ in the block $B(\gamma, w_k(\gamma))$ is projective and that there is at least one such summand. Let $P^{\nu_1}, \ldots, P^{\nu_c}$ be the summands of $H^{(2^m;k)}$ in $B(\gamma, w_k(\gamma))$. Using Lemma 3.4.2 to lift these summands to summands of $H^{(2^m;k)}_{\mathbb{Z}_p}$ we see that the ordinary character of the summand of $H^{(2^m;k)}_{\mathbb{Z}_p}$ lying in the p-block of S_{2m+k} with core γ and weight $w_k(\gamma)$ is $\psi^{\nu_1} + \cdots + \psi^{\nu_c}$. By Lemma 3.2.1 we have

$$\psi^{\nu_1} + \cdots + \psi^{\nu_c} = \sum_{\mu \in \mathcal{E}_k(\gamma)} \chi^\mu. \qquad (\dagger)$$

By hypothesis λ is a maximal partition in the dominance order on $\mathcal{E}_k(\gamma)$, and by $(\star)$ each ψ^{ν_j} is a sum of ordinary irreducible characters χ^μ for partitions μ dominated by ν_j. Therefore one of the partitions ν_j must equal λ, as required. $\square$

We are now ready to prove Theorem 3.1.1. We repeat the statement below for the reader's convenience.

THEOREM 3.1.1 *Let p be an odd prime. Let γ be a p-core and let $k \in \mathbb{N}_0$. Let $n = |\gamma| + w_k(\gamma)p$. If $k \geqslant p$ suppose that*

$$w_{k-p}(\gamma) \neq w_k(\gamma) - 1.$$

Then $\mathcal{E}_k(\gamma)$ is equal to the disjoint union of subsets $\mathcal{X}_1, \ldots, \mathcal{X}_c$ such that each $\mathcal{X}_j$ has a unique maximal partition ν_j in the dominance order. Each ν_j is p-regular and the column of the decomposition matrix of S_n in characteristic p labelled by ν_j has 1s in the rows labelled by partitions in $\mathcal{X}_j$, and 0s in all other rows.

Proof: Suppose that the indecomposable projective summands of $H^{(2^m;k)}$ lying in the block $B(\gamma, w_k(\gamma))$ are $P^{\nu_1}, \ldots, P^{\nu_c}$. Then by (†) above, $\mathcal{E}_k(\gamma)$ has a partition into disjoint subsets $\mathcal{X}_1, \ldots, \mathcal{X}_c$ such that $\nu_j \in \mathcal{X}_j$ and

$$\psi^{\nu_j} = \sum_{\mu \in \mathcal{X}_j} \chi^{\mu}$$

for each j. It now follows from ($\star$) that the column of the decomposition matrix of S_n in characteristic p labelled by ν_j has 1s in the rows labelled by partitions in $\mathcal{X}_j$, and 0s in all other rows. $\square$

3.5 Applications of Theorem 3.1.1 and Proposition 3.1.3

We begin with a precise statement of the result on diagonal Cartan numbers mentioned in the introduction of this chapter after Proposition 3.1.3.

THEOREM 3.5.1 ([66, THEOREM 2.8] OR [3, PROPOSITION 4.6(I)]) *Let ν be a p-regular partition of n such that ν has weight w; then $d_{\mu\nu} \neq 0$ for at least $w+1$ distinct partitions μ.*

If $|\mathcal{E}_k(\gamma)| \leq 2w_k(\gamma) + 1$ then it follows from Theorems 3.1.1 and 3.5.1 that $\mathcal{E}_k(\gamma)$ has a unique maximal partition, say λ, and the only non-zero entries of the column of the decomposition matrix of S_n labelled by λ are 1s in rows labelled by partitions in $\mathcal{E}_k(\gamma)$. In these cases Theorem 3.1.1 becomes a sharp result.

EXAMPLE 3.5.2 *Firstly let $p = 3$ and let $\gamma = (3, 1, 1)$. We leave it to the reader to check that $w_0(\gamma) = 3$ and*

$$\mathcal{E}_0(\gamma) = \{(8, 4, 2), (6, 6, 2), (6, 4, 4), (6, 4, 2, 2)\}.$$

Hence the column of the decomposition matrix of S_{12} in characteristic 3 labelled by $(8, 4, 2)$ has 1s in the rows labelled by the four partitions in $\mathcal{E}_0(\gamma)$ and no other non-zero entries.

Secondly let $p = 7$ and let $\gamma = (4, 4, 4)$. Then $w_6(\gamma) = 2$ and $\mathcal{E}_6(\gamma) = \mathcal{X} \cup \mathcal{X}'$ where

$$\begin{aligned}
\mathcal{X} &= \{(11, 4, 4, 3, 1^4), (11, 4, 4, 2, 1^5), (10, 5, 4, 3, 1^4), (10, 5, 4, 2, 1^5)\},\\
\mathcal{X}' &= \{(9, 5, 5, 5, 1, 1), (9, 5, 5, 4, 1, 1, 1), (8, 5, 5, 5, 1, 1, 1)\}.
\end{aligned}$$

The partitions in $\mathcal{X}$ and $\mathcal{X}'$ are mutually incomparable under the dominance order. Thus Theorem 3.1.1 determines the columns of the decomposition matrix of S_{26} in characteristic 7 labelled by $(11, 4, 4, 3, 1^4)$ and $(9, 5, 5, 5, 1, 1)$.

Finally let $p = 5$ and let $\gamma = (5, 4, 2, 1^4)$. Then $w_6(\gamma) = 3$, and

$$\mathcal{E}_6(\gamma) = \left\{ \begin{array}{c} (15, 9, 2, 1^4), (15, 6, 5, 1^4), (13, 11, 2, 1^4) \\ (13, 6, 5, 3, 1^3), (10, 9, 7, 1^4), (10, 9, 5, 3, 1^3) \end{array} \right\}.$$

It is easily seen that $w_1(\gamma) > 2$. (In fact $w_1(\gamma) = 8$.) Therefore Theorem 3.1.1 determines the column of the decomposition matrix of S_{30} in characteristic 5 labelled by $(15, 9, 2, 1^4)$.

We now use the following combinatorial lemma to prove that the bound in Theorem 3.5.1 is attained in blocks of every weight. Note that when $p = 3$ and $e = 2$ the core used is $(3, 1, 1)$, as in the first example above.

LEMMA 3.5.3 *Let p be an odd number, let $e \in \mathbb{N}_0$, and let γ be the p-core represented by the p-abacus with two beads on runner 1, $e+1$ beads on runner $p-1$, and one bead on every other runner. If $0 \leq k \leq e+1$ then $w_k(\gamma) = e+1-k$ and $|\mathcal{E}_k(\gamma)| = w_k(\gamma)+1$.*

Proof: The p-core γ is represented by the abacus A shown in Figure 3.1 overleaf. Moving the lowest $e+1-k$ beads on runner $p-1$ down one step leaves a partition with exactly k odd parts. Therefore $w_k(\gamma) \leq e+1-k$.

Suppose that λ is a partition with exactly k odd parts that can be obtained by a sequence of single step bead moves on A in which exactly $e-r$ beads are moved on runner $p-1$ and at most $e+1-k$ moves are made in total. We may suppose that

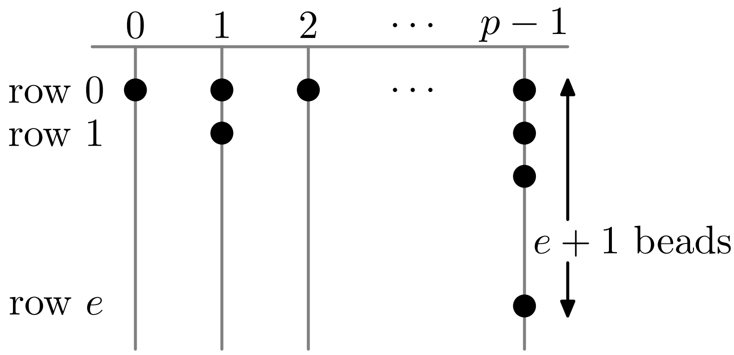

Figure 3.1: Abacus A representing the p-core γ in Lemma 3.5.3.

$r \geqslant k$ and that the beads on runner $p-1$ are moved first, leaving an abacus $A^\star$. Numbering rows as in Figure 3.1, so that row 0 is the highest row, let row ℓ be the lowest row (i.e. labelled by the greatest number) of $A^\star$ to which any bead is moved in the subsequent moves. Let B be the abacus representing λ that is obtained from $A^\star$ by making these moves. The number of spaces before each beads on runner $p-1$ in rows $\ell, \ell+1, \ldots, r$ is the same in both $A^\star$ and B, and is clearly odd in $A^\star$. Hence the parts corresponding to these beads are odd. Therefore $\ell \geqslant r-k+1$.

If B has a bead in row ℓ on a runner other than runner 1 or runner $p-1$, then this bead has been moved down from row 0, and so has been moved at least ℓ times. The total number of moves made is at least $(e-r)+\ell \geqslant e-k+1$, and so $\ell = r-k+1$. But now B has beads corresponding to odd parts of λ on runner $p-1$ in row 0, as well as rows $\ell, \ell+1, \ldots, r$, giving $k+1$ odd parts in total, a contradiction.

It follows that the sequence of bead moves leading to B may be reordered so that the first $e-r$ moves are made on runner $p-1$, and then the lowest bead on runner 1 is pushed down $r-k$ times to row $r-k+1$. The partition after these moves has $k+1$ odd parts. Moving the bead on runner 1 down one step from row $r-k+1$ reduces the number of odd parts by one, and is the only such move that does not move a bead on runner $p-1$. Therefore $\mathcal{E}_k(\gamma)$ contains the partition constructed at the start of the proof, and one further partition for each $r \in \{0, 1, \ldots, e-k\}$. $\square$

Given an arbitrary weight $w \in \mathbb{N}$ and $k \in \mathbb{N}_0$, Lemma 3.5.3 gives an explicit partition λ satisfying the hypothesis of Theorem 3.1.1 and such that $w_k(\gamma) = w$. We use this in the following proposition.

Proposition 3.5.4 *Let p be an odd prime and let k, $w \in \mathbb{N}_0$ be given. There exists a p-core γ and a partition λ with p-core γ and weight w such that λ has exactly k odd parts and the only non-zero entries in the column of the decomposition matrix labelled by λ are 1s lying in the $w+1$ rows labelled by elements of $\mathcal{E}_k(\gamma)$.*

Proof: If $w = 0$ and $k = 0$ then take $\lambda = (2)$. Otherwise let γ be the p-core in Lemma 3.5.3 when $e = w + k - 1$. By this lemma we have $w_k(\gamma) = w$. Moreover, if $k \geqslant p$ then $w_{k-p}(\gamma) = w + p$. Taking λ to be a maximal element of $\mathcal{E}_k(\gamma)$, the proposition follows from Theorem 3.1.1 and Theorem 3.5.1. $\square$

We now turn to an application of Proposition 3.1.3. Write $H_R^{(2^m;k)}$ for the twisted Foulkes module defined over a commutative ring R. Since the ordinary character of $H_{\mathbb{Q}}^{(2^m;k)}$ is multiplicity-free, the endomorphism algebra of $H_{\mathbb{F}}^{(2^m;k)}$ is commutative whenever the field $\mathbb{F}$ has characteristic zero. Hence the endomorphism ring of $H_{\mathbb{Z}}^{(2^m;k)}$ is commutative. This ring has a canonical $\mathbb{Z}$-basis indexed by the double cosets of the subgroup $S_2 \wr S_m \times S_k$ in S_{2m+k}. This basis makes it clear that the canonical map

$$\mathrm{End}_{\mathbb{Z}S_{2m+k}}(H_{\mathbb{Z}}^{(2^m;k)}) \to \mathrm{End}_{\mathbb{F}_p S_{2m+k}}(H_{\mathbb{F}_p}^{(2^m;k)})$$

is surjective, and so $\mathrm{End}_{\mathbb{F}S_{2m+k}}(H_{\mathbb{F}}^{(2^m;k)})$ is commutative for *any* field $\mathbb{F}$. This fact has some strong consequences for the structure of twisted Foulkes modules.

PROPOSITION 3.5.5 *Let U and V be distinct summands in a decomposition of $H^{(2^m;k)}$, defined over a field $\mathbb{F}$, into direct summands. Then $\mathrm{End}_{\mathbb{F}S_{2m+k}}(U)$ is commutative and $\mathrm{Hom}_{\mathbb{F}S_{2m+k}}(U, V) = 0$.*

Proof: Let π_U be the projection map from $H^{(2^m;k)}$ onto U along V and let ι_U and ι_V be the inclusion maps of U and V respectively into $H^{(2^m;k)}$. Suppose that $\phi \in \mathrm{Hom}_{\mathbb{F}S_{2n}}(U, V)$ is a non-zero homomorphism. Then $\iota_V \phi \pi_U$ does not commute with $\iota_U \pi_U$. Moreover sending $\theta \in \mathrm{End}_{\mathbb{F}S_{2m+k}}(U)$ to $\iota_U \theta \pi_U$ defines an injective multiplicative (but not unital) map from $\mathrm{End}_{\mathbb{F}S_{2m+k}}(U)$ into the commutative algebra $\mathrm{End}_{\mathbb{F}S_{2m+k}}(H^{(2^m;k)})$. $\square$

Proposition 3.5.5 implies that if λ is a p-regular partition and P^λ is a direct summand of $H^{(2^m;k)}$, defined over a field of characteristic p, then there are no non-zero homomorphisms from P^λ to any other summand of $H^{(2^m;k)}$. Thus every composition factor of $H^{(2^m;k)}$ isomorphic to D^λ must come from P^λ. We also obtain the following corollary.

COROLLARY 3.5.6 *Let $\mathbb{F}$ be a field of odd characteristic. Given any $w \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and an indecomposable projective module P^λ for $\mathbb{F}S_n$ lying in a block of weight w such that $\mathrm{End}_{FS_n}(P^\lambda)$ is commutative.*

Proof: Let γ be the p-core in Lemma 3.5.3 when $e + 1 = w$. Taking $k = 0$ we see that $w_0(\gamma) = w$. If λ is a maximal element of $\mathcal{E}_0(\gamma)$ then, by Proposition 3.1.3, P^λ is a direct summand of $H^{(2^m)}$, where $2m = |\lambda|$ and both modules are defined over the field $\mathbb{F}$. The result now follows from Proposition 3.5.5. $\square$

Chapter 4

The modular structure of the Foulkes module

4.1 Introduction and outline

The main goal of this chapter is to generalize part of the work done in Chapter 3 on the permutation module $H^{(2^n)}$, in order to obtain a complete description of the Green vertices and Green correspondents of the complete family of Foulkes modules $H^{(a^n)}$ for any $n \in \mathbb{N}$ and any $a < p$ where p is the fixed prime characteristic of the underlying field $\mathbb{F}$. In particular in Section 4.2 we will prove the following theorem:

THEOREM 4.1.1 *Let p be an odd prime and let a and n be natural numbers such that $a < p \leqslant n$. Let U be an indecomposable non-projective summand of the $\mathbb{F}S_{an}$-module $H^{(a^n)}$ and let Q be a vertex of U. Then there exists $s \in \{1, 2, \ldots, \lfloor \frac{n}{p} \rfloor\}$ such that Q is conjugate to a Sylow p-subgroup of $S_a \wr S_{sp}$. Moreover the Green correspondent of U admits a tensor factorization $V \boxtimes Z$ as a module for $\mathbb{F}((N_{S_{asp}}(Q)/Q) \times S_{a(n-sp)})$, where V is isomorphic to the projective cover of the trivial $\mathbb{F}(N_{S_{asp}}(Q)/Q)$-module and Z is an indecomposable, projective summand of $H^{(a^{n-sp})}$.*

This result is a generalization of Theorem 3.1.2. The structure of the proof is similar to the one given in Section 3.3 but different ideas and ad hoc arguments will be needed in this case.

In Section 4.3 we will use this new information on the modular structure of the Foulkes modules to derive corollaries on the decomposition matrices of the symmetric groups. To present our main result we need to introduce the following definition. Let γ be a p-core partition and let $\phi^{(a^n)}$ be the ordinary character afforded by the Foulkes module $H^{(a^n)}$. Denote by $\mathcal{F}_0(\gamma)$ the set containing all the partitions μ of an

such that the p-core $\gamma(\mu)$ of μ is equal to γ and such that the irreducible ordinary character of S_{an} labelled by μ has non-zero multiplicity in the decomposition of $\phi^{(a^n)}$ as a sum of irreducible characters. In general, for any $s \in \{0, 1, \ldots, \lfloor \frac{n}{p} \rfloor\}$ denote by $\mathcal{F}_{sp}(\gamma)$ the set defined by

$$\mathcal{F}_{sp}(\gamma) = \{\mu \vdash a(n-sp) \ : \ \gamma(\mu) = \gamma \ \text{ and } \ \left\langle \chi^{\mu}, \phi^{(a^{n-sp})} \right\rangle \neq 0\}.$$

The new results obtained in Theorem 4.1.1 will lead us to prove the following theorem.

Theorem 4.1.2 *Let a and n be natural numbers and let p be an odd prime such that $a < p \leqslant n$. Let λ be a p-regular partition of na. Denote by γ the p-core of λ. Suppose that for all $s \in \{1, 2, \ldots, \lfloor \frac{n}{p} \rfloor\}$, $\mathcal{F}_{sp}(\gamma) = \emptyset$. If λ is maximal under the dominance order in $\mathcal{F}_0(\gamma)$, then the only non-zero entries in the column labelled by λ of the decomposition matrix of S_{an} are in the rows labelled by partitions $\mu \in \mathcal{F}_0(\gamma)$. Moreover*

$$[S^{\mu} : D^{\lambda}] \leqslant \left\langle \phi^{(a^n)}, \chi^{\mu} \right\rangle.$$

Dealing with partitions of *small* p-weight we obtain the following sharper result.

