Maximal Subgroups of Branch Groups

Submitted by

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Declaration

I, Theofanis Alexoudas, hereby declare that this thesis and the work presented in it is entirely my own. Where I have consulted the work of others, this is always clearly stated.

Signed .... (Theofanis Alexoudas)

Date: September 17, 2013

To my parents

Abstract

This thesis is concerned with the study of maximal subgroups of torsion multi-edge spinal groups. We prove that every torsion multi-edge spinal group has maximal subgroups only of finite index. This implies that every such group does not contain dense proper subgroups with respect to the profinite topology.

Moreover, for a torsion multi-edge spinal group, we show that all its maximal subgroups are normal of finite index p, where p is the odd prime such that the group acts on the p-adic regular rooted tree.

Multi-edge spinal groups are modelled after the GGS-groups, named after R. Grigorchuk, N. Gupta and S. Sidki. Every group in the class of GGS-groups is generated by a rooted automorphism and a recursively defined directed automorphism.

In contrast to the GGS-groups, every multi-edge spinal group is generated by a rooted automorphism and an arbitrary finite number of directed automorphisms.

The thesis follows the work of E. Pervova, who showed that every torsion GGS-group has maximal subgroups only of finite index. A key ingredient in the description of maximal subgroups of GGS-groups is the existence of a map Θ_1 under which the length of elements in the derived group decreases.

We introduce a length function on elements of multi-edge spinal groups, and also a second map Θ_2 to use the recursive structure of the groups more effectively. For a multi-edge spinal group G, we prove that the length of every element of its derived group of length greater or equal to 3, decreases under repeated applications of a combination of the maps Θ_1 and Θ_2. By introducing this second map Θ_2 we also simplify the methods developed by E. Pervova.

Acknowledgement

First and foremost, I would like to thank my supervisor, Benjamin Klopsch. I would like to thank him for being a constant source of inspiration, for guiding me throughout my doctoral studies with such endurance, for helping me constantly in my research and for being next to me through all the difficulties I faced these years. Above all, I would like to thank him for being such a great teacher.

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Last but not least, I would like to thank from the bottom of my heart my parents. I simply owe everything to them.

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Chapter 1

Introduction

1.1 Historical remark

Branch groups were explicitly defined by R. Grigorchuk at the conference Groups St Andrews 1997 in Bath. Even though they were defined as a class of groups only recently, examples of what are now called branch groups appeared in the literature significantly earlier, starting with the article of J. Wilson [34] on just infinite groups. Important examples within the class of branch groups were produced by R. Grigorchuk [10] and by N. Gupta and S. Sidki [13] in the early 1980's. The group constructed in [10] is a three generated infinite torsion 2-group that was initially defined as a group permuting the halves of the unit interval and is now known as the first Grigorchuk group. It is the first example of a finitely generated infinite group of intermediate growth, answering J. Milnor's question [20] on the existence of such groups. The first Grigorchuk group was later realised as a subgroup of the full automorphism group of the binary tree. A second class of examples was constructed by N. Gupta and S. Sidki and became known as the Gupta-Sidki groups; see [13]. In contrast to the Grigorchuk group, each Gupta-Sidki group, one group for every odd prime p, is a two generated infinite torsion group acting on the p-adic regular rooted tree for every odd prime p.

The class of branch groups is related to the General Burnside Problem on torsion groups. Even though the first example of a finitely generated infinite torsion group was constructed by E. Golod [9], the class of branch groups provides many examples of finitely generated residually finite infinite torsion groups which are considerably easier to describe; see [10] and [13].

Branch groups play an important role in the theory of just infinite groups (i.e. infinite groups all of whose proper quotients are finite). In [34], J. Wilson developed a structure theory for abstract just infinite groups. In that paper J. Wilson showed that the class of just infinite groups splits into two subclasses, the groups with finite and the groups with infinite structure lattice. However, J. Wilson's dichotomy on just infinite groups [34] was reformulated by R. Grigorchuk [11] in the form of a trichotomy according to which every finitely generated just infinite group is either a branch group or can easily be constructed from a simple group or from a hereditarily just infinite group (i.e. an infinite group all of whose subgroups of finite index are just infinite).

There are two approaches to the definition of branch groups; see [3]. The first one is purely algebraic, defining branch groups as groups whose lattice of subnormal subgroups is similar to the structure of a spherically homogeneous rooted tree. The second approach is based on a geometric point of view according to which a branch group is a group acting transitively on a spherically homogeneous rooted tree T and admits a structure of subnormal subgroups similar to the corresponding structure in the full automorphism group $\mathrm{Aut}(T)$ of the tree T.