Corollary 4.1.3 *Let a and n be natural numbers and let p be an odd prime such that $a < p \leqslant n$. Let λ be a p-regular partition of na such that λ has p-weight $w < a$. Denote by γ the p-core of λ. If λ is maximal in $\mathcal{F}_0(\gamma)$, then the only non-zero entries in the column labelled by λ of the decomposition matrix of S_{an} are in the rows labelled by partitions $\mu \in \mathcal{F}_0(\gamma)$. Moreover*

$$[S^{\mu} : D^{\lambda}] \leqslant \left\langle \phi^{(a^n)}, \chi^{\mu} \right\rangle.$$

Theorem 4.1.2 and Corollary 4.1.3 allow us to detect new information on decomposition numbers from the study of the ordinary structure of the Foulkes character. In particular the study of the zero-multiplicity characters in the decomposition of $\phi^{(a^n)}$ leads to some new non-obvious zeros in certain columns of the decomposition matrix of S_{an} (see Corollary 4.3.2 and Example 4.3.3).

4.2 The indecomposable summands of $H^{(a^n)}$

This section is devoted to the proof of Theorem 4.1.1. We start by fixing some notation. Let p be an odd prime number and let a and n be natural numbers such that $a < p \leqslant n$. Let $\mathbb{F}$ be a field of prime characteristic p and let $S_a \wr S_n$ be the subgroup of S_{an} acting transitively and imprimitively on $\{1, 2, \ldots, an\}$ and having as blocks of imprimitivity the sets

$$T_j = \{j, n+j, 2n+j, \ldots, (a-1)n+j\}$$

for all $j \in \{1, \ldots, n\}$. In this setting we have that for any Sylow p-subgroup P of $S_a \wr S_n$ there exists a Sylow p-subgroup Q of $S_{\{1,\ldots,n\}}$, such that P is conjugate to

$$\overline{Q} = \{\overline{x} \mid x \in Q\},$$

where $(j+kn)\overline{x} = (j)x + kn$ for all $j \in \{1, \ldots, n\}$ and all $k \in \{0, 1, \ldots, a-1\}$. Let ρ be an element of order p in $S_a \wr S_n$. By the above remarks there exists $s \in \{1, 2, \ldots, \lfloor \frac{n}{p} \rfloor\}$ such that ρ has sa orbits of order p and $a(n-sp)$ fixed points in its natural action on $\{1, 2, \ldots, an\}$.

For all $j \in \mathbb{N}$ such that $pj \leqslant an$ let z_j be the p-cycle of S_{an} defined by

$$z_j = (p(j-1)+1, p(j-1)+2, \ldots, pj).$$

Denote by R_ℓ the cyclic subgroup of S_{an} of order p generated by $z_1 z_2 \cdots z_\ell$. We will call $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ the *p-orbits* of R_ℓ in order to have $\mathcal{O}_j = \mathrm{supp}(z_j)$ for all $j \in \{1, 2, \ldots, \ell\}$.

In the following lemma we will study the Broué correspondence for $H^{(a^n)}$ with respect to R_ℓ.

LEMMA 4.2.1 *Let a and n be natural numbers and p an odd prime such that $a < p \leqslant n$. Let ℓ be a natural number such that $\ell p \leqslant an$. If $\ell = as$ for some natural number s, then*

$$H^{(a^n)}(R_{as}) \cong H^{(a^{sp})}(R_{as}) \boxtimes H^{(a^{n-sp})}$$

as $\mathbb{F}N_{S_{an}}(R_{as})$-modules. If ℓ is not an integer multiple of a then $H^{(a^n)}(R_\ell) = 0$.

Proof: We already noticed that the number of p-orbits of an element of order p in $S_a \wr S_n$ must be a multiple of a. Therefore if ℓ is not an integer multiple of a then R_ℓ is not conjugate to any subgroup of $S_a \wr S_n$. This implies that $H^{(a^n)}(R_\ell) = 0$ by Theorem 1.2.18.

Suppose now that $\ell = as$ for some $s \in \mathbb{N}$. Let $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \in \Omega^{(a^n)}$ be fixed by R_{as}. Then there exist $\omega_{j_1}, \ldots, \omega_{j_{sp}}$ sets of ω such that

$$\bigcup_{i=1}^{sp} \omega_{j_i} = \mathrm{supp}(R_{as})$$

since no set of a fixed set partition can contain two numbers x and y such that $x \in \operatorname{supp}(R_{as})$ and $y \notin \operatorname{supp}(R_{as})$. So we can write each fixed set partition $\omega \in \Omega^{(a^n)}(R_{as})$ as $\omega = u_\omega \cup v_\omega$ where

$$u_\omega = \left\{\omega_{j_1}, \ldots, \omega_{j_{sp}}\right\} \in \Omega^{(a^{sp})}(R_{as}),$$

and v_ω is a set partition in $\Omega_+^{(a^{n-sp})}$, that is the collection of all the set partitions of $\{asp+1, \ldots, an\}$ into $n - sp$ sets of size a. We will also denote by $H_+^{(a^{n-sp})}$ the $\mathbb{F}S_{\{asp+1,\ldots,an\}}$-permutation module generated by $\Omega_+^{(a^{n-sp})}$ as a vector space. The map

$$\psi : \Omega^{(a^n)}(R_{as}) \longrightarrow \Omega^{(a^{sp})}(R_{as}) \times \Omega_+^{(a^{n-sp})}$$

that associates to each $\omega \in \Omega^{(a^n)}(R_{as})$ the element $u_\omega \times v_\omega \in \Omega^{(a^{sp})}(R_{as}) \times \Omega_+^{(a^{n-sp})}$ is a well defined bijection. This factorization of the linear basis of $H^{(a^n)}(R_{as})$ induces an isomorphism of vector spaces between $H^{(a^n)}(R_{as})$ and $H^{(a^{sp})}(R_{as}) \boxtimes H_+^{(a^{n-sp})}$, that is compatible with the action of $N_{S_{an}}(R_{as}) \cong N_{S_{asp}}(R_{as}) \times S_{a(n-sp)}$. Therefore we have that

$$H^{(a^n)}(R_{as}) \cong H^{(a^{sp})}(R_{as}) \boxtimes H_+^{(a^{n-sp})}$$

as $\mathbb{F}(N_{S_{an}}(R_{as}))$-modules. The proposition follows after identifying $S_{\{asp+1,\ldots,an\}}$ with $S_{a(n-sp)}$ and $H_+^{(a^{n-sp})}$ with $H^{(a^{n-sp})}$. $\square$

Lemma 4.2.1 allow us to restrict for the moment our attention to the study of the Broué correspondent $H^{(a^{sp})}(R_{as})$ of $H^{(a^{sp})}$. In particular we will now give a precise description of its canonical basis $\Omega^{(a^{sp})}(R_{as})$ constitued by the set partitions fixed under the action of R_{as}. In order to do this we need to introduce a new important concept.

Let $\delta = \{\delta_1, \delta_2, \ldots, \delta_s\}$ be a set partition of $\{1, 2, \ldots, as\}$ into s sets of size a (namely $\delta \in \Omega^{(a^s)}$). Let $A_1, A_2, \ldots, A_s$ be subsets of $\{1, 2, \ldots, asp\}$ of size a such that

$$|A_i \cap \mathcal{O}_j| = \begin{cases} 1 & \text{if } \ j \in \delta_i. \\ 0 & \text{if } \ j \notin \delta_i. \end{cases}$$

In particular each set A_i contains at most one element of a given orbit of R_{as}. Consider now ω to be the element of $\Omega^{(a^{sp})}(R_{as})$ of the form

$$\omega = \{A_1, A_1\sigma, A_1\sigma^2, \ldots, A_1\sigma^{p-1}, A_2, \ldots, A_2\sigma^{p-1}, \ldots\ldots, A_s, \ldots, A_s\sigma^{p-1}\},$$

where $\sigma = z_1 z_2 \cdots z_{as}$. In this situation we will say that the set partition ω is of *type* δ. Notice that from the type we can read how the orbits of R_{as} are relatively distributed in the sets of the set partition ω.

In the following lemma we will show that the set partitions of $\Omega^{(a^{sp})}$ that are fixed by the action of R_{as} are precisely the ones of the form described above.

LEMMA 4.2.2 *Let the set partition* $\omega = \{\omega_1, \ldots, \omega_{sp}\}$ *be an element of* $\Omega^{(a^{sp})}$. *Then* ω *is fixed by* R_{as} *if and only if there exists a corresponding set partition* $\delta = \{\delta_1, \ldots, \delta_s\} \in \Omega^{(a^s)}$ *and* s *sets* $A_1, \ldots, A_s$ *of* ω *such that*

$$|A_i \cap \mathcal{O}_j| = \begin{cases} 1 & \textit{if} \;\; j \in \delta_i \\ \\ 0 & \textit{if} \;\; j \notin \delta_i \end{cases}$$

and

$$\omega = \{A_1, A_1\sigma, A_1\sigma^2, \ldots, A_1\sigma^{p-1}, A_2, \ldots, A_2\sigma^{p-1}, \ldots\ldots, A_s, \ldots, A_s\sigma^{p-1}\},$$

where $\sigma = z_1 z_2 \cdots z_{as}$.

Proof: Suppose that ω is fixed by $R_{as} = \langle\sigma\rangle$. Let $\mathcal{O}_j$ be an orbit of R_{as} for some $j \in \{1, 2, \ldots, as\}$ and let $\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_l}$ be the sets of ω such that $\omega_{j_i} \cap \mathcal{O}_j \neq \emptyset$. Clearly $l \leqslant p$ because $|\mathcal{O}_j| = p$. Since $\omega\sigma = \omega$ we have that for all $x \in \{j_1, \ldots, j_l\}$ there exists $y \in \{j_1, \ldots, j_l\}$ such that $\omega_{j_x}\sigma = \omega_{j_y}$ and no ω_{j_i} is fixed by σ because $a < p$. In particular we have that R_{as} acts without fixed points on the set $\{\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_l}\}$, therefore there exists a number $k \geqslant 1$ such that

$$kp = |\{\omega_{j_1}, \omega_{j_2}, \ldots, \omega_{j_l}\}| = l \leqslant p.$$

This immediately implies that $l = p$ and therefore that $|\omega_{j_i} \cap \mathcal{O}_j| = 1$ for all $i \in \{1, 2, \ldots, l\}$. This argument holds for all the R_{as}-orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{as}$. Hence for all $x \in \{1, 2, \ldots, sp\}$ the set ω_x of ω contains a numbers no two of which are in the same R_{as}-orbit. Consider one of those sets, say A_1, of ω. Define the correspondent set δ_1 of size a as follows: for all $i \in \{1, 2, \ldots, as\}$ let $i \in \delta_1$ if and only if $|A_1 \cap \mathcal{O}_i| = 1$. Observe that since $\omega\sigma = \omega$ we have that $A_1, A_1\sigma, \ldots, A_1\sigma^{p-1}$ are p distinct sets of ω such that for all $k \in \{0, 1, \ldots, p-1\}$ we have that

$$|A_1\sigma^k \cap \mathcal{O}_j| = \begin{cases} 1 & \text{if} \;\; j \in \delta_1 \\ \\ 0 & \text{if} \;\; j \notin \delta_1. \end{cases}$$

We repeat now the above construction by considering a set A_2 of ω such that $A_2 \neq A_1\sigma^k$ for any $k \in \{0, 1, \ldots, p-1\}$ and defining the corresponding set δ_2 exactly as above. After s iterations of the process we obtain the claim, where the set partition $\delta \in \Omega^{(a^s)}$ corresponding to ω is $\delta = \{\delta_1, \delta_2, \ldots, \delta_s\}$.

The converse is trivial since an ω of the form described in the hypothesis is clearly fixed by the action of σ. $\square$

From Lemma 4.2.2 we obtain that every $\omega \in \Omega^{(a^{sp})}(R_{as})$ is of a well defined type $\delta \in \Omega^{(a^s)}$. In the next lemma we will fix a $\delta \in \Omega^{(a^s)}$ and we will calculate explicitly the number of set partitions of type δ in $\Omega^{(a^{sp})}(R_{as})$.

Lemma 4.2.3 *For every $\delta \in \Omega^{(a^s)}$ there are exactly $p^{(a-1)s}$ distinct set partitions in $\Omega^{(a^{sp})}(R_{as})$ of type δ.*

Proof: Define $\delta^\star \in \Omega^{(a^s)}$ by

$$\delta^\star = \big\{\{1, 1+s, \ldots, 1+(a-1)s\}, \{2, 2+s, \ldots, 2+(a-1)s\}, \cdots, \{s, 2s, \ldots, as\}\big\}.$$

By Lemma 4.2.2 we have that given any set partition $\omega = \{\omega_1, \ldots, \omega_{sp}\} \in \Omega^{(a^{sp})}(R_s)$ of type $\delta^\star$, each set ω_j contains exactly one element lying in $\{1, 2, \ldots, sp\}$, the union of the first s orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_s$ of R_{as}. Therefore, without loss of generality, we can relabel the indices of the sets of ω in order to have $\omega = \{\omega_1, \ldots, \omega_{sp}\}$ and for all $j \in \{1, 2, \ldots, sp\}$

$$\omega_j = \{j, x_j^1, x_j^2, \ldots, x_j^{a-1}\},$$

where x_j^i is a number lying in the R_{as}-orbit of $j + isp$ for all $i \in \{1, 2, \ldots, a-1\}$. Notice that this implies that there are p possible different choices for each x_j^i. If we fix $j \in \{1, 2, \ldots, sp\}$ such that j is not divisible by p then there exist unique natural numbers t and k in $\{0, 1, \ldots, s-1\}$ and $\{1, 2, \ldots, p-1\}$ respectively, such that $j = tp + k$. Moreover, by definition of σ it follows that $((t+1)p)\sigma^k = j$. Since $\omega\sigma^k = \omega$, we must have $\omega_{(t+1)p}\sigma^k = \omega_j$. Therefore for all $i \in \{1, 2, \ldots, a-1\}$ we have that

$$x_j^i = x_{(t+1)p}^i \sigma^k$$

Hence the set partition ω is uniquely determined by its sets $\omega_p, \omega_{2p}, \ldots, \omega_{sp}$. This implies that there are exactly $p^{(a-1)s}$ different set partitions of type $\delta^\star$ in $\Omega^{(a^{sp})}(R_{as})$. It is an easy exercise to verify that, changing the labels, the argument above works for any other type δ in $\Omega^{(a^s)}$. $\square$

Consider now the subgroup C of $N_{S_{asp}}(R_{as})$ defined by

$$C = \langle z_1 \rangle \times \langle z_2 \rangle \times \ldots \times \langle z_{as} \rangle .$$

Notice that C preserves the type in its action on $\Omega^{(a^{sp})}(R_{as})$. Therefore we have that the subvectorspace K_δ of $H^{(a^{sp})}(R_{as})$ generated by all the fixed set partitions of type δ is an $\mathbb{F}C$-submodule of $H^{(a^{sp})}(R_{as})$ for any given δ. Moreover, we deduce the following result:

Proposition 4.2.4 *The following isomorphism of $\mathbb{F}C$-modules holds:*

$$\mathrm{Res}_C\left(H^{(a^{sp})}(R_{as})\right) \cong \bigoplus_{\delta \in \Omega^{(a^s)}} K_\delta .$$

Proof: For any given $\delta \in \Omega^{(a^s)}$ denote by B_δ the subset of $\Omega^{(a^{sp})}(R_{as})$ consisting of all the set partitions of type δ. Clearly $H^{(a^{sp})}(R_{as})$ decomposes as a vector space into the direct sum of all the K_δ for $\delta \in \Omega^{(a^s)}$. Moreover we observe that the orbits of C on $\{1, 2, \ldots, asp\}$ are exactly the same as the orbits of R_{as}, therefore if $\omega \in B_\delta$ then $\omega c \in B_\delta$ for any $c \in C$. This implies that

$$\mathrm{Res}_C(H^{(a^{sp})}(R_{as})) \cong \bigoplus_{\delta \in \Omega^{(a^s)}} K_\delta$$

as $\mathbb{F}C$-modules, as desired. $\square$

We will now define three p-subgroups of S_{asp} that will play a central role in the next part of the section. For all $j \in \{1, 2, \ldots, s\}$ denote by π_j the p-element of C given by

$$\pi_j = z_j z_{j+s} z_{j+2s} \cdots z_{j+(a-1)s}.$$

Let E_s be the elementary abelian subgroup of C of order p^s defined by

$$E_s = \langle \pi_1 \rangle \times \cdots \times \langle \pi_s \rangle.$$

Let P_s be a Sylow p-subgroup of $S_{\{1,\ldots,sp\}}$ with base group $\langle z_1, \ldots, z_s \rangle$, chosen so that $z_1 z_2 \cdots z_s$ is in its centre. (The existence of such Sylow p-subgroup follows from the construction of Sylow p-subgroups of symmetric groups described in Section 1.3.3. In particular we have that $z_1 z_2 \cdots z_s$ is a product of the central elements identified after Definition 1.3.8. Moreover we observe that the non-zero powers of $z_1, z_2, \ldots, z_s$ are the unique p-cycles in P_s, by Remark 1.3.11.)

Let Q_s be the group consisting of all permutations $\overline{g}$ where g lies in P_s. For the reader's convenience we recall that for all $k \in \{0, 1, \ldots, a-1\}$ and all $j \in \{1, 2, \ldots, sp\}$, we have that

$$(j + ksp)\overline{g} = (j)g + ksp.$$

In particular we observe that this implies that $\overline{g} = g_0 g_1 \cdots g_{a-1}$, where for all $k \in \{0, 1, \ldots a-1\}$, g_k is the element of S_{asp} that fixes all the numbers outside $\{ksp+1, ksp+2, \ldots, (k+1)sp\}$ and such that $(j+ksp)g_k = (j)g + ksp$ for all $j \in \{1, 2, \ldots, sp\}$. In particular we have that $\overline{z_i} = \pi_i$ for all $i \in \{1, 2, \ldots, s\}$. Notice that Q_s has E_s as normal subgroup by construction and clearly $R_{as} \trianglelefteq E_s \trianglelefteq C$ and $R_{as} \trianglelefteq Q_s$.

We are now very close to deduce the indecomposability of $H^{(a^{sp})}(R_{as})$ as an $\mathbb{F}N_{S_{asp}}(R_{as})$-module. In order to prove this we need to observe an important structural property of the $\mathbb{F}C$-modules K_δ for all $\delta \in \Omega^{(a^s)}$.

Proposition 4.2.5 *For any $\delta \in \Omega^{(a^s)}$ there exists $g \in N_{S_{asp}}(R_{as})$ such that*

$$K_\delta \cong \mathrm{Ind}_{E_s^g}^{C}(\mathbb{F})$$

Proof: As usual, define $\delta^\star \in \Omega^{(a^s)}$ by

$$\delta^\star = \{\delta_1, \delta_2, \ldots, \delta_s\},$$

where $\delta_i = \{i, i+s, i+2s, \ldots, i+(a-1)s\}$ and let $\omega^\star$ be any fixed element of $B_{\delta^\star}$. Then by Lemma 4.2.2 we have that

$$\omega^\star = \{A_1, A_1\sigma, \ldots, A_1\sigma^{p-1}, A_2, A_2\sigma, \ldots, A_2\sigma^{p-1}, \ldots\ldots, A_s, A_s\sigma \ldots, A_s\sigma^{p-1}\},$$

for some sets $A_1, A_2, \ldots, A_s$ such that

$$|A_i \cap \mathcal{O}_j| = \begin{cases} 1 & \text{if } \ j \in \delta_i^\star \\ 0 & \text{if } \ j \notin \delta_i^\star. \end{cases}$$

This implies that we can equivalently rewrite $\omega^\star$ as

$$\omega^\star = \{A_1, A_1\pi_1, \ldots, A_1\pi_1^{p-1}, A_2, A_2\pi_2 \ldots A_2\pi_2^{p-1}, \ldots\ldots, A_s, \ldots, A_s\pi_s^{p-1}\}.$$

Therefore $\omega^\star$ is fixed by the action of E_s. Moreover if we denote by L the stabilizer in C of $\omega^\star$ we have that as $\mathbb{F}C$-modules

$$K_{\delta^\star} \cong \mathrm{Ind}_L^C(\mathbb{F})$$

since C acts transitively on the elements of $B_{\delta^\star}$.

Lemma 4.2.3 implies that $\dim_{\mathbb{F}}(K_{\delta^\star}) = p^{s(a-1)}$, therefore by [1, Corollary 3, page 56] we have that

$$p^{s(a-1)} = |C : L| \leqslant |C : E_s| = p^{s(a-1)}.$$

Hence $E_s = L$ and $K_{\delta^\star} \cong \mathrm{Ind}_{E_s}^C(\mathbb{F})$ as $\mathbb{F}C$-modules. Since $N_{S_{asp}}(R_{as})$ acts as the full symmetric group on the set $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{as}\}$, we obtain that for any $\delta \in \Omega^{(a^s)}$ there exists $g \in N_{S_{asp}}(R_{as})$ such that any set partition of type δ in $\Omega^{(a^{sp})}(R_{as})$ is fixed by E_s^g. With an argument completely similar to the one used above we deduce that $K_\delta \cong \mathrm{Ind}_{E_s^g}^C(\mathbb{F})$. □

The following corollary of Proposition 4.2.5 will be extremely useful in the last part of the section.

Corollary 4.2.6 *Every indecomposable summand of $H^{(a^{sp})}(R_{as})$ has vertex containing E_s.*

Proof: Let U be an indecomposable summand of $H^{(a^{sp})}(R_{as})$. By Lemma 1.2.28, Proposition 4.2.4 and Proposition 4.2.5 we observe that the restriction of U to C is isomorphic to a direct sum of indecomposable p-permutation $\mathbb{F}C$-modules with vertices conjugate in $N_{S_{asp}}(R_{as})$ to E_s. Therefore by the first part of Corollary 1.2.22 we obtain that E_s is contained in a vertex of U. □

It is now possible to prove that $H^{(a^{sp})}(R_{as})$ is indecomposable and to determine a vertex of $H^{(a^{sp})}(R_{as})$ as an $\mathbb{F}N_{S_{asp}}(R_{as})$-module.

Proposition 4.2.7 *The $\mathbb{F}N_{S_{asp}}(R_{as})$-module $H^{(a^{sp})}(R_{as})$ is indecomposable and has vertex $Q_s \in \mathrm{Syl}_p(S_a \wr S_{sp})$.*

Proof: Let $\delta^\star$ be the set partition of $\Omega^{(a^s)}$ defined at the beginning of the proof of Proposition 4.2.5. Since $\omega \in \Omega^{(a^{sp})}(R_{as})$ is fixed by E_s if and only if $\omega \in B_{\delta^\star}$ and since $E_s \trianglelefteq C$, we have that

$$\mathrm{Res}_C\left((H^{(a^{sp})}(R_{as}))(E_s)\right) = \mathrm{Res}_C\left(H^{(a^{sp})}(R_{as})\right)(E_s) = K_{\delta^\star}(E_s) \cong \mathrm{Ind}_{E_s}^C(\mathbb{F}),$$

as $\mathbb{F}C$-modules. By Lemma 1.2.28 we have that $\mathrm{Res}_C\left((H^{(a^{sp})}(R_{as}))(E_s)\right)$ is indecomposable, hence also $(H^{(a^{sp})}(R_{as}))(E_s)$ is indecomposable. Therefore by Proposition 1.2.25 there exists a unique summand of $H^{(a^{sp})}(R_{as})$ with vertex containing E_s, but this implies that $H^{(a^{sp})}(R_{as})$ is indecomposable by Corollary 4.2.6.

Let $Q \leqslant N_{S_{asp}}(R_{as})$ be a vertex of $H^{(a^{sp})}(R_{as})$. Consider $\omega^\star$ to be the set partition in $\Omega^{(a^{sp})}(R_{as})$ defined by

$$\omega^\star = \{\omega_1, \omega_2, \dots, \omega_{sp}\},$$

where $\omega_j = \{j, j + sp, j + 2sp, \dots, j + (a-1)sp\}$, for all $j \in \{1, 2, \dots, sp\}$. By construction we have that Q_s fixes $\omega^\star$. Therefore a conjugate of Q_s is a subgroup of Q. On the other hand by Corollary 1.2.19 there exists $\omega \in \Omega^{(a^{sp})}(R_{as})$ such that Q fixes ω. Since the stabilizer of ω in S_{asp} is isomorphic to $S_a \wr S_{sp}$, we deduce that Q is isomorphic to a subgroup of a Sylow p-subgroup of $S_a \wr S_{sp}$. In particular this implies that $|Q| \leqslant |Q_s|$ and therefore we obtain that Q_s is a vertex of $H^{(a^{sp})}(R_{as})$. $\square$

Corollary 4.2.8 *The Foulkes module $H^{(a^{sp})}$ has a unique indecomposable summand U with vertex $Q_s \in \mathrm{Syl}_p(S_a \wr S_{sp})$. Moreover, any other indecomposable summand of $H^{(a^{sp})}$ has vertex conjugate to a subgroup of Q_s. In particular, U is the Scott module $\mathrm{Sc}(S_{asp}, S_a \wr S_{sp})$ and we have that*

$$H^{(a^{sp})}(Q_s) = U(Q_s) \cong P_{\mathbb{F}}$$

as $\mathbb{F}(N_{S_{asp}}(Q_s)/Q_s)$-modules, where $P_{\mathbb{F}}$ denotes the projective cover of the trivial module.

Proof: Since $H^{(a^{sp})}$ is isomorphic to the permutation module induced by the action of S_{asp} on the cosets of $S_a \wr S_{sp}$, it is clear that any vertex of an indecomposable summand is contained in a Sylow p-subgroup of $S_a \wr S_{sp}$ and therefore is conjugate to a subgroup of Q_s. From Proposition 4.2.7 we have that $H^{(a^{sp})}(R_{as})$ is indecomposable. Therefore by Proposition 1.2.25 we deduce that exists a unique indecomposable summand U of $H^{(a^{sp})}$ such that R_{as} is contained in a vertex of U. We know that $R_{as} \leqslant Q_s$ and for sure the Scott module $\mathrm{Sc}(S_{asp}, S_a \wr S_{sp}) = \mathrm{Sc}(S_{asp}, Q_s)$ is an indecomposable summand of $H^{(a^{sp})}$, with vertex Q_s. Therefore $U = \mathrm{Sc}(S_{asp}, Q_s)$ and by Theorem 1.2.29 we have that

$$H^{(a^{sp})}(Q_s) = U(Q_s) \cong P_{\mathbb{F}},$$

as $\mathbb{F}(N_{S_{asp}}(Q_s)/Q_s)$-modules. $\square$

In order to prove Theorem 4.1.1 we need the following technical lemma. Denote by D_s the group $C \cap N_{S_{asp}}(Q_s)$.

LEMMA 4.2.9 *Let p be a prime and let a and s be natural numbers such that $a < p$. Then the unique Sylow p-subgroup of $N_{S_{asp}}(Q_s)$ is the subgroup $\langle D_s, Q_s \rangle$ of $N_{S_{asp}}(R_{as})$.*

Proof: Keeping the notation introduced after Proposition 4.2.4, for $j \in \{1, 2, \dots, as\}$ let $\mathcal{O}_j = \{(j-1)p + 1, \dots, jp\}$ and for $k \in \{1, \dots, s\}$ let

$$X_k = \bigcup_{l=0}^{a-1} \mathcal{O}_{ls+k}.$$

Since Q_s normalizes E_s, it permutes the sets $X_1, \dots, X_s$ as blocks for its action. Moreover given $x \in N_{S_{asp}}(Q_s)$ we have that $\pi_j^x \in \langle \pi_1, \dots, \pi_s \rangle$ for all $j \in \{1, \dots, s\}$ (this follows from Remark 1.3.11). Therefore also $N_{S_{asp}}(Q_s)$ permutes as blocks for its action the sets $X_1, \dots, X_s$.

Let g be a p-element of $N_{S_{asp}}(Q_s)$. The group $\langle Q_s, g \rangle$ permutes the sets in $X := \{X_1, \dots, X_s\}$ as blocks for its action. Let

$$\pi : \langle Q_s, g \rangle \to S_X$$

be the corresponding group homomorphism. By construction Q_s acts on the sets $X_1, \dots, X_s$ as a Sylow p-subgroup of $S_{\{X_1,\dots,X_s\}}$; hence $Q_s\pi$ is a Sylow p-subgroup of S_X. Therefore, since $\langle Q_s, g \rangle$ is a p-group, there exists $\tilde{g} \in Q_s$ such that $g\pi = \tilde{g}\pi$. Let $y = g\tilde{g}^{-1}$. Since y acts trivially on the sets in X, we may write

$$y = g_1 \dots g_s$$

where $g_j \in S_{X_j}$ for each j. The p-group $\langle Q_s, y \rangle$ has as a subgroup $\langle \pi_j, y \rangle$. The permutation group induced by the subgroup on X_j, namely $\langle \pi_j, g_j \rangle$, is a p-group acting on a set of size ap. Since $p > a$, the unique Sylow p-subgroup of S_{X_j} containing π_j is $\langle z_j, z_{j+s}, \dots, z_{j+(a-1)s} \rangle$. Hence $g_j \in \langle z_j, z_{j+s}, \dots, z_{j+(a-1)s} \rangle$ for each $j \in \{1, \dots, s\}$. Therefore $y \in \langle z_1, z_2 \dots, z_{as} \rangle = C$. We also know that $y \in \langle Q_s, g \rangle \leqslant N_{S_{asp}}(Q_s)$. Therefore $y \in D_s$, and since $\tilde{g} \in Q_s$, it follows that $g \in \langle D_s, Q_s \rangle$, as required. Conversely, the subgroup $\langle D_s, Q_s \rangle$ is contained in $N_{S_{asp}}(Q_s)$ because both D_s and Q_s are. It follows that $\langle D_s, Q_s \rangle$ is the unique Sylow p-subgroup of $N_{S_{asp}}(Q_s)$. $\square$

We are now ready to prove Theorem 4.1.1.

Proof: [Theorem 4.1.1] In order to simplify the notation we denote by K_s the group

$N_{S_{asp}}(R_{as})$. Let U be an indecomposable summand of $H^{(a^n)}$ with vertex Q. Let $\ell \in \{1, 2, \ldots, \lfloor \frac{an}{p} \rfloor\}$ be maximal with respect to the property that R_ℓ is a subgroup of (a conjugate of) the vertex Q. The Broué correspondent $U(R_\ell)$ is a non-zero direct summand of $H^{(a^n)}(R_\ell)$ by Theorem 1.2.12. Therefore we deduce by Lemma 4.2.1 that there exist a natural number s such that $\ell = as$ and Z a non-zero summand of $H^{(a^{n-sp})}$ such that

$$U(R_{as}) \cong H^{(a^{sp})}(R_{as}) \boxtimes Z$$

as $\mathbb{F}(K_s \times S_{a(n-sp)})$-modules (where $K_s = N_{S_{asp}}(R_{as})$). Since R_{as} is normal in Q_s, it follows from Lemmas 1.2.23 and 1.2.24 that there is an isomorphism of $\mathbb{F}(N_{K_s}(Q_s) \times S_{a(n-sp)})$-modules

$$U(Q_s) \cong (U(R_{as}))(Q_s) \cong (H^{(a^{sp})}(R_{as}))(Q_s) \boxtimes Z.$$

By Proposition 4.2.7 we deduce that $\big(H^{(a^{sp})}(R_{as})\big)(Q_s) \boxtimes Z \neq 0$. Hence we have that $Q_s \leqslant Q$.

Let $\mathcal{B}$ be a p-permutation basis for the $\mathbb{F}K_s$-module $H^{(a^{sp})}(R_{as})$ with respect to a Sylow p-subgroup of K_s containing Q_s. It follows from Corollary 1.2.19 and Lemma 4.2.9 that $\mathcal{C} = \mathcal{B}^{Q_s}$ is a p-permutation basis for the $\mathbb{F}N_{K_s}(Q_s)$-module $(H^{(a^{sp})}(R_{as}))(Q_s)$ with respect to the unique Sylow p-subgroup $P := \langle D_s, Q_s \rangle$ of $N_{K_s}(Q_s)$. Let $\mathcal{C}'$ be a p-permutation basis for Z with respect to P', a Sylow p-subgroup of $S_{\{asp+1,\ldots,an\}} \cong S_{a(n-sp)}$. Hence

$$\mathcal{C} \times \mathcal{C}' = \{v \otimes v' : v \in \mathcal{C}, v' \in \mathcal{C}'\}$$

is a p-permutation basis for $(H^{(a^{sp})}(R_{as}))(Q_s) \boxtimes Z$ with respect to the Sylow p-subgroup $P \times P'$ of $N_{K_s}(Q_s) \times S_{a(n-sp)}$.

Suppose, for a contradiction, that Q strictly contains Q_s. Since Q is a p-group there exists a p-element $g \in N_Q(Q_s)$ such that $g \notin Q_s$. Notice that Q_s has orbits of length at least p on $\{1, \ldots, asp\}$ and fixes $\{asp+1, \ldots, an\}$. Since g permutes these orbits as blocks for its action, we may factorize g as $g = hh^+$ where $h \in N_{S_{asp}}(Q_s)$ and $h^+ \in S_{a(n-sp)}$. By Lemma 4.2.9 we have that $\langle Q_s, h \rangle \leq N_{K_s}(Q_s)$.

Corollary 1.2.19 now implies that $(\mathcal{C} \times \mathcal{C}')^{\langle Q_s, g \rangle} \neq \varnothing$. Let $v \otimes v' \in \mathcal{C} \times \mathcal{C}'$ be such that $(v \otimes v')g = v \otimes v'$. Then $v \in \mathcal{B}^{\langle Q_s, h \rangle}$. But Q_s is a vertex of $H^{(a^{sp})}(R_{as})$, so it follows from Corollary 1.2.19 that $h \in Q_s$. Hence h' is a non-identity element of Q. By taking an appropriate power of h' we find that Q contains a product of one or more p-cycles with support contained in $\{asp+1, \ldots, an\}$. This contradicts our assumption that $l = as$ was maximal such that R_{as} is contained in a vertex of

U. Therefore U has vertex Q_s.

We saw above that there is an isomorphism

$$U(Q_s) \cong (H^{(a^{sp})}(R_{as}))(Q_s) \boxtimes Z$$

of $\mathbb{F}(N_{K_s}(Q_s) \times S_{a(n-sp)})$-modules. This identifies $U(Q_s)$ as a vector space on which $N_{S_{an}}(Q_s) = N_{S_{asp}}(Q_s) \times S_{a(n-sp)}$ acts. It is clear from the isomorphism in Lemma 4.2.1 that $N_{S_{asp}}(Q_s)$ acts on the first tensor factor and $S_{a(n-sp)}$ acts on the second. Hence the action of $N_{K_s}(Q_s)$ on $(H^{(a^{sp})}(R_{as}))(Q_s)$ extends to an action of $N_{S_{asp}}(Q_s)$ on $(H^{(a^{sp})}(R_{as}))(Q_s)$ and we obtain a tensor factorization $V \boxtimes Z$ of $U(Q_s)$ as an $\mathbb{F}\big(N_{S_{asp}}(Q_s) \times S_{a(n-sp)}\big)$-module. An outer tensor product of modules is projective if and only if both factors are projective, so by Theorem 1.2.20 and Corollary 4.2.8, V is isomorphic to the projective cover of the trivial $\mathbb{F}(N_{S_{asp}}(Q_s)/Q_s)$-module, Z is a projective $\mathbb{F}S_{a(n-sp)}$-module, and $U(Q_s)$ is the Green correspondent of U. $\square$

An interesting consequence of Theorem 4.1.1 is proved in the following corollary.

Corollary 4.2.10 *Let $\mathbb{F}$ be a field of prime characteristic p. Let a and n be natural numbers such that $a < p \leqslant n$. If U is an indecomposable and non-projective direct summand of the $\mathbb{F}S_{an}$-module $H^{(a^n)}$, then U is not a Young module.*

Proof: From Theorem 4.1.1 there exists $t \in \{1, 2, \ldots, \lfloor \frac{n}{p} \rfloor\}$ such that U has vertex Q_t, a Sylow p-subgroup of $S_a \wr S_{tp}$. It is easy to notice that Q_t has support of size atp. Moreover, since $a < p$ we have that Q_t is isomorphic as an abstract group to a Sylow p-subgroup of S_{tp}. Suppose, for a contradiction, that U is a non-projective Young module. By Theorem 1.4.1 there exists a partition ρ of the form

$$\rho = (p^k, \ldots, p^k, p^{k-1}, \ldots, p^{k-1}, \ldots, p, \ldots, p),$$

such that $S_\rho \leqslant S_{an}$ and Q_t is conjugate to a Sylow p-subgroup of S_ρ. This implies that S_ρ and S_{tp} have isomorphic Sylow p-subgroups. Therefore the support of the Young subgroup S_ρ must have size tp, as a subgroup of S_{an}. This is a contradiction since $\mathrm{supp}(Q_t) = atp$ and $a \geqslant 2$. $\square$

Corollary 4.2.10 immediately implies that the the modular version of Foulkes' Conjecture is false.