Various generalisations of the two families of groups led to the first distinction within the class of branch groups; namely the Grigorchuk type of groups and GGS-groups (named after R. Grigorchuk, N. Gupta and S. Sidki). In [1], S. Alešin found a family of finitely generated infinite p-groups, arising as groups of automatic transformations. In [19], Y. Merzlyakov showed that the groups introduced in [1] are very closely related to the Grigorchuk and the Gupta-Sidki groups. In the Russian literature the groups introduced in [1] have become known as Alešin type of groups (or AT-groups). In [4], L. Bartholdi and Z. Šunić proposed a generalisation of the class of branch groups and initiated a systematic study of the new groups that they named spinal groups.

At the moment of writing this thesis, the theory of spinal groups is a very active area of research. Of course, it is beyond our scope to report on all recent developments in this area. Perhaps, the most important recent developments concerning spinal groups can be found in A. Zugadi-Reizabal's Ph.D thesis [36]; see also [8] and [32]. The main subject of study in [36] is the order of congruence quotients of groups in several families of spinal groups. As a consequence, the author determines the Hausdorff dimension of groups in several families of spinal groups. In particular, the family of GGS-groups.

It is also worth mentioning that in [12], R. Grigorchuk and J. Wilson established some interesting results concerning abstract commensurability of subgroups for the first Grigorchuk group. More precisely, the authors showed that if a group L is commensurable with the first Grigorchuk group, then all maximal subgroups of L have finite index in L. Their result applies to GGS-groups, and it would be very interesting to investigate this property for the class of spinal groups.

In this thesis we deal with multi-edge spinal groups, that is with spinal groups for which the corresponding spines have more than one vertex at every level; see Section 3.1. The term *multi-edge spinal group* has been proposed by A. Zugadi-Reizabal; see [36]. The groups that we study are modelled after the GGS-groups, but in our case each group is generated by a rooted automorphism and an arbitrary finite number of directed automorphisms.

1.2 Motivation

In [3], R. Grigorchuk, L. Bartholdi and Z. Šunić asked the following question.

Question 1. Is every maximal subgroup of a finitely generated branch group G necessarily of finite index in G?

In [23] and [26], E. Pervova showed that this is indeed the case for the Grigorchuk group and torsion GGS-groups respectively. On the other hand, I. Bondarenko [5] constructed a finitely generated branch group which has maximal subgroups of infinite index, answering in the negative Question 1.

The main aim of this thesis is to investigate the same question in the context of multi-edge spinal groups. In particular, we extend and simplify the results obtained in [26] to the class of torsion multi-edge spinal groups, and we show that these groups have maximal subgroups only of finite index.

As indicated in [26] by E. Pervova, one motivation for our investigation comes from a conjecture of Passman concerning the group algebra $K[G]$ of a finitely generated group G over a field K with $\operatorname{char} K = p$. The conjecture states that, if the Jacobson radical $\mathcal{J}(K[G])$ coincides with the augmentation ideal $\mathcal{A}(K[G])$ then G is a finite p-group; see [21, Conjecture 6.1]. In [21], Passman showed that if $\mathcal{J}(K[G]) = \mathcal{A}(K[G])$ then G is a p-group and every maximal subgroup of G is normal of index p. Hence multi-edge spinal groups that are torsion yield natural candidates for testing Passman's conjecture. It is important to widen this class of candidates, as even the Gupta-Sidki group for $p = 3$ does not satisfy $\mathcal{J}(K[G]) = \mathcal{A}(K[G])$; this follows from [31].

1.3 Structure of the thesis

In Chapter 2 we give a short introduction to branch groups and establish some notation and prerequisites for the rest of the thesis. For a detailed account on abstract branch groups the reader can consult the survey article [3], by L. Bartholdi, R. Grigorchuk and Z. Šunić. For an account on profinite branch groups see the article [11], by R. Grigorchuk. In this thesis we deal almost exclusively with abstract branch groups.

In Chapter 3 we introduce the class of multi-edge spinal groups acting on the p-adic regular rooted tree T for the odd prime p. We prove the following.

Lemma A. *Every multi-edge spinal group G is fractal.*

For a multi-edge spinal group G, this means that the restriction of every vertex stabiliser, denoted by $\mathrm{Stab}_G(u)$, to the subtree rooted at the vertex u coincides with the group G. The above property turns out to be a very useful tool in applying inductive arguments.

For G a multi-edge spinal group that is not $\mathrm{Aut}(T)$-conjugate to the GGS-group $\mathcal{G}$, arising from a constant defining vector we prove the following.

Proposition B. *Let G be a multi-edge spinal group that is not $\mathrm{Aut}(T)$-conjugate to the GGS-group $\mathcal{G}$. Then G is a branch group.*

Moreover, we show that

Proposition C. *Every torsion multi-edge spinal group G is just infinite.*

In addition, Chapter 3 contains several technical results that are later used in the rest of the thesis.