Proposition 4.2.11 *Let $\mathbb{F}$ be a field of prime characteristic p. Let a and n be natural numbers such that $a < p \leqslant n$. The $\mathbb{F}S_{an}$-module $H^{(n^a)}$ is not a direct summand of the $\mathbb{F}S_{an}$-module $H^{(a^n)}$.*

Proof: By definition we have that $H^{(n^a)}$ is isomorphic to $\mathrm{Ind}_{S_n \wr S_a}^{S_{an}}(\mathbb{F})$. Therefore the Scott module $Y := \mathrm{Sc}(S_{an}, S_n \wr S_a)$ is a non-projective direct summand of $H^{(n^a)}$. By Proposition 1.4.2 we deduce that Y is a Young module, therefore we have that Y can not be a direct summand of $H^{(a^n)}$, by Corollary 4.2.10. This completes the proof. $\square$

4.3 One corollary on decomposition numbers

In this section we will give upper bounds to the entries of some columns of the decomposition matrix of $\mathbb{F}_p S_{an}$ when $a < p$. In particular we will prove Theorem 4.1.2.

For the rest of the section let $\mathbb{F}_p$ be the finite field of size p and let a be a natural number such that $a < p$. Let $B := B(\gamma, w)$ be a block of the group algebra $\mathbb{F}_p S_{an}$ such that $\mathcal{F}_0(\gamma) \neq \emptyset$ and $\mathcal{F}_{sp}(\gamma) = \emptyset$ for all $s \in \{1, 2, \ldots, \lfloor \frac{n}{p} \rfloor\}$, where $\mathcal{F}_{sp}(\gamma)$ are the sets defined in section 4.1, before the statement of Theorem 4.1.2. For every p-regular partition ν of an, we will denote by P^ν the projective cover of the simple $\mathbb{F}_p S_{an}$-module D^ν.

An essential step towards the proof of Theorem 4.1.2 is the following proposition.

Proposition 4.3.1 *The block component of $H^{(a^n)}$ for the block B is projective.*

Proof: We start by considering the special case where $n = rp$ for some $r \in \mathbb{N}$ and $B = B(\emptyset, ar)$. In this situation the hypotheses of Theorem 4.1.2 are never satisfied since the trivial partition $(a(n - sp))$ is always in $\mathcal{F}_{sp}(\emptyset)$. Hence we can rule out this case. For all the other possibilities we use the following argument: let U be an indecomposable summand of $H^{(a^n)}$ lying in the block B. Suppose that U is non-projective. Then by Theorem 4.1.1 there exists $t \in \{1, \ldots, \lfloor \frac{n}{p} \rfloor\}$, such that $Q_t \in \mathrm{Syl}_p(S_a \wr S_{tp})$ is a vertex of U. Moreover, we have that $U(Q_t) \cong P_{\mathbb{F}_p} \boxtimes V$, where V is an indecomposable projective summand of $H^{(a^{n-tp})}$ lying in the block $B(\gamma, w - at)$, by Theorem 1.3.13. Using Theorem 1.2.27 to lift V to a summand of $H_{\mathbb{Z}_p}^{(a^{n-tp})}$ lying in the p-block of $S_{a(n-tp)}$ with p-core γ, we deduce that the ordinary character of the summand $V_{\mathbb{Z}_p}$ of $H_{\mathbb{Z}_p}^{(a^{n-tp})}$ is of the form

$$\chi_V = \chi^{\mu_1} + \cdots + \chi^{\mu_s},$$

for some $\mu_1, \ldots, \mu_s$ partitions of $a(n-tp)$ such that $\left\langle \chi^{\mu_j}, \phi^{(a^{n-tp})} \right\rangle \neq 0$ and $\gamma(\mu_j) = \gamma$ for all $j \in \{1, \ldots, s\}$. This is a contradiction since $\mathcal{F}_{tp}(\gamma) = \emptyset$. $\square$

We are now ready to prove Theorem 4.1.2.

Proof: [Theorem 4.1.2] Since $\mathcal{F}_0(\gamma) \neq \emptyset$ the block component W of $H^{(a^n)}$ lying B is non zero. By Proposition 4.3.1 we have that W is projective. Let $\zeta_1, \ldots, \zeta_s$ be the p-regular partitions of an such that

$$W = P^{\zeta_1} \oplus P^{\zeta_2} \oplus \cdots \oplus P^{\zeta_s}.$$

We can now use Theorem 1.2.27 to lift W to the summand $W_{\mathbb{Z}_p}$ of $H^{(a^n)}_{\mathbb{Z}_p}$. It follows that the ordinary character of the summand of $H^{(a^n)}_{\mathbb{Z}_p}$ lying in the p-block of S_{an} with p-core γ is

$$\psi^{\zeta_1} + \cdots + \psi^{\zeta_s} = \sum_{\mu \in \mathcal{F}_0(\gamma)} \big(\sum_{i=1}^{s} d_{\mu\zeta_i}\big)\chi^\mu.$$

By hypothesis λ is a maximal partition in the dominance order on $\mathcal{F}_0(\gamma)$. Therefore, arguing exactly as in the proof of Proposition 3.1.3 in Section 3.4, we deduce that one of the partitions ζ_j must equal λ, as required.

Therefore P^λ is a direct summand of $H^{(a^n)}$ and ψ^λ is a summand of the Foulkes character $\phi^{(a^n)}$. Hence

$$d_{\mu\lambda} = \left\langle \psi^\lambda, \chi^\mu \right\rangle \leqslant \left\langle \phi^{(a^n)}, \chi^\mu \right\rangle,$$

for all $\mu \vdash an$. In particular if $\mu \notin \mathcal{F}_0(\gamma)$ then $d_{\mu\lambda} = 0$. □

In order to prove Corollary 4.1.3 it will be enough to show that whenever the partition $\lambda \in \mathcal{F}_0(\gamma)$ has p-weight $w < a$ then $\mathcal{F}_{sp}(\gamma) = \emptyset$ for all $s \in \{1, \ldots, \lfloor \frac{n}{p} \rfloor\}$.

Proof: [Corollary 4.1.3] The p-core γ of λ is a partition of $an - wp$. Suppose for a contradiction that exist $s \in \{1, \ldots, \lfloor \frac{n}{p} \rfloor\}$ and $\mu \vdash a(n - sp)$ such that $\mu \in \mathcal{F}_{sp}(\gamma)$. It follows that

$$|\gamma| = an - wp > a(n - p) \geqslant a(n - sp) = |\mu|.$$

Hence the p-core $\gamma(\mu)$ of μ can not be equal to γ. This yields a contradiction. Since λ is maximal in $\mathcal{F}_0(\gamma)$, by Theorem 4.1.1 we obtain the statement. □

As already mentioned in the introduction, Theorem 4.1.2 and Corollary 4.1.3 allow us to recover new information on the decomposition numbers via the study of the ordinary Foulkes character $\phi^{(a^n)}$. An example of this possibility is the following result.

Corollary 4.3.2 *Let λ be a p-regular partition of na. Denote by γ the p-core of λ. If λ is maximal in $\mathcal{F}_0(\gamma)$ and $\mathcal{F}_{sp}(\gamma) = \emptyset$ for all $s \in \{1, \ldots, \lfloor \frac{n}{p} \rfloor\}$, then $[S^\mu : D^\lambda] = 0$ for all $\mu \vdash na$ such that μ has more than n parts.*

Proof: It is a well known fact (see for instance Proposition 2.2.1) that if μ has more than n parts then

$$\left\langle \phi^{(a^n)}, \chi^{\mu} \right\rangle = 0.$$

The statement now follows from Theorem 4.1.2. $\square$

We conclude with an explicit example.

EXAMPLE 4.3.3 *Let $a = 4$, $n = p = 5$ and let $\lambda = (18, 2)$ be a weight 3 partition of 20. The 5-core of λ is $\gamma = (3, 2)$ and the multiplicity of χ^{λ} as an irreducible constituent of $\phi^{(4^5)}$ is 1, by Corollary 2.2.6. Therefore $\lambda \in \mathcal{F}_0(\gamma)$ and it is clearly maximal under the dominance order on $\mathcal{F}_0(\gamma)$. By Corollary 4.3.2 we obtain a number of non-trivial zeros in the column labelled by λ of the decomposition matrix of S_{20} in characteristic 5. For instance, all the partitions μ obtained from $(3, 2, 1^5)$ by adding two 5-hooks have 5-core γ and are such that $[S^{\mu} : D^{\lambda}] = 0$.*

Chapter 5

The vertices of Specht modules

5.1 Introduction and outline

One of the mainstream themes in the representation theory of finite groups has been to determine global information about the p-modular representation theory of a group G by studying its local structure, namely representations of its p-subgroups and their normalizers. An interesting topic is the investigation of the vertices of indecomposable modules over group algebras. In the case of the symmetric group the study of the modular structure of Specht modules is one of the important open problems in the area.

The vertices of Specht modules were first considered by Murphy and Peel in [60]; their work focused on hook Specht modules in the case $p = 2$. In [76], Wildon made some progress on the topic by characterizing the vertices of hook Specht modules for $\mathbb{F}S_n$ when $\mathbb{F}$ is a field of prime characteristic p and n is not divisible by p. Müller and Zimmermann described vertices and sources of some hook Specht and simple modules in [59]. In [50] Lim gave a necessary condition for a Specht module to have an abelian vertex and characterized the possible abelian vertices of Specht modules. The vertices of irreducible Specht modules are completely described by the work of Hemmer [36] and Donkin [16]. In particular, in [36] it is shown that every irreducible Specht module is a signed Young module and in [16] a complete characterization of the vertices of signed Young modules is given. Danz and Erdmann in [13] described the vertices of $S^{(n-2,2)}$ and $D^{(n-2,2)}$ defined over a field of characteristic 2. Wildon gave a general structural description of the vertices of all Specht modules; more precisely in [77] he proved the following result.

THEOREM 5.1.1 *Let λ be a partition of n, let t be a λ-tableau and denote by $H(t)$ the subgroup of $R(t)$ (i.e. the row-stabilising group of t) which permutes, as blocks* *for its action, the columns of equal length of t. If the Specht module S^{λ}, defined over a field of characteristic p, is indecomposable, then it has a vertex containing a subgroup isomorphic to a Sylow p-subgroup of $H(t)$.*

For example, if $\lambda = (5, 5, 2, 2, 2, 2)$ and t is the λ-tableau shown in Figure 5.1 below,

$$t = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & \multicolumn{3}{c}{} \\ \cline{1-2} 13 & 14 & \multicolumn{3}{c}{} \\ \cline{1-2} 15 & 16 & \multicolumn{3}{c}{} \\ \cline{1-2} 17 & 18 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

Figure 5.1: The most dominant (5,5,2,2,2,2)-tableau

then $R(t) = S_{\{1,2,3,4,5\}} \times S_{\{6,7,8,9,10\}} \times S_{\{11,12\}} \times S_{\{13,14\}} \times S_{\{15,16\}} \times S_{\{17,18\}}$ and $H(t)$ is the subgroup generated by the permutations

$$(3, 4, 5)(8, 9, 10), (3, 4)(8, 9) \text{ and } (1, 2)(6, 7)(11, 12)(13, 14)(15, 16)(17, 18).$$

Considered as an abstract group we have that $H(t) \cong S_3 \times S_2$.

In the first part of this chapter we will generalize and improve the lower bound on the vertex given in 5.1.1 for Specht modules S^{λ} defined over any field of prime characteristic p. Given a partition λ of a natural number n and a λ-tableau t, we denote by t' the transposed tableau of t (as defined in Section 1.3.1). In Section 5.2 we will show that the subgroup of S_n generated by $H(t)$ and $H(t')$ is in fact equal to the direct product $H(t) \times H(t')$. This is one of the key ideas that will lead to the proof of the following theorem.

Theorem 5.1.2 *Let n be a natural number and let $\mathbb{F}$ be a field of prime characteristic p. Let λ be a partition of n and let t be a λ-tableau. If the Specht module S^{λ} defined over $\mathbb{F}$ is indecomposable, then each of its vertices contains a subgroup conjugate to a Sylow p-subgroup of $H(t) \times H(t')$.*

In Section 5.3, we will use dimensional arguments to determine the vertices of the particular family of Specht modules labelled by partitions λ of n of the form

$$\lambda = (m, x_1, x_2, \ldots, x_k),$$

where the partition $\gamma := (x_1, x_2, \ldots, x_k)$ is a p-core partition of $n - m$.

THEOREM 5.1.3 *Let n be a natural number and let $\lambda = (m, x_1, \ldots, x_k)$ be a partition of n such that the partition $\gamma = (x_1, \ldots, x_k)$ is a p-core partition of $n - m$. Denote by ρ and w the p-core and the p-weight of λ respectively. Then the vertex of S^λ is equal to the defect group of the corresponding block $B(\rho, w)$.*

5.2 A lower bound on the vertices of Specht modules

The main goal of this section is to prove Theorem 5.1.2. In order to do this, we need to prove some preliminary results. We start with a general lemma.

LEMMA 5.2.1 *Let G be a finite group and let H and K be subgroups of G such that $H \leqslant N_G(K)$ and $K \leqslant N_G(H)$. If $H \cap K = \{1\}$, then*

$$\langle H, K \rangle = H \times K.$$

Proof: It is sufficient to prove that for all $h \in H$ and for all $k \in K$ we have that $hk = kh$. Consider the commutator $[h, k] = h^{-1}k^{-1}hk$. By hypothesis we have that $[h, k] \in H \cap K = \{1\}$. $\square$

We immediately use Lemma 5.2.1 to prove the proposition below.

PROPOSITION 5.2.2 *Let λ be a partition of a natural number n and let t be a λ-tableau. Then the following equality between subgroups of S_n holds:*

$$\big\langle H(t), H(t') \big\rangle = H(t) \times H(t').$$

Proof: Let r be the number of rows of t and let s be the number of columns of t. For every $i \in \{1, \ldots, r\}$ let R_i be the set consisting of all the entries of the i^{th} row of t. Similarly, for all $j \in \{1, \ldots, s\}$ let C_j be the set consisting of all the entries of the j^{th} column of t. Denote by $\mathcal{R}$ and $\mathcal{C}$ the sets defined by

$$\mathcal{R} = \{R_1, \ldots, R_r\} \text{ and } \mathcal{C} = \{C_1, \ldots, C_s\}.$$

It is easy to observe that $H(t')$ is the collection of all the elements of $C(t)$ that permute the rows of equal length of t as blocks for their action. In particular $H(t')$ permutes $\mathcal{R}$ and acts trivially on $\mathcal{C}$. Similarly $H(t)$ permutes the set $\mathcal{C}$ and acts trivially on $\mathcal{R}$. Consider $g \in H(t)$ and $h \in H(t')$. For all $i \in \{1, \ldots, r\}$ there exists a unique $k \in \{1, \dots, r\}$ such that $R_i h = R_k$. In particular we observe that

$$R_i(ghg^{-1}) = R_i(hg^{-1}) = R_k(g^{-1}) = R_k.$$

Hence ghg^{-1} permutes $\mathcal{R}$. Moreover for all $j \in \{1, \dots, s\}$ there exists a unique $\ell \in \{1, \dots, s\}$ such that $C_j g = C_\ell$. In particular we have that

$$C_j(ghg^{-1}) = C_\ell(hg^{-1}) = C_j(g^{-1}) = C_j.$$

Therefore ghg^{-1} acts trivially on $\mathcal{C}$. This implies that $ghg^{-1} \in H(t')$ and so that $H(t) \leqslant N_{S_n}(H(t'))$. In a complete similar way we obtain that $H(t') \leqslant N_{S_n}(H(t))$.

Let now g be an element of $H(t) \cap H(t')$. For all $x \in \{1, 2, \dots, n\}$ there exist $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$ such that $R_i \cap C_j = \{x\}$. Therefore $xg \in R_i g \cap C_j g = R_i \cap C_j = \{x\}$. Hence $g = 1_{S_n}$.

The proof is now an easy consequence of Lemma 5.2.1. □

Example 5.2.3 *Let $\lambda = (5, 5, 2, 2, 2, 2)$ and let t be the λ-tableau shown in Figure 5.1. In this case we have that $H(t)$ and $H(t')$ are the subgroups of S_{18} defined by*

$$H(t) = \langle (3,4,5)(8,9,10), (3,4)(8,9) \rangle \times \langle (1,2)(6,7)(11,12)(13,14)(15,16)(17,18) \rangle$$

and

$$H(t') = \langle (13,15,17)(14,16,18), (11,13)(12,14) \rangle \times \langle (1,6)(2,7)(3,8)(4,9)(5,10) \rangle.$$

In particular we have that, as an abstract group, $H(t) \times H(t') \cong S_3 \times S_2 \times S_4 \times S_2$.

We observe that, by construction, $H(t) \times H(t')$ permutes both rows and columns of t as blocks for its action. Notice also that for all u and t λ-tableaux, we have that $H(t) \times H(t')$ is a conjugate of $H(u) \times H(u')$ in S_n. In fact if $u = tg$, for some $g \in S_n$, then $H(u) \times H(u') = (H(t) \times H(t'))^g$.

The following lemma is a fundamental step towards the proof of Theorem 5.1.2.

Lemma 5.2.4 *Let λ be a partition of n and t a λ-tableau. Let P be a Sylow p-subgroup of $H(t) \times H(t')$. Then $e_t y = e_t$, for all $y \in P$.*

Proof: Since P is a Sylow p-subgroup of $H(t) \times H(t')$, there exist Q and Q' Sylow p-subgroups of $H(t)$ and $H(t')$ respectively such that $P = Q \times Q'$. Therefore for any element $y \in P$ there exist unique $h \in Q$ and $k \in Q'$ such that $y = hk$. Hence it suffices to prove that $e_t h = e_t$ for all $h \in Q$ and $e_t k = e_t$ for all $k \in Q'$.

Let $h \in Q \leqslant H(t) \leqslant R(t)$. By definition h permutes the columns of t as blocks for its action, therefore $C(t)^h = C(t)$ and of course $\{t\}h = \{t\}$. Hence

$$e_t h = \sum_{g \in C(t)} \mathrm{sgn}(g)\{t\}gh = \sum_{x \in C(t)^h} \mathrm{sgn}(x)\{t\}hx = \sum_{x \in C(t)} \mathrm{sgn}(x)\{t\}x = e_t.$$

Let $k \in Q'$. Since $k \in H(t') \leqslant C(t)$ we have that $e_t k = \mathrm{sgn}(k)e_t$. If the characteristic p of the underlying field $\mathbb{F}$ is 2 then clearly $e_t k = e_t$. On the other hand, if the prime characteristic $p > 2$ then by definition $Q' \leqslant C(t) \cap A_n$, hence we have again that $e_t k = e_t$, as required. $\square$

To proceed with the proof of Theorem 5.1.2 we will denote by $t^\star$ the greatest standard λ-tableau in the dominance order (as defined at the end of Section 1.3.1). In particular if $\lambda = (\rho_1, \ldots, \rho_k)$, we have that for all $i \in \{1, 2, \ldots, k\}$ the entries in the i^{th} row of $t^\star$ are

$$R_i = \{1 + \sum_{j=0}^{i-1} \rho_j, 2 + \sum_{j=0}^{i-1} \rho_j, \ldots, \sum_{j=0}^{i} \rho_j\},$$

where we take $\rho_0 = 0$. For example, the standard tableau shown in Figure 5.1 is the most dominant $(5, 5, 2, 2, 2, 2)$-tableau.

Proof: [Theorem 5.1.2] Since the subgroups $H(t) \times H(t')$ for different tableaux t are all conjugate in S_n, without loss of generality, it suffices to prove that a Sylow p-subgroup of $H(t^\star) \times H((t^\star)')$ is contained in a vertex of S^λ. Let P be a Sylow p-subgroup of $H(t^\star) \times H((t^\star)')$. Lemma 5.2.4 implies that $e_{t^\star} \in (S^\lambda)^P$. In order to apply Proposition 1.2.13 and complete the proof, we need to show that $e_{t^\star} \notin \mathrm{Tr}^P(S^\lambda)$, as defined in Section 1.2.3. Consider V to be the subspace of S^λ generated by all the elements of the form

$$e_s + e_s g + \cdots + e_s g^{p-1},$$

where s is any standard λ-tableau and g is any element of P. Since any maximal subgroup of P has index p in P, we have that $\mathrm{Tr}^P(S^\lambda) \leqslant V$, therefore it will suffice to show that $e_{t^\star} \notin V$. Suppose by contradiction that

$$e_{t^\star} = \sum_{s,g} a_{s,g}(e_s + \cdots + e_s g^{p-1}), \quad \text{for some } a_{s,g} \in \mathbb{F}.$$

By Theorem 1.3.1 we have that the standard polytabloids are linearly independent, therefore there exists a standard tableau s and some $g \in P$ such that, when $e_s +$

$\cdots + e_s g^{p-1}$ is expressed as a linear combination of standard polytabloids, $e_{t^\star}$ appears with non zero coefficient. This implies that there exists $i \in \{0, 1, \ldots, p-1\}$ such that $e_{t^\star}$ appears in the expression of $e_s g^i$. Let u be the column-standard tableau whose columns are setwise equal to the columns of sg^i. Clearly $e_{sg^i} = \pm e_u$ and by Theorem 1.3.1 we have that $e_u = e_{\overline{u}} + x$, where $\overline{u}$ is the row-straightening of u and x is a linear combination of standard polytabloids e_v with $v \lhd \overline{u} \unlhd t^\star$. We deduce that $t^\star = \overline{u}$ because $t^\star$ is the greatest standard tableau in the dominance order.

Observe that if $a, b \in \{1, 2, \ldots, n\}$ are in the same row of $t^\star$ then they lie in the same row of u and since the columns of u agree setwise with the columns of sg^i we obtain that a and b lie in different columns of sg^i. Let a, b be two elements of $\{1, 2, \ldots, n\}$ lying in the same row R_j of $t^\star$. Suppose for a contradiction that a and b are also lying in the same column of s. Since $g^i \in P \leqslant H(t^\star) \times H((t^\star)')$ permutes the rows of $t^\star$ as blocks for its action, we have that ag^i, bg^i belong to the same row $R_j g^i$ of $t^\star$. In particular, ag^i and bg^i lie in the same row of u and therefore in different columns of sg^i. This is in clear contradiction with the assertion that a and b are lying in the same column of s. We have just proved that no two numbers in the same row of $t^\star$ can possibly lie in the same column of s. More precisely we have that for each $j \in \{1, 2, \ldots, p(\lambda)\}$, the elements of R_j lie in different columns of s. Since s is standard, we deduce that the first row of s must contain exactly the elements of R_1. Similarly we deduce that row j of s equals R_j for all j. Hence we obtain that $s = t^\star$. Therefore $e_s = e_{t^\star}$ and by Lemma 5.2.4 we deduce that $e_{t^\star} g = e_{t^\star}$. It follows that

$$e_s + e_s g + \cdots + e_s g^{p-1} = p e_{t^\star} = 0.$$

This contradicts our initial assumption. Therefore $e_{t^\star} \notin V$, as required. We have proved that $S^\lambda(P) \neq 0$ and therefore we have that P is contained in a vertex of S^λ. □

Theorem 5.1.2 clearly generalizes Theorem 5.1.1. In particular we observe that every Sylow p-subgroup of $H(t)$ is contained in a Sylow p-subgroup of $H(t) \times H(t')$ therefore we obtain Theorem 5.1.1 as a corollary of Theorem 5.1.2. Moreover, for all the partitions λ of n such that p divides the order of $H(t')$ we have that every Sylow p-subgroup of $H(t) \times H(t')$ properly contains a Sylow p-subgroup of $H(t)$. In all these cases our Theorem 5.1.2 strictly improves the lower bound on a vertex of S^λ given by Wildon. One explicit example for the prime 3 is $\lambda = (5, 5, 2, 2, 2, 2)$ and t the λ-tableau shown in Figure 5.1.

In the following remark we show that in the case of hook-partitions of a natural number n that is not divisible by p, our theorem gives a complete description of the vertices of the corresponding Specht modules.

Remark 5.2.5 *Let $\lambda = (n-k, 1^k)$ be a hook partition of a natural number n such that p does not divide n and such that the corresponding Specht module S^λ is indecomposable. The lower bound on the vertex of S^λ obtained from Theorem 5.1.2 is attained. In fact, in this case we have*

$$H(t) \times H(t') \cong S_k \times S_{n-k-1}.$$

By [76, Theorem 2] we have that the vertex of S^λ is isomorphic to a Sylow p-subgroup of $S_k \times S_{n-k-1}$.

It is also interesting to notice that using Theorem 5.1.2 we are able to give an independent and alternative proof of the above mentioned Theorem 2 of [76], in the case where p does not divide $n-k$. It is enough to observe that when p does not divide both n and $n-k$ the Specht module $S^{(n-1,1)}$ is a direct summand of the natural $\mathbb{F}S_n$-module $M^{(n-1,1)}$. Moreover by [59, Proposition 2.3] we have that $S^{(n-k,1^k)} \cong \bigwedge^k(S^{(n-1,1)})$, for all $k \in \{0, 1, \ldots, n-1\}$. Therefore we have that

$$S^{(n-k,1^k)} \cong \bigwedge^k(S^{(n-1,1)}) \mid \bigwedge^k(M^{(n-1,1)}) \cong \mathrm{Ind}_{S_{n-k}\times S_k}^{S_n}(S^{(n-k)} \boxtimes S^{(1^k)}).$$

Hence $S^{(n-k,1^k)}$ is relatively $(S_{n-k} \times S_k)$-projective. By Lemma 1.2.3 we deduce that $S^{(n-k,1^k)}$ is relatively P-projective for some

$$P \in \mathrm{Syl}_p(S_{n-k} \times S_k) = \mathrm{Syl}_p(S_{n-k-1} \times S_k).$$

If Q is a vertex of $S^{(n-k,1^k)}$ contained in P, by Theorem 5.1.2 we deduce that

$$P^g \leqslant Q \leqslant P,$$

for some $g \in S_n$. This clearly implies that $Q = P$, as required.

5.3 A family of Specht modules with maximal vertex

In this section we will prove Theorem 5.1.3. In order to do this we need to introduce some further notation and definitions. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n and let $[\lambda]$ be the associated Young diagram. For all $a \in \{1, 2, \ldots, k\}$ and $b \in \{1, 2, \ldots, \lambda_a\}$ denote by $h_{(a,b)}$ the length of the hook associated to the box of $[\lambda]$ lying in row a and column b. More precisely we have that

$$h_{(a,b)} = 1 + (\lambda_a - b) + (\lambda'_b - a).$$

In [45, 2.7.40] the following fact is proved.

Proposition 5.3.1 *The p-weight of a partition λ of n is equal to the number of boxes in $[\lambda]$ whose associated hook has length divisible by p.*

We immediately use Proposition 5.3.1 to prove the following key lemma.

Lemma 5.3.2 *Let λ be a partition of n and let w be the weight of λ. Denote by $(a_1, b_1), (a_2, b_2), \ldots, (a_w, b_w)$ the boxes of $[\lambda]$ such that p divides $h_{(a_i,b_i)}$. Suppose that $h_{(a_i,b_i)} < h_{(a_{i+1},b_{i+1})}$ for all $i \in \{1, 2, \ldots, w-1\}$. Then $h_{(a_i,b_i)} = ip$ for all $i \in \{1, 2, \ldots, w\}$ and the vertices of S^λ coincides with the defect groups of the p-block $B(\gamma(\lambda), w)$.*

Proof: Let Q be a vertex of S^λ, let D be a defect group of $B(\gamma(\lambda), w)$ and let P_n be a Sylow p-subgroup of S_n such that $Q \leqslant D \leqslant P_n$. For all $i \in \{1, \ldots, w\}$ let $ph_i = h_{(a_i,b_i)}$. By Theorem 1.3.14 we have that

$$|D| = (wp!)_p = p^w (w!)_p.$$

Moreover, by Theorem 1.2.7 and Theorem 1.3.2 we deduce that

$$|P_n : D| \cdot |D : Q| = |P_n : Q| \mid (\dim(S^\lambda))_p = \frac{(n!)_p}{p^w \prod_{i=1}^{w} (h_i)_p}.$$

Therefore there exists $r \in \mathbb{N}_0$ such that

$$\frac{(n!)_p}{p^w (w!)_p} \cdot \frac{|D|}{|Q|} \cdot p^r = \frac{(n!)_p}{p^w \prod_{i=1}^{w} (h_i)_p}.$$

This in particular implies that $(w!)_p \geqslant \prod_i^w (h_i)_p$. On the other hand for all $i \in \{1, \ldots, w\}$ we have that $h_i \geqslant i$, hence $(w!)_p = \prod_i^w (h_i)_p$. Therefore we deduce that $h_i = i$ and $h_{(a_i,b_i)} = ip$ for all $i \in \{1, 2, \ldots, w\}$. We conclude that $|D : Q| p^r = 1$ that necessarily implies $D = Q$ as desired. $\square$

We are now ready to prove Theorem 5.1.3.

Proof: [Theorem 5.1.3] Let $\lambda = (m, \gamma)$, where $m \in \mathbb{N}$ and γ is a p-core partition of $n - m$. Let w be the weight of λ. Since γ is p-core partition all the boxes (a, b) of $[\lambda]$ such that $p \mid h_{(a,b)}$ must lie in the first row. Let $1 \leqslant b_1 < b_2 < \ldots < b_w \leqslant m$ be such that $p \mid h_{(1,b_i)}$ for all $i \in \{1, \ldots, w\}$. Let now $i, j \in \{1, \ldots, w\}$ with $i < j$. Then

$$h_{(1,b_i)} - h_{(1,b_j)} = \underbrace{(b_j - b_i)}_{>0} + \underbrace{(\lambda'_{b_i} - \lambda'_{b_j})}_{\geqslant 0} > 0.$$

We just proved that λ satisfies the hypothesis of Lemma 5.3.2. $\square$

We conclude the section with a small example.

EXAMPLE 5.3.3 *Let* $\mathbb{F}$ *be a field of characteristic* 3 *and let* λ *be the partition of* 13 *defined by*

$$\lambda = (5, 4, 2, 1, 1).$$

Notice that λ *has weight equal to* 3 *and core equal to* $(3, 1)$. *By Theorem 1.3.14 the block* $B((3, 1), 3)$ *of* $\mathbb{F}S_{13}$ *has defect group* D *equal to a Sylow* 3*-subgroup of* S_9. *In particular* $D \cong C_3 \wr C_3$. *Since* $(4, 2, 1, 1)$ *is a* 3*-core partition of* 8*, our Theorem 5.1.3 implies that* S^λ *has vertices conjugate to* D *in* S_{13}.

Chapter 6

Vertices of simple modules labelled by hook partitions

This chapter is based on the paper [14]. The results were obtained in collaboration with Prof. Susanne Danz. We equally contributed to achieve all the main theorems of the chapter. Susanne Danz's knowledge of the structure of Sylow p-subgroups of S_n was fundamental to prove Proposition 6.3.5

6.1 Introduction and outline

The aim of this chapter is to complete the description of the vertices of a distinguished class of simple modules of symmetric groups. Then, as already mentioned in Section 1.3.5, the isomorphism classes of simple $\mathbb{F}S_n$-modules are labelled by the p-regular partitions of n. We denote the simple $\mathbb{F}S_n$-module corresponding to a p-regular partition λ by D^λ. If $\lambda = (n-r, 1^r)$, for some $r \in \{0, \ldots, p-1\}$, then λ is called a p-regular hook partition of n. Whilst, in general, even the dimensions of the simple $\mathbb{F}S_n$-modules are unknown, one has a neat description of an $\mathbb{F}$-basis of $D^{(n-r,1^r)}$; we shall comment on this in 6.2 below.

The problem of determining the vertices of the simple $\mathbb{F}S_n$-module $D^{(n-r,1^r)}$ has been studied before by Wildon in [76], by Müller and Zimmermann in [59], and by Danz in [12]. In consequence of these results, the vertices of $D^{(n-r,1^r)}$ have been known, except in the case where $p > 2$, $r = p-1$ and $n \equiv p \pmod{p^2}$. In Section 6.4 we shall prove the following theorem.

Theorem 6.1.1 *Let $p > 2$, let $\mathbb{F}$ be a field of characteristic p, and let $n \in \mathbb{N}$ be such that $n \equiv p \pmod{p^2}$. Then the vertices of the simple $\mathbb{F}S_n$-module $D^{(n-p+1,1^{p-1})}$ are* *precisely the Sylow p-subgroups of S_n.*

In [59], Müller and Zimmermann stated the following conjecture.

CONJECTURE 6.1.2 *Let p be an odd prime, let r and n be a natural numbers such that n is divisible by p, $n \neq p$ and $r < p$. Denote by P_n a Sylow p-subgroup of S_n. Let $\lambda = (n-r, 1^r)$.*

(a) If $r = p-1$, then P_n is a vertex of D^{λ}.

(b) For all $r > 1$ we have that $\mathrm{Res}_{P_n}(D^{\lambda})$ is a source of D^{λ}.

Theorem 6.1.1 together with [12, Corollary 5.5] proves part (a) of Conjecture 6.1.2.

Our key ingredients for proving Theorem 6.1.1 will be the Brauer construction as described in Section 1.2.3 and Wildon's result in [76]. Both of these will enable us to obtain lower bounds on the vertices of $D^{(n-p+1,1^{p-1})}$, which together will then provide sufficient information to deduce Theorem 6.1.1.

To summarize, the abovementioned results in [12, 59, 76] and Theorem 6.1.1 lead to the following exhaustive description of the vertices of the modules $D^{(n-r,1^r)}$:

THEOREM 6.1.3 *Let $\mathbb{F}$ be a field of characteristic $p > 0$, and let $n \in \mathbb{N}$. Let further $r \in \{0, 1 \ldots, p-1\}$, and let Q be a vertex of the simple $\mathbb{F}S_n$-module $D^{(n-r,1^r)}$.*

(a) *If $p \nmid n$ then Q is S_n-conjugate to a Sylow p-subgroup of $S_{n-r-1} \times S_r$.*

(b) *If $p = 2$, $p \mid n$ and $(n, r) \neq (4, 1)$ then Q is a Sylow 2-subgroup of S_n.*

(c) *If $p = 2$, $n = 4$ and $r = 1$ then Q is the unique Sylow 2-subgroup of A_4.*

(d) *If $p > 2$ and $p \mid n$ then Q is a Sylow p-subgroup of S_n.*

In the case where $p \nmid n$, the simple module $D^{(n-r,1^r)}$ is isomorphic to the Specht $\mathbb{F}S_n$-module $S^{(n-r,1^r)}$, by work of Peel [65]. Thus assertion (a) follows immediately from [76, Theorem 2]. Assertions (b) and (c) have been established by Müller and Zimmermann [59, Theorem 1.4]. Moreover, if $p > 2$, $p \mid n$ and $r < p-1$ then assertion (d) can also be found in [59, Theorem 1.2]. The case where $p > 2$, $p \mid n$, $r = p-1$ was treated in [12, Corollary 5.5], except when $n \equiv p \pmod{p^2}$, which is covered by Theorem 6.1.1 above.

We should also like to comment on the sources of the simple $\mathbb{F}S_n$-modules $D^{(n-r,1^r)}$. For $r = 0$, we get the trivial $\mathbb{F}S_n$-module $D^{(n)}$, which has of course trivial source. If $p \mid n$, then the module $D^{(n-1,1)}$ restricts indecomposably to its vertices, by [59, Theorems 1.3, 1.5], except when $p = 2$ and $n = 4$. For $p = 2$, the simple $\mathbb{F}S_4$-module $D^{(3,1)}$ has trivial source, by [59, Theorem 1.5]. If $p \nmid n$ then $D^{(n-r,1^r)} \cong S^{(n-r,1^r)}$ has always trivial sources; see, for instance [59, Theorem 1.3].

However, in the case where $p > 2$, $p \mid n$ and $r > 1$, we do not know the sources of $D^{(n-r,1^r)}$. In these latter cases, the restrictions of $D^{(n-r,1^r)}$ to its vertices should, conjecturally, be indecomposable, hence should be sources of $D^{(n-r,1^r)}$; see part (b) of Conjecture 6.1.2. This conjecture has been verified computationally in several cases, see [12, 59], but remains still open in general.

6.2 Exterior powers of the natural $\mathbb{F}S_n$-module.

Throughout this chapter, let $\mathbb{F}$ be a field of characteristic $p > 0$. We begin by introducing some basic notation that we shall use repeatedly throughout subsequent sections. Whenever H and K are subgroups of G such that H is G-conjugate to a subgroup of K, we write $H \leqslant_G K$. If H and K are G-conjugate then we write $H =_G K$.

In this section we shall recall some well-known properties of the simple $\mathbb{F}S_n$-modules labelled by hook partitions $(n-r, 1^r)$, for $r \in \{0, \ldots, p-1\}$, that we shall need repeatedly in the proof of Theorem 6.1.1. In particular, we shall fix a convenient $\mathbb{F}$-basis of $D^{(n-r,1^r)}$. In light of Theorem 6.1.1 we shall only be interested in the case where $p \mid n$ and $p > 2$.

Let $p > 2$, let $n \in \mathbb{N}$ be such that $p \mid n$, and let $M := M^{(n-1,1)}$ be the natural Young permutation $\mathbb{F}S_n$-module, with natural permutation basis $\Omega = \{\omega_1, \ldots, \omega_n\}$. Since $p \mid n$, the module M is uniserial with composition series $\{0\} \subset M_2 \subset M_1 \subset M$, where $M_1 = \{\sum_{i=1}^n a_i\omega_i : a_1, \ldots, a_n \in \mathbb{F}, \sum_{i=1}^n a_i = 0\}$ and $M_2 = \{a\sum_{i=1}^n \omega_i : a \in \mathbb{F}\}$; see, for instance, [39, Example 5.1].

Furthermore, $M_1 = S^{(n-1,1)}$, and

$$M_1/M_2 = S^{(n-1,1)}/\mathrm{rad}(S^{(n-1,1)}) =: \mathrm{Hd}(S^{(n-1,1)}) \cong D^{(n-1,1)};$$

in particular, $\dim_{\mathbb{F}}(D^{(n-1,1)}) = n-2$. One sometimes calls $D^{(n-1,1)}$ the *natural simple module.*

An $\mathbb{F}$-basis of M_1 is given by the elements $\omega_i - \omega_1$, where $i \in \{2, \ldots, n\}$. In the following, we shall identify the module $D^{(n-1,1)}$ with M_1/M_2. Consider the natural epimorphism $^- : M_1 \to M_1/M_2$, and set $e_i := \overline{\omega_i - \omega_1}$, for $i \in \{1, \ldots, n\}$. Then

$$e_n = -e_2 - e_3 - \cdots - e_{n-1},$$

and the elements $e_2, \ldots, e_{n-1}$ form an $\mathbb{F}$-basis of $D^{(n-1,1)}$.

Let $r \in \{0, \ldots, n-1\}$. By [59, Proposition 2.3], there is an $\mathbb{F}S_n$-isomorphism

$S^{(n-r,1^r)} \cong \bigwedge^r S^{(n-1,1)}$. Moreover, if $r \leqslant n-2$ then, in consequence of [65],

$$\mathrm{Hd}(S^{(n-r,1^r)}) \cong \mathrm{Hd}(\bigwedge^r S^{(n-1,1)}) \cong \bigwedge^r \mathrm{Hd}(S^{(n-1,1)}) \cong \bigwedge^r D^{(n-1,1)} =: D_r$$

is simple. Thus D_r has $\mathbb{F}$-basis

$$\mathcal{B}_r := \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} : 2 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1\}. \tag{6.1}$$

If $r \leqslant p-1$ then $\bigwedge^r D^{(n-1,1)} \cong D^{(n-r,1^r)}$.

6.3 The p-subgroups of symmetric groups

The aim of this section is to prove a number of properties of the Sylow p-subgroups of S_n and their subgroups. In particular, the characterisation given in Proposition 6.3.5 will be particularly useful to prove our main Theorem 6.1.1. In order to do this we will focus on the study of some particular elementary abelian p-subgroups of S_n. Where necessary we will recall and use the notation introduced in Section 1.3.3.

6.3.1 Elementary abelian groups.

Let d be a natural number and let P_{p^d} be the Sylow p-subgroup of S_{p^d} defined in Section 1.3.3. We have that

$$P_{p^d} = P_{p^{d-1}} \wr P_p = (Q_1 \times \cdots \times Q_p) \rtimes C_p,$$

where Q_i is an S_{p^d}-conjugate to $S_{p^d}P_{p^{d-1}}$, for all $i \in \{1, 2, \ldots, p\}$. For the reader's convenience we recall that P_{p^d} is generated by the elements $g_1, g_2, \ldots, g_d$ defined by

$$g_j = \prod_{k=1}^{p^{j-1}} (k, k+p^{j-1}, k+2p^{j-1}, \ldots, k+(p-1)p^{j-1}),$$

for all $j \in \{1, 2, \ldots, d\}$. Moreover, readopting the notation introduced in Section 1.3.3, for $j \in \{1, \ldots, d-1\}$, let

$$g_{j,j+1} := \prod_{i=0}^{p-1} g_{j+1}^{-i} g_j g_{j+1}^{i},$$

and for $l \in \{1, \ldots, d-j-1\}$, we inductively set

$$g_{j,j+1,\ldots,j+l+1} := \prod_{i=0}^{p-1} g_{j+l+1}^{-i} \cdot g_{j,j+1,\ldots,j+l} \cdot g_{j+l+1}^{i} \,.$$

We denote by E_{p^d} the following elementary abelian p-subgroup of P_{p^d} that acts regularly (transitively and fixed point freely) on $\{1, 2, \ldots, p^d\}$:

$$E_{p^d} = \langle g_{1,\ldots,d}, g_{2,\ldots,d}, \ldots, g_{d-1,d}, g_d \rangle \,.$$

EXAMPLE 6.3.1 *Suppose that $p = 3$ and $n = 27$. Then $E_n = E_{27}$ is generated by the elements*

$$\begin{aligned} g_{1,2,3} &= (1,2,3)(4,5,6)(7,8,9)(10,11,12)\cdots(22,23,24)(25,26,27)\,, \\ g_{2,3} &= (1,4,7)(2,5,8)(3,6,9)(10,13,16)\cdots(20,23,26)(21,24,27)\,, \\ g_3 &= (1,10,19)(2,11,20)(3,12,21)(4,13,22)\cdots(8,17,26)(9,18,27)\,. \end{aligned}$$

In the following lemma we will slightly extend the description of the lattice of subgroups of the Sylow p-subgroups of S_n given in Lemma 1.3.10. In order to do this will be very useful to denote by P_n the fixed Sylow p-subgroup of S_n defined in Section 1.3.3 by

$$P_n := P_{p,1} \times \cdots \times P_{p,n_1} \times \cdots \times P_{p^r,1} \times \cdots \times P_{p^r,n_r},$$

where

$$P_{p^i,j_i} := (1, 1+k(j_i)) \cdots (p^i, p^i + k(j_i)) \cdot P_{p^i} \cdot (1, 1+k(j_i)) \cdots (p^i, p^i + k(j_i))$$

and $k(j_i) := \sum_{l=0}^{i-1} n_l p^l + (j_i - 1)p^i$ for $i \in \{t \in \mathbb{N} \mid n_t \neq 0\}$ and $1 \leqslant j_i \leqslant n_i$. Given this convention, we shall then also write $P_n = \prod_{i=0}^{r} (P_{p^i})^{n_i}$, for simplicity.

LEMMA 6.3.2 *Let n be a natural number with p-adic expansion $n = \sum_{i=0}^{r} n_i p^i$. Suppose that $E \leqslant P_n$ is such that $E =_{S_n} E_{p^i}$, for some $i \in \{1, \ldots, r\}$. Then $E \leqslant P_{p^l,j_l}$, for some $l \in \{i, \ldots, r\}$ and some $1 \leqslant j_l \leqslant n_l$. Moreover, we deduce that E is contained in one of the p^{l-i} subgroups of P_{p^l,j_l} that are S_n-conjugate to P_{p^i}.*

Proof: Since E has precisely one non-trivial orbit in its action on $\{1, 2, \ldots, n\}$ we deduce that $E \leqslant P_{p^l,j_l}$ for some $l \in \{1, 2, \ldots, r\}$. The size of the non-trivial orbit of E is p^i, hence $l \in \{i, \ldots, r\}$.

To prove the second assertion we proceed by induction on $l-i$. If $l=i$ then the statement is clearly true since $E \leqslant P_{p^i,j_i} =_{S_n} P_{p^i}$. Suppose now that $l>i$. As explained in Section 1.3.3 we have that

$$P_{p^l,j_l} = (Q_1 \times \cdots \times Q_p) \rtimes C_p,$$

where $Q_1, Q_2, \ldots, Q_p$ are the only subgroups of P_{p^l,j_l} that are S_n-conjugate to $P_{p^{l-1}}$, by Lemma 1.3.10. Denote by B the base group of P_{p^l,j_l}, $B = Q_1 \times \cdots \times Q_p$. Since $l>i$ we have that $\mathrm{supp}(E) \subsetneq \mathrm{supp}(P_{p^l,j_l})$, therefore we deduce that E has at least one fixed point in its action on $\mathrm{supp}(P_{p^l,j_l})$. It follows that $E \leqslant B$. Moreover, as already mentioned, E has a unique non-trivial orbit of size p^i, hence $E \leqslant Q_s$ for some $s \in \{1, 2, \ldots, p\}$. Since $Q_s =_{S_n} P_{p^{l-1}}$, we can apply the inductive hypothesis to deduce that E is contained in R, where R one of the p^{l-1-i} subgroups of Q_s that are conjugate to P_{p^i} in S_n. Again by Lemma 1.3.10 we deduce that R is one of the p^{l-i} subgroups of P_{p^l,j_l} that are conjugate to P_{p^i} in S_n. The proof is now complete. $\square$

Lemma 6.3.3 *Let $n, d \in \mathbb{N}$, and let $P \leqslant P_{p^d} \leqslant S_n$. Suppose that P contains an S_n-conjugate of $P_{p^{d-1}}$. Suppose further that P contains an elementary abelian group E of order p^d acting regularly on $\{1, \ldots, p^d\}$. Then $P = P_{p^d}$.*

Proof: If $d=1$ then $P_{p^d} = P_p = E$. From now on we may suppose that $d \geqslant 2$. Recall that P_{p^d} is generated by the elements $g_1, \ldots, g_d$ introduced in Section 1.3.3 (and recalled at the beginning of this section). Moreover, P_{p^d} acts imprimitively on the set $\{1, \ldots, p^d\}$, a system of imprimitivity being given by $\Delta := \{\Delta_1, \ldots, \Delta_p\}$, where $\Delta_s := \{(s-1)p^{d-1}+1, \ldots, sp^{d-1}\}$, for $s \in \{1, \ldots, p\}$. Since E acts transitively on $\{1, \ldots, p^d\}$, there is some $g \in E$ such that $(1)g = p^{d-1}+1$; in particular, $\Delta_1 \cdot g = \Delta_2$. Since $p^{d-1}+1 \neq 1$, we have $g \neq 1$, hence g is an element of order p. Moreover, the group $\langle g \rangle$ acts on Δ, so that we obtain a group homomorphism $\varphi : \langle g \rangle \to \mathrm{Sym}(\Delta) \cong S_p$. Since $\Delta_1 \cdot g = \Delta_2 \neq \Delta_1$, φ must be injective. Thus $\varphi(g)$ has order p, implying $\Delta_1 \cdot g = \Delta_2$, $\Delta_2 \cdot g = \Delta_{i_3}, \ldots, \Delta_{i_p} \cdot g = \Delta_1$, for some $i_3, \ldots, i_p$ such that $\{1, 2, i_3, \ldots, i_p\} = \{1, \ldots, p\}$.

Let $R := P_{p^{d-1}}^{\sigma} \leqslant P$, for some $\sigma \in S_n$. By Lemma 1.3.10, we know that $R = g_d^{-i} P_{p^{d-1}} g_d^{i}$, for some $i \in \{0, \ldots, p-1\}$. Thus $\mathrm{supp}(R) = \Delta_{i+1}$. So, for $s \in \{0, \ldots, p-1\}$, the group R^{g^s} has support $\Delta_{i+1} \cdot g^s$. As we have just seen, the sets $\Delta_{i+1}, \Delta_{i+1} \cdot g, \ldots, \Delta_{i+1} \cdot g^{p-1}$ are pairwise disjoint. Consequently, the groups R, $R^g, \ldots, R^{g^{p-1}}$ are precisely the different subgroups of P_{p^d} that are P_{p^d}-conjugate to $P_{p^{d-1}}$, and are all contained in P. Therefore $B := \prod_{s=0}^{p-1} R^{g^s}$ is the base group of P_{p^d}, and is contained in P. Clearly $g \notin B$, since $(1)g \notin \Delta_1$. Since $[P_{p^d} : B] = p$, this implies $P_{p^d} = \langle B, g\rangle \leqslant P \leqslant P_{p^d}$, and the proof is complete. $\square$

We need now to introduce an important class of elementary abelian p-subgroups of the symmetric group. The following definitions could be given for arbitrary n, but we restrict our attention to the case where $p \mid n$, that is the setting required to prove Theorem 6.1.1. Let $n \in \mathbb{N}$ be arbitrary with $p \mid n$, and let $t, m_1, \ldots, m_t \in \mathbb{N}_0$ be such that $n = \sum_{i=1}^{t} m_i p^i$. For $i \in \{s \in \mathbb{N} \mid m_s \neq 0\}$ and $1 \leqslant j_i \leqslant m_i$, we set $k(j_i) := \sum_{l=0}^{i-1} m_l p^l + (j_i - 1)p^i$ and

$$E_{p^i,j_i} := (1, 1+k(j_i)) \cdots (p^i, p^i + k(j_i)) \cdot E_{p^i} \cdot (1, 1+k(j_i)) \cdots (p^i, p^i + k(j_i)) \leqslant P_{p^i,j_i}\,.$$

Denote by $E(m_1, \ldots, m_t)$ the elementary abelian subgroup of S_n defined by

$$E(m_1, \ldots, m_t) = E_{p,1} \times \cdots \times E_{p,m_1} \times \cdots \times E_{p^t,1} \times \cdots \times E_{p^t,m_t}\,.$$

Notice that for $i \in \{1, \ldots, t\}$ and $j_i \in \{1, \ldots, m_i\}$, the direct factor E_{p^i,j_i} of $E(m_1, \ldots, m_t)$ is determined by i and its support $S(i, j_i)$.

We emphasize that the integers $m_1, \ldots, m_t$ need not be strictly less than p.

LEMMA 6.3.4 *Let $n, t \in \mathbb{N}$ and let $m_1, \ldots, m_t \in \mathbb{N}_0$ be such that $m_t \neq 0$ and $n = \sum_{i=1}^{t} m_i p^i$. Suppose that $m_1 = 1$ and $t \geqslant 2$. Let P be a maximal subgroup of $E(m_1, \ldots, m_t)$ such that $E_{p,1} \nleqslant P$. Then P contains a subgroup $Q \leqslant \prod_{i=2}^{t} \prod_{j=1}^{m_i} E_{p^i,j}$ that acts fixed point freely on $\{p+1, \ldots, n\}$.*

Proof: For convenience, set $E' := \prod_{i=2}^{t} \prod_{j=1}^{m_i} E_{p^i,j}$, so that

$$E(m_1, \ldots, m_t) = E_p \times E' \geqslant P.$$

By Goursat's Lemma (see [53, Page 75]), we may identify P with the quintuple $(P_1, K_1, \eta, P_2, K_2)$, where P_1 and P_2 are the projections of P onto E_p and onto E', respectively, $K_1 := \{g \in E_p : (g, 1) \in P\} \trianglelefteq P_1$, $K_2 := \{h \in E' : (1, h) \in P\} \trianglelefteq P_2$, and $\eta : P_2/K_2 \to P_1/K_1$ is a group isomorphism. Since $|E_p| = p$, there are precisely three possibilities for the section (P_1, K_1) of E_p:

(i) $P_1 = K_1 = E_p$,
(ii) $P_1 = K_1 = \{1\}$,
(iii) $P_1 = E_p$ and $K_1 = \{1\}$.

Case (i) cannot occur, since we are assuming $E_p \nleqslant P$. In case (ii) we get $P = E'$, so that the assertion then holds with $Q := P$. So suppose that $P_1 = E_p$ and $K_1 = \{1\}$, so that also $[P_2 : K_2] = p$. Next recall that $P/(K_1 \times K_2) \cong P_1/K_1 \cong P_2/K_2$; see, for instance, [6, 2.3.21]. This forces

$$|E'| = |P| = |K_2| \cdot |P_1| = |K_1| \cdot |P_2| = |P_2|.$$

Thus $P_2 = E'$, and K_2 is a maximal subgroup of E'. Assume that K_2 has a fixed point x on $\{p+1, \dots, n\}$. Then $x \in \mathrm{supp}(E_{p^i,j})$, for some $i \geqslant 2$ with $m_i \neq 0$ and some $j \in \{1, \dots, m_i\}$. But then K_2 has to fix the entire support of $E_{p^i,j}$, since $E_{p^i,j}$ acts regularly on its support. This implies $[P_2 : K_2] \geqslant p^i \geqslant p^2$, a contradiction. Consequently, K_2 must act fixed point freely on $\{p+1, \dots, n\}$, and the assertion of the lemma follows with $Q := \{1\} \times K_2 \leqslant P$. $\square$

The next result will be one of the key ingredients of our proof of Theorem 6.1.1 in Section 6.4 below.

Proposition 6.3.5 *Let $n \in \mathbb{N}$ with p-adic expansion $n = p + \sum_{i=2}^{r} n_i p^i$, where $r \geqslant 2$ and $n_r \neq 0$. Let $Q \leqslant P_n$ be such that $P_{n-2p} \leqslant_{S_n} Q$ and $E(1, n_2, \dots, n_r) \leqslant_{S_n} Q$. Then $Q = P_n$.*

Proof: Let $2 \leqslant s \leqslant r$ be minimal such that $n_s \neq 0$. Then $n - 2p$ has the following p-adic expansion:

$$n - 2p = \sum_{j=1}^{s-1} (p-1)p^j + (n_s - 1)p^s + \sum_{i=s+1}^{r} n_i p^i.$$

Moreover, we have

$$P_n = P_{p,1} \times \prod_{i=s}^{r} \prod_{j=1}^{n_i} P_{p^i,j} \quad \text{and} \quad E_n = E(1, n_2, \dots, n_r) = E_{p,1} \times \prod_{i=s}^{r} \prod_{j=1}^{n_i} E_{p^i,j}\,.$$

By our hypothesis, there is some $g \in S_n$ such that

$$E_{p,1}^g \times \prod_{i=s}^{r} \prod_{j=1}^{n_i} E_{p^i,j}^g \leqslant Q \leqslant P_n.$$

In consequence of Lemma 1.3.10 and Lemma 6.3.2, we may suppose that $E_{p^i,j}^g \leqslant P_{p^i,j}$, for $i \geqslant 2$ and $1 \leqslant j \leqslant n_i$, as well as $E_{p,1}^g = E_{p,1} = P_{p,1}$. Since also $P_{n-2p} \leqslant_{S_n} Q$, there exists some $R \leqslant Q \leqslant P_n$ of the form

$$R = \prod_{i=1}^{s-1} \prod_{j=1}^{p-1} R_{p^i,j} \times \prod_{j=1}^{n_s - 1} R_{p^s,j} \times \prod_{i=s+1}^{r} \prod_{j=1}^{n_i} R_{p^i,j}\,,$$

where $R_{p^k,l} =_{S_n} P_{p^k,l}$, for all possible k and l. By Lemma 1.3.10, we must have

$$\prod_{i=s+1}^{r}\prod_{j=1}^{n_i} R_{p^i,j} = \prod_{i=s+1}^{r}\prod_{j=1}^{n_i} P_{p^i,j} \leqslant P_n.$$

Moreover, there is some $k \in \{1, \ldots, n_s\}$ and some $m \in \{1, \ldots, p-1\}$ such that

$$\prod_{j=1}^{n_s-1} R_{p^s,j} = \prod_{j=1}^{k-1} P_{p^s,j} \times \prod_{l=k+1}^{n_s} P_{p^s,l} \leqslant P_n \quad \text{and} \quad R_{p^{s-1},m} \leqslant P_{p^s,k}.$$

By Lemma 1.3.10, $R_{p^{s-1},m}$ is thus $P_{p^s,k}$-conjugate to one of the p subgroups of $P_{p^s,k}$ that are S_n-conjugate to $P_{p^{s-1}}$. Since Q also contains the regular elementary abelian group $E^g_{p^s,k} \leqslant P_{p^s,k}$, Lemma 6.3.3 now implies that $P_{p^s,k} \leqslant Q$. Altogether this shows that indeed $P_n \leqslant Q$, and the assertion of the proposition follows. □

6.4 The proof of Theorem 6.1.1

The aim of this section is to establish a proof of Theorem 6.1.1. To this end, let $\mathbb{F}$ be a field of characteristic $p > 2$, and let $n \in \mathbb{N}$ be such that $n \equiv p \pmod{p^2}$. The simple $\mathbb{F}S_n$-module $D^{(n-p+1,1^{p-1})}$ will henceforth be denoted by D. If $p = n$ then the Sylow p-subgroups of S_n are abelian, and are thus the vertices of D, by Theorem 1.2.4. From now on we shall suppose that $n \geqslant p^2 + p$. Let P_n be the Sylow p-subgroup of S_n introduced in Section 1.3.3 (and recalled in Section 6.3.1). In order to show that P_n is a vertex of D, we shall proceed as follows: suppose that $Q \leqslant P_n$ is a vertex of D. Then:

(i) Building on Wildon's result in [76, Theorem 2], it was shown by Danz in [12, Proposition 5.2] that $P_{n-2p} = P_{n-(p-1)-2} \times P_{p-1} <_{S_n} Q$.

(ii) Let $n = \sum_{i=2}^{r} n_i p^i + p$ be the p-adic expansion of n, where $r \geqslant 2$ and $n_r \neq 0$. It is a corollary of Proposition 6.4.6 below that $D(E(1, n_2, \ldots, n_r)) \neq \{0\}$. Here $E(1, n_2, \ldots, n_r)$ denotes the elementary abelian subgroup of P_n defined in 6.3.1 and $D(E(1, n_2, \ldots, n_r))$ denotes the Brauer quotient of D with respect to $E(1, n_2, \ldots, n_r)$ as defined in Section 1.2.3. Thus, $E(1, n_2, \ldots, n_r) \leqslant_{S_n} Q$, by Proposition 1.2.13.

(iii) Once we have verified (ii), we can apply Proposition 6.3.5, which then shows that $Q = P_n$.

We begin by establishing (ii). To this end we fix some notation first.

Let $\mathcal{B} := \mathcal{B}_{p-1}$ be the $\mathbb{F}$-basis of D defined in (6.1), and let $u \in D$ be such that

$$u = \sum_{b \in \mathcal{B}} \lambda_b \cdot b, \quad \text{for} \quad \lambda_b \in \mathbb{F}.$$

The basis element $e_2 \wedge e_3 \wedge \cdots \wedge e_p \in \mathcal{B}$ will from now on be denoted by e. Moreover, suppose that $k, x \in \{2, \ldots, n-1\}$, and $k \leqslant p$. Then we denote by $\hat{e}_k \wedge e_x$ the element of D defined by

$$\hat{e}_k \wedge e_x = e_2 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_p \wedge e_x.$$

In the case where $\hat{e}_k \wedge e_x \in \mathcal{B}$, the coefficient $\lambda_{e_2 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_p \wedge e_x}$ will be denoted by $\lambda_{\hat{k},x}$.

Similarly, if $2 \leqslant k < l \leqslant p$ and $x, y \in \{2, \ldots, n-1\}$, then we set

$$\hat{e}_{k,l} \wedge e_x \wedge e_y := e_2 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_{l-1} \wedge e_{l+1} \wedge \cdots \wedge e_p \wedge e_x \wedge e_y \in D.$$

In the case where $\hat{e}_{k,l} \wedge e_x \wedge e_y \in \mathcal{B}$, then we denote by $\lambda_{\widehat{k,l},x,y}$ the coefficient at $\hat{e}_{k,l} \wedge e_x \wedge e_y$ in u.

Let $u \in D$ be such that $u = \sum_{b \in \mathcal{B}} \lambda_b \cdot b$, with $\lambda_b \in \mathbb{F}$. We say that the basis element $b \in \mathcal{B}$ *occurs in* u with coefficient λ_b. If $\lambda_b \neq 0$ then we also simply say that b *occurs in* u.

For $k_1, k_2 \in \{2, \ldots, n-1\}$, we set

$$s(k_1, k_2) := \begin{cases} k_2 - (k_1 - 1) & \text{if } k_1 \leqslant k_2\,, \\ 0 & \text{if } k_2 < k_1\,. \end{cases} \tag{6.2}$$

Thus, if $k_1 \leqslant k_2$ then

$$s(k_1, k_2) \equiv \begin{cases} 0 \pmod 2 & \text{if } k_1 \not\equiv k_2 \pmod 2\,, \\ 1 \pmod 2 & \text{if } k_1 \equiv k_2 \pmod 2\,. \end{cases}$$

From now on, let $t, m_2, \ldots, m_t \in \mathbb{N}$ be such that $t2$, $m_t \neq 0$, and $n = p + \sum_{i=2}^{t} m_i p^i$. The elementary abelian group $E(1, m_2, \ldots, m_t) \leqslant S_n$ will be denoted by E. Note that, by our convention in Section 6.3.1, we have $(1, 2, \ldots, p) \in E$. In the case where $t = r$ and $m_i = n_i$, for $i = 2, \ldots, r$, we, in particular, get $E = E(1, n_2, \ldots, n_r)$.

In the course of this section we shall have to compute explicitly the actions of elements in E on our chosen basis $\mathcal{B}$ of D. The following lemmas will be used repeatedly in this section.

LEMMA 6.4.1 *Let* $\alpha := (1, 2, \ldots, p) \in S_n$. *Let further* $\beta := (x_1, \ldots, x_p) \in S_n$ *be such that* $\{x_1, \ldots, x_p\} \cap \{1, \ldots, p\} = \emptyset$.

(a) *For* $i \in \{2, \ldots, n-1\}$, *one has*

$$e_i\alpha = \begin{cases} e_{i+1} - e_2 & \textit{if } 2 \leqslant i \leqslant p-1\,, \\ -e_2 & \textit{if } i = p\,, \\ e_i - e_2 & \textit{if } i \geqslant p+1\,. \end{cases}$$

Moreover, $e_{p-(i-1)}\alpha^i = -e_{i+1}$, *for all* $i \in \{1, 2, \ldots, p-1\}$.

(b) *If* $n \notin \mathrm{supp}(\beta)$ *then, for* $i \in \{2, \ldots, n-1\}$, *one has*

$$e_i\beta = \begin{cases} e_i & \textit{if } i \notin \mathrm{supp}(\beta)\,, \\ e_{(i)\beta} & \textit{if } i \in \mathrm{supp}(\beta)\,. \end{cases}$$

(c) *If* $x_p = n$ *then, for* $i \in \{2, \ldots, n-1\}$, *one has*

$$e_i\beta = \begin{cases} e_i & \textit{if } i \notin \mathrm{supp}(\beta)\,, \\ e_{(i)\beta} & \textit{if } i \in \{x_1, \ldots, x_{p-2}\}\,, \\ -\sum_{j=2}^{n-1} e_j & \textit{if } i = x_{p-1}\,. \end{cases}$$

Proof: (a) If $2 \leqslant i \leqslant p-1$, then

$$\begin{aligned} e_i\alpha =& (\overline{\omega_i - \omega_1})\alpha = \overline{(\omega_i - \omega_1)\alpha} \\ =& \overline{\omega_{(i)\alpha} - \omega_{(1)\alpha}} = \overline{\omega_{i+1} - \omega_2} \\ =& \overline{(\omega_{i+1} - \omega_1) - (\omega_2 - \omega_1)} = e_{i+1} - e_2. \end{aligned}$$

If $i = p$, then $e_i\alpha = (\overline{\omega_p - \omega_1})\alpha = \overline{\omega_1 - \omega_2} = -e_2$.

Finally, if $i \geqslant p+1$, then we have

$$e_i\alpha = \overline{\omega_i - \omega_2} = \overline{(\omega_i - \omega_1) - (\omega_2 - \omega_1)} = e_i - e_2\,.$$

The proofs of (b) and (c) are similar, and are left to the reader. $\square$

LEMMA 6.4.2 *Let $k, l \in \{2, \ldots, p\}$, and let $x \in \{p+1, \ldots, n-1\}$. Then one has*

(a) $e_{k+1} \wedge \cdots \wedge e_p \wedge e_2 \wedge \cdots \wedge e_{k-1} \wedge e_x = (-1)^{s(k+1,p)(k-2)} \hat{e}_k \wedge e_x$;

(b) $\hat{e}_k \wedge e_k = (-1)^{s(k+1,p)} e$;

(c) *if* $k < l$ *then* $\hat{e}_{k,l} \wedge e_x \wedge e_l = (-1)^{s(l+1,p)+1} \hat{e}_k \wedge e_x$;

(d) *if* $k < l$ *then* $\hat{e}_{k,l} \wedge e_x \wedge e_k = (-1)^{s(k+1,p)} \hat{e}_l \wedge e_x$.

Proof: (a) For $k \in \{2, \ldots, p\}$ and $x \in \{p+1, \ldots, n-1\}$, we have

$$\overbrace{e_{k+1} \wedge \cdots \wedge e_p \wedge e_x}^{s(k+1,p)+1} \wedge \underbrace{e_2 \wedge \cdots \wedge e_{k-1}}_{k-2} =$$

$$= (-1)^{(s(k+1,p)+1)} e_2 \wedge e_{k+1} \wedge \cdots \wedge e_p \wedge e_x \wedge e_3 \wedge \cdots \wedge e_{k-1}$$

$$= (-1)^{(s(k+1,p)+1)(k-2)} \cdot \hat{e}_k \wedge e_x.$$

The proofs of (b), (c) and (d) are similar, and are left to the reader. $\square$

COROLLARY 6.4.3 *For $e := e_2 \wedge e_3 \wedge \cdots \wedge e_p$, we have $e \in D^{P_n}$; in particular, $e \in D^P$, for every $P \leqslant P_n$.*

Proof: With the notation introduced in Section 1.3.3 we have $P_n = P_p \times \prod_{i=2}^{r} (P_{p^i})^{n_i}$, and $P_p = \langle \alpha \rangle$, where $\alpha := (1, 2, \ldots, p)$. If $\beta \in \prod_{i=2}^{r} (P_{p^i})^{n_i}$ then we clearly have $e\beta = e$. By Lemma 6.4.1 and Lemma 6.4.2(b), we also have

$$\begin{aligned} e\alpha &= (e_3 - e_2) \wedge (e_4 - e_2) \wedge \cdots \wedge (e_p - e_2) \wedge (-e_2) \\ &= (-1)^{s(3,p)+1} e = (-1)^2 e = e. \quad \square \end{aligned}$$

The following lemma will be a fundamental step in the proof of Case (2.2) of Lemma 6.4.5.

LEMMA 6.4.4 *Let $1 \neq \sigma \in E$, and let $q \in \mathbb{N}$ be such that*

$$\sigma = (x_1^1, \ldots, x_p^1) \cdots (x_1^q, \ldots, x_p^q),$$

where $\{x_i^s : 1 \leqslant i \leqslant p, 1 \leqslant s \leqslant q\} = \mathrm{supp}(\sigma) \subseteq \{p+1, \ldots, n\}$ and $x_p^q = n$. Let further $u \in D$ be such that $u = \sum_{b \in \mathcal{B}} \lambda_b \cdot b$, for $\lambda_b \in \mathbb{F}$. Suppose that $u \cdot \sigma = u$. Then one has the following:

(a) $\sum_{k=2}^{p}(-1)^k \cdot \lambda_{\hat{k},x_i^q} = 0$, *for every* $i \in \{1,\ldots,p-1\}$*;*

(b) $\sum_{k=2}^{p}(-1)^{k+1} \cdot \lambda_{\hat{k},x_i^s} = \sum_{k=2}^{p}(-1)^{k+1}\lambda_{\hat{k},x_1^s}$, *for* $i \in \{1,\ldots,p\}$ *and* $1 \leqslant s \leqslant q-1$.

Proof: Let $x \in \{x_i^s : 1 \leqslant i \leqslant p,\ 1 \leqslant s \leqslant q-1\}$, and let $k \in \{2,\ldots,p\}$. Suppose that $b \in \mathcal{B}$ is such that $\hat{e}_k \wedge e_x$ occurs with non-zero coefficient in $b \cdot \sigma$. Then

(i) $b = \hat{e}_k \wedge e_{(x)\sigma^{-1}}$, or

(ii) $b = \hat{e}_k \wedge e_{x_{p-1}^q}$, or

(iii) $b = \hat{e}_{k,k_2} \wedge e_{(x)\sigma^{-1}} \wedge e_{x_{p-1}^q}$ and $(x)\sigma^{-1} < x_{p-1}^q$, for some $k < k_2 \leqslant p$, or

(iv) $b = \hat{e}_{k_1,k} \wedge e_{(x)\sigma^{-1}} \wedge e_{x_{p-1}^q}$ and $(x)\sigma^{-1} < x_{p-1}^q$, for some $2 \leqslant k_1 < k$, or

(v) $b = \hat{e}_{k,k_2} \wedge e_{x_{p-1}^q} \wedge e_{(x)\sigma^{-1}}$ and $(x)\sigma^{-1} > x_{p-1}^q$, for some $k < k_2 \leqslant p$, or

(vi) $b = \hat{e}_{k_1,k} \wedge e_{x_{p-1}^q} \wedge e_{(x)\sigma^{-1}}$ and $(x)\sigma^{-1} > x_{p-1}^q$, for some $2 \leqslant k_1 < k$.

If b is one of the basis elements in (i)–(vi) then the following table records $b\sigma$ as well as the coefficient at $\hat{e}_k \wedge e_x$ in $b\sigma$, which is obtained using Lemma 6.4.2.

b	$b \cdot \sigma$	coefficient
$\hat{e}_k \wedge e_{(x)\sigma^{-1}}$	$\hat{e}_k \wedge e_x$	1
$\hat{e}_k \wedge e_{x_{p-1}^q}$	$\hat{e}_k \wedge \sum_{y=2}^{n-1}(-e_y)$	-1
$\hat{e}_{k,k_2} \wedge e_{(x)\sigma^{-1}} \wedge e_{x_{p-1}^q}$	$\hat{e}_{k,k_2} \wedge e_x \wedge \sum_{y=2}^{n-1}(-e_y)$	$(-1)^{1+s(k_2+1,p)+1}$
$\hat{e}_{k_1,k} \wedge e_{(x)\sigma^{-1}} \wedge e_{x_{p-1}^q}$	$\hat{e}_{k_1,k} \wedge e_x \wedge \sum_{y=2}^{n-1}(-e_y)$	$(-1)^{1+s(k_1+1,p)}$
$\hat{e}_{k,k_2} \wedge e_{x_{p-1}^q} \wedge e_{(x)\sigma^{-1}}$	$\hat{e}_{k,k_2} \wedge \sum_{y=2}^{n-1}(-e_y) \wedge e_x$	$(-1)^{1+s(k_2+1,p)}$
$\hat{e}_{k_1,k} \wedge e_{x_{p-1}^q} \wedge e_{(x)\sigma^{-1}}$	$\hat{e}_{k_1,k} \wedge \sum_{y=2}^{n-1}(-e_y) \wedge e_x$	$(-1)^{s(k_1+1,p)}$

Now note that

$$(-1)^{1+s(k_2+1,p)+1} = (-1)^{1+p-k_2+1} = (-1)^{k_2+1}$$

and

$$(-1)^{1+s(k_1+1,p)} = (-1)^{1+p-k_1} = (-1)^{k_1}.$$

Since $u \cdot \sigma = u$, this shows that

$$\lambda_{\hat{k},x} = \lambda_{\hat{k},(x)\sigma^{-1}} - \lambda_{\hat{k},x_{p-1}^q} + \sum_{k_2=k+1}^{p}(-1)^{k_2+1}\lambda_{\widehat{k,k_2},(x)\sigma^{-1},x_{p-1}^q} + \sum_{k_1=2}^{k-1}(-1)^{k_1}\lambda_{\widehat{k_1,k},(x)\sigma^{-1},x_{p-1}^q} \tag{6.3}$$

if $(x)\sigma^{-1} < x^q_{p-1}$ and

$$\lambda_{\hat{k},x} = \lambda_{\hat{k},(x)\sigma^{-1}} - \lambda_{\hat{k},x^q_{p-1}} - \sum_{k_2=k+1}^{p} (-1)^{k_2+1} \lambda_{\widehat{k,k_2},x^q_{p-1},(x)\sigma^{-1}} - \sum_{k_1=2}^{k-1} (-1)^{k_1} \lambda_{\widehat{k_1,k},x^q_{p-1},(x)\sigma^{-1}} \tag{6.4}$$

if $\sigma^{-1}(x) > x^q_{p-1}$. Moreover,

$$\begin{aligned} &\sum_{k=2}^{p}(-1)^{k+1}\left(\sum_{k_2=k+1}^{p}(-1)^{k_2+1}\lambda_{\widehat{k,k_2},(x)\sigma^{-1},x^q_{p-1}} + \sum_{k_1=2}^{k-1}(-1)^{k_1}\lambda_{\widehat{k_1,k},(x)\sigma^{-1},x^q_{p-1}}\right) \\ &= \sum_{k=2}^{p}\sum_{l=k+1}^{p}((-1)^{k+1}(-1)^{l+1} + (-1)^{k}(-1)^{l+1}) \cdot \lambda_{\widehat{k,l},(x)\sigma^{-1},x^q_{p-1}} = 0 \end{aligned}$$

if $(x)\sigma^{-1} < x^q_{p-1}$, and

$$\begin{aligned} &\sum_{k=2}^{p}(-1)^{k+1}\left(-\sum_{k_2=k+1}^{p}(-1)^{k_2+1}\lambda_{\widehat{k,k_2},x^q_{p-1},(x)\sigma^{-1}} - \sum_{k_1=2}^{k-1}(-1)^{k_1}\lambda_{\widehat{k_1,k},x^q_{p-1},(x)\sigma^{-1}}\right) \\ &= -\sum_{k=2}^{p}\sum_{l=k+1}^{p}((-1)^{k+1}(-1)^{l+1} + (-1)^{k}(-1)^{l+1})\lambda_{\widehat{k,l},x^q_{p-1},(x)\sigma^{-1}} = 0 \end{aligned}$$

if $(x)\sigma^{-1} > x^q_{p-1}$. Hence, from (6.3) and (6.4) we get

$$\sum_{k=2}^{p}(-1)^{k+1}\lambda_{\hat{k},x^s_i} = \sum_{k=2}^{p}(-1)^{k+1}\lambda_{\hat{k},(x^s_i)\sigma^{-1}} + \sum_{k=2}^{p}(-1)^{k}\lambda_{\hat{k},x^q_{p-1}}\,, \tag{6.5}$$

for every $i \in \{1,\dots,p\}$ and $1 \leqslant s \leqslant q-1$.

We also have $u \cdot \sigma^i = u$, for $i = 1,\dots,p-1$. To compare the coefficient at e in u and in $u \cdot \sigma^i$, let $i \in \{1,\dots,p-1\}$ and suppose that $b \in \mathcal{B}$ is such that e occurs in $b \cdot \sigma^i$ with non-zero coefficient. Then either $b = e$ and $e = e \cdot \sigma^i$, or $b = \hat{e}_k \wedge e_{(x^q_p)\sigma^{-i}}$, for some $k \in \{2,\dots,p\}$. Moreover, in the latter case we have $b \cdot \sigma^i = \hat{e}_k \wedge (-e_2 - e_3 - \cdots - e_{n-1})$, where e occurs with coefficient

$$(-1)^{s(k+1,p)+1} = \begin{cases} 1 & \text{if } 2 \mid k\,, \\ -1 & \text{if } 2 \nmid k\,, \end{cases}$$

by Lemma 6.4.2. So we obtain $\lambda_e = \lambda_e + \sum_{k=2}^{p}(-1)^k \cdot \lambda_{\hat{k},(x^q_p)\sigma^{-i}}$, for $i \in \{1,\dots,p-1\}$, that is,

$$0=\sum_{k=2}^{p}(-1)^k\lambda_{\hat{k},x_j^q}\,, \tag{6.6}$$

for $j\in\{1,\ldots,p-1\}$, which proves assertion (a). Now assertion (b) follows from (6.5) and (6.6) with $j=p-1$. $\square$

Next we shall show that $D(E)\neq\{0\}$, where E is the elementary abelian group defined by

$$E=E(1,m_2,\ldots,m_t)\ \text{ for some }\ m_2,\ldots,m_t\in\mathbb{N}.$$

In order to do so, we want to apply Proposition 1.2.15 with $b_0=e$.

LEMMA 6.4.5 *Let P be a maximal subgroup of E. If $u\in D^P$ then e occurs in* $\mathrm{Tr}_P^E(u)$ *with coefficient* 0.

Proof: Set $\alpha:=(1,2,\ldots,p)$. Let $u\in D^P$, and write $u=\sum_{b\in\mathcal{B}}\lambda_b\cdot b$, where $\lambda_b\in\mathbb{F}$. We shall treat the case where $\alpha\in P$ and the case where $\alpha\notin P$ separately.

Case 1: $\alpha\in P$. Then there is some $1\neq g\in\prod_{i=2}^{t}\prod_{j=1}^{m_i}E_{p^i,j}$ with $g\notin P$. Thus $\{1,g,g^2,\ldots,g^{p-1}\}$ is a set of representatives of the right cosets of P in E, so that we get

$$\mathrm{Tr}_P^E(u)=u+ug+\cdots+ug^{p-1}=\sum_{b\in\mathcal{B}}\lambda_b\cdot(\sum_{i=0}^{p-1}bg^i).$$

Since $g\neq 1$ and $t\geqslant 2$, we have

$$g=(x_1^1,\ldots,x_p^1)\cdots(x_1^q,\ldots,x_p^q)\,,$$

for some $q\geqslant p$ and $\{x_i^s:2\leqslant i\leqslant p,\,1\leqslant s\leqslant q\}=\mathrm{supp}(g)\subseteq\{p+1,\ldots,n\}$.

Suppose first that $n\notin\mathrm{supp}(g)$, and let $b\in\mathcal{B}$. Let further $i\in\{0,\ldots,p-1\}$, and suppose that e occurs in bg^i with non-zero coefficient. Then we must have $b=e$, in which case $\sum_{i=0}^{p-1}bg^i=pe=0$, by Corollary 6.4.3; in particular, e occurs in $\mathrm{Tr}_P^E(u)$ with coefficient 0.

So we may now suppose that $n\in\mathrm{supp}(g)$. Moreover, we may suppose that $x_p^q=n$. Let $i\in\{0,\ldots,p-1\}$, and let $b\in\mathcal{B}$ be such that e occurs in bg^i with non-zero coefficient. If $i=0$ then we must of course have $b=e=eg^0$. If $i\geqslant 1$ then $b=e$, or $b=\hat{e}_k\wedge e_{(x_p^q)g^{-i}}$, for some $k\in\{2,\ldots,p\}$. In the latter case, we have

$(\hat{e}_k \wedge e_{(x_p^q)g^{-i}}) \cdot g^i = \hat{e}_k \wedge (-e_2 - e_3 - \cdots - e_{n-1})$, in which e occurs with coefficient

$$(-1)^{s(k+1,p)+1} = \begin{cases} 1 & \text{if } 2 \mid k\,, \\ -1 & \text{if } 2 \nmid k\,, \end{cases}$$

by Lemma 6.4.2. Consequently, the coefficient at e in $\mathrm{Tr}_P^E(u)$ equals

$$p\lambda_e + \sum_{i=1}^{p-1} \left(\sum_{\substack{k=2 \\ 2|k}}^{p} \lambda_{\hat{k},x_i^q} - \sum_{\substack{l=2 \\ 2\nmid l}}^{p} \lambda_{\hat{l},x_i^q} \right) = \sum_{i=1}^{p-1} \sum_{k=2}^{p} (-1)^k \lambda_{\hat{k},x_i^q}\,. \tag{6.7}$$

Next we use the fact that $u \in D^P$ to show that this coefficient is indeed 0. Since $\alpha \in P$, we, in particular, have $u = u\alpha^i$, for every $i \in \{1, \ldots, p-1\}$. So let $i \in \{1, \ldots, p-1\}$, and let $x \in \{x_1^q, \ldots, x_p^q\}$. Suppose that $b \in \mathcal{B}$ is such that $\hat{e}_{i+1} \wedge e_x$ occurs in $b\alpha^i$. Then from Lemma 6.4.1 we deduce that $b = \hat{e}_{(1)\alpha^{-i}} \wedge e_x$. Moreover, we have

$$\hat{e}_{(1)\alpha^{-i}} \wedge e_x \cdot \alpha^i = (e_{i+2} - e_{i+1}) \wedge \cdots \wedge (e_p - e_{i+1}) \wedge (e_2 - e_{i+1}) \wedge \cdots \wedge (e_i - e_{i+1}) \wedge (e_x - e_{i+1})\,.$$

Thus, by part (a) of Lemma 6.4.2, the coefficient of $\hat{e}_{i+1} \wedge e_x$ in $(\hat{e}_{(1)\alpha^{-i}} \wedge e_x) \cdot \alpha^i$ equals

$$(-1)^{s(i+2,p)(i-1)} = 1.$$

Letting i vary over $\{1, \ldots, p-1\}$ and comparing the coefficient of $\hat{e}_{i+1} \wedge e_x$ in u and in $u\alpha^i$, we deduce that $\lambda_{\hat{k},x} = \lambda_{\widehat{p-k+2},x}$, for $k \in \{2, \ldots, (p+1)/2\}$ and every $x \in \{x_1^q, \ldots, x_p^q\}$. Since k is even if and only if $p-k+2$ is odd, we conclude that the right-hand side of (6.7) is 0, as claimed. This completes the proof in case 1.

<u>Case 2:</u> $\alpha \notin P$, so that $\{1, \alpha, \alpha^2, \ldots, \alpha^{p-1}\}$ is a set of representatives for the cosets of P in E, and we get $\mathrm{Tr}_P^E(u) = u + u\alpha + \cdots + u\alpha^{p-1}$. We determine the coefficient at e in $\mathrm{Tr}_P^E(u) = u + u\alpha + \cdots + u\alpha^{p-1}$. Let $i \in \{0, \ldots, p-1\}$, and let $b \in \mathcal{B}$ be such that e occurs in $b\alpha^i$ with non-zero coefficient. If $i = 0$ then $b = e = e\alpha^0$. So let $i \geqslant 1$. Then, by Lemma 6.4.1, we either have $b = e$, or $b = \hat{e}_{(1)\alpha^{-i}} \wedge e_x$, for some $x \in \{p+1, \ldots, n-1\}$. Moreover, in the latter case,

$$b \cdot \alpha^i = (e_{i+2} - e_{i+1}) \wedge (e_{i+3} - e_{i+1}) \wedge \cdots \wedge (e_p - e_{i+1}) \wedge (e_2 - e_{i+1}) \wedge \cdots \wedge (e_i - e_{i+1}) \wedge (e_x - e_{i+1})\,.$$

So the coefficient at e in $(\hat{e}_{(1)\alpha^{-i}} \wedge e_x) \cdot \alpha^i$ equals

$$(-1)^{s(i+2,p)(i-1)+s(i+2,p)+1} = \begin{cases} 1 & \text{if } 2 \nmid i\,, \\ -1 & \text{if } 2 \mid i\,. \end{cases}$$

Since i is even if and only if $(1)\alpha^{-i}$ is even, we deduce from this that the coefficient at e in $u + u\alpha + \cdots + u\alpha^{p-1}$ equals

$$p\lambda_e + \sum_{x=p+1}^{n-1} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x} = \sum_{x=p+1}^{n-1} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x}\,. \tag{6.8}$$

To show that this coefficient is 0, we again exploit the fact that $u \in D^P$. In fact, we shall show that

$$\sum_{\substack{x \in \mathrm{supp}(E_{p^l,j_l}) \\ x<n}} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x} = 0\,, \tag{6.9}$$

for every $l \in \{2, \ldots, t\}$ and $1 \leqslant j_l \leqslant m_l$. For each such l and j_l, there is, by Lemma 6.3.4, some element $\sigma(l, j_l) \in P$ such that $\mathrm{supp}(E_{p^l,j_l}) \subseteq \mathrm{supp}(\sigma(l, j_l)) \subseteq \{p+1, \ldots, n\}$. Fixing l and j_l, we write

$$\sigma := \sigma(l, j_l) = (x_1^1, \ldots, x_p^1) \cdots (x_1^q, \ldots, x_p^q)\,,$$

for some $q \geqslant |E_{p^l,j_l}|/p$ and $\mathrm{supp}(\sigma) = \{x_i^j : 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q\} \subseteq \{p+1, \ldots, n\}$.

Case 2.1: $n \notin \mathrm{supp}(\sigma)$, or equivalently, $\mathrm{supp}(\sigma) \cap \mathrm{supp}(E_{p^t,m_t}) = \emptyset$. Let $x \in \mathrm{supp}(\sigma)$, let $k \in \{2, \ldots, p\}$, and let $b \in \mathcal{B}$ be such that $\hat{e}_k \wedge e_x$ occurs in $b\sigma$ with non-zero coefficient. This forces $b = \hat{e}_k \wedge e_{(x)\sigma^{-1}}$, and $(\hat{e}_k \wedge e_{(x)\sigma^{-1}}) \cdot \sigma = \hat{e}_k \wedge e_x$. Thus, $\lambda_{\hat{k},x} = \lambda_{\hat{k},(x)\sigma^{-1}}$. This shows that $\lambda_{\hat{k},x_1^s} = \lambda_{\hat{k},x_i^s}$, for all $i \in \{1, \ldots, p\}$ and $s \in \{1, \ldots, q\}$. By rearranging commuting p-cycles in σ, we may assume that there is some $1 \leqslant q_0 \leqslant q$ such that $\mathrm{supp}(E_{p^l,j_l}) = \{x_i^s : 1 \leqslant i \leqslant p,\ 1 \leqslant s \leqslant q_0\}$. Then

$$\begin{aligned} \sum_{x \in \mathrm{supp}(E_{p^l,j_l})} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x} &= \sum_{i=1}^{p} \sum_{s=1}^{q_0} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x_i^s} \\ &= \sum_{s=1}^{q_0} p \cdot \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x_1^s} = 0\,, \end{aligned} \tag{6.10}$$

as desired.

Case 2.2: $n \in \mathrm{supp}(\sigma)$. Then we may suppose that $x_p^q = n$. If $(l, j_l) \neq (t, m_t)$, then we may further suppose that there is some $1 \leqslant q_1 < q$ such that $\mathrm{supp}(E_{p^l,j_l}) = \{x_i^s : 1 \leqslant i \leqslant p,\, 1 \leqslant s \leqslant q_1\}$. By Lemma 6.4.4(b), we then get

$$\begin{aligned} \sum_{x \in \mathrm{supp}(E_{p^l,j_l})} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x} &= \sum_{i=1}^{p} \sum_{s=1}^{q_1} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x_i^s} \\ &= \sum_{s=1}^{q_1} p \cdot \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x_1^s} = 0\,. \end{aligned} \tag{6.11}$$

If $(l, j_l) = (t, m_t)$ then we may suppose that there is $1 \leqslant q_2 \leqslant q$ such that $\mathrm{supp}(E_{p^t,m_t}) = \{x_i^s : 1 \leqslant i \leqslant p,\, q_2 \leqslant s \leqslant q\}$. In this case, Lemma 6.4.4 gives

$$\begin{aligned} \sum_{\substack{x \in \mathrm{supp}(E_{p^t,m_t}) \\ x<n}} \sum_{k=2}^{p} (-1)^{k+1} \cdot \lambda_{\hat{k},x} &= \sum_{i=1}^{p} \sum_{s=q_2}^{q-1} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x_i^s} + \sum_{i=1}^{p-1} \sum_{k=2}^{p} (-1)^{k+1} \lambda_{\hat{k},x_i^q} \\ &= \sum_{s=q_2}^{q-1} \sum_{k=2}^{p} (-1)^{k+1} \cdot p \cdot \lambda_{\hat{k},x_1^s} - \sum_{i=1}^{p-1} \sum_{k=2}^{p} (-1)^{k} \lambda_{\hat{k},x_i^q} = 0\,. \end{aligned}$$

To summarize, we have now verified equation (6.9), which together with (6.7) shows that the coefficient at e in $\mathrm{Tr}_P^E(u)$ is 0. This now completes the proof in case 2 and, thus, of the lemma. □

As a direct consequence of Lemma 6.4.5, Proposition 1.2.15, and Proposition 1.2.13 we thus have proved the following

Proposition 6.4.6 *Let $n \in \mathbb{N}$ be such that $n = p + \sum_{i=2}^{t} m_i p^i$, for some $t \geqslant 2$, $m_2, \ldots, m_t \in \mathbb{N}_0$ with $m_t \neq 0$. Let further $D := D^{(n-p+1,1^{p-1})}$, and let $Q \leqslant S_n$ be a vertex of D. Then $D(E(1, m_2, \ldots, m_t)) \neq \{0\}$; in particular, $E(1, m_2, \ldots, m_t) \leqslant_{S_n} Q$.*

Remark 6.4.7 *Again consider the p-adic expansion $n = p + \sum_{i=2}^{r} n_i p^i$, where $r \geqslant 2$ and $n_r \neq 0$. Note that Proposition 6.4.6, in particular, holds for $t = r$ and $m_1 = 1, \ldots, m_r = n_r$. Thus the elementary abelian subgroup $E(1, n_2, \ldots, n_r) \leqslant P_n$ is S_n-conjugate to a subgroup of every vertex of D. This settles item (ii) at the beginning of this section. This also completes the proof of Theorem 6.1.1.*

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