In Chapter 4 we restrict our attention to torsion multi-edge spinal groups for the rest of the thesis. We describe the abelianisation $G/[G,G]$ of a multi-edge spinal group G, and define a length function on elements of G. In addition, we introduce the theta maps $\Theta_1, \Theta_2 : [G,G] \to [G,G]$ and we prove that the length of every element of $[G,G]$ of length at least 3 decreases down to 0 or 2 under repeated applications of a combination of these maps; see Theorem 4.2.1. The maps Θ_1 and Θ_2 are defined in such a way to investigate maximal subgroups of torsion multi-edge spinal groups. More precisely, we are interested in descending to upper companion groups further down in the tree T in certain coordinates. In this context, the theta maps Θ_1 and Θ_2 enable us to use induction to prove our main results.

Chapter 5 constitutes another important part of this thesis. First we establish several intermediate propositions, some of which are direct generalisations of respective results in [26] to the context of torsion multi-edge spinal groups.

Then based on these propositions we establish our main result.

Theorem D. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a just infinite multi-edge spinal group. Suppose G is not $\mathrm{Aut}(T)$-conjugate to a group in $\boldsymbol{\mathcal{E}}$. Then G does not contain any proper dense torsion subgroups, with respect to the profinite topology.*

As a corollary we obtain the following.

Corollary E. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a torsion multi-edge spinal group. Suppose G is not $\mathrm{Aut}(T)$-conjugate to a group in $\boldsymbol{\mathcal{E}}$. Then G does not contain any maximal subgroups of infinite index.*

Finally, for G a just infinite multi-edge spinal group, we show that all its maximal subgroups are normal of index p, where p is the odd prime such that G acts on the p-adic regulat rooted tree.

Theorem F. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a just infinite multi-edge spinal group. Suppose G is not $\mathrm{Aut}(T)$-conjugate to a group in $\boldsymbol{\mathcal{E}}$. Then every maximal subgroup of G is normal of index p, where p is the odd prime such that G acts on the p-adic regulat rooted tree.*

In Appendix **A** we provide a hands on computation using the theta maps Θ_1 and Θ_2. We "test" Theorem 4.2.1 using a three generated torsion multi-edge spinal group $G = \langle a, b_1, b_2 \rangle$ acting on the p-adic regular rooted tree T for an odd prime p. More precisely, we show that every element of its derived group of length greater or equal to 3 decreases under repeated applications of a combination of the maps Θ_1 and Θ_2.

Finally, in Appendix **B** we provide the code of a MAGMA [6] program we wrote during the early stages of our research. This program allows to investigate the map Θ_1 introduced by E. Pervova in [26]. The code is adjusted for the case of the Gupta-Sidki group for $p = 3$. We also provide the output of a "small" computation using as an example the Gupta-Sidki group for $p = 3$. This example shows that one cannot expect to reduce the length of elements in the derived group of the Gupta-Sidki group in all cases, without using the map Θ_2 we introduce in Section 4.2.

Chapter 2

Preliminaries

In this chapter we give a short introduction to (abstract) branch groups and establish some notation and prerequisites for the rest of the thesis. All definitions in this chapter are drawn directly from existing published literature; see [3] and [11].

2.1 p-adic trees

A *tree* is a connected graph with no cycles. There is a type of tree which is of particular interest due to its rich group-theoretical properties that appear in its group of automorphisms. That is the *regular p-adic rooted tree*, for p a prime number; *rooted* because it has a distinguished root vertex and *regular* because every vertex has the same out-degree p. In this thesis all trees are regular p-adic with distinct root, for p an odd prime.

Let X be an alphabet on p letters, e.g. $X = \{1, 2, \ldots, p\}$, and denote by $\overline{X}$ the associated free monoid. A *positive word of length n* over X is any formal product of the form $w = x_1x_2 \cdots x_n$, where $n \in \mathbb{N}_0$ and $x_i \in X$ for all $i \in \{1, \ldots, n\}$. The unique word of length 0, the *empty word*, is denoted by $\emptyset$. The length n of the word w is denoted by $|w|$.

We introduce a partial order on the set of all words over X by the *prefix relation* $\leqslant$. Namely, $u \leqslant v$ if u is an initial segment of the sequence v, i.e. if $u = u_1 \ldots u_n$, $v = v_1 \ldots v_k$, where $n \leqslant k$ and $u_i = v_i$ for $i \in \{1, \ldots, n\}$. The partially ordered set of words over X, denoted by T^X, is called the *regular rooted tree* over X. In order to simplify the notation we denote T^X by T.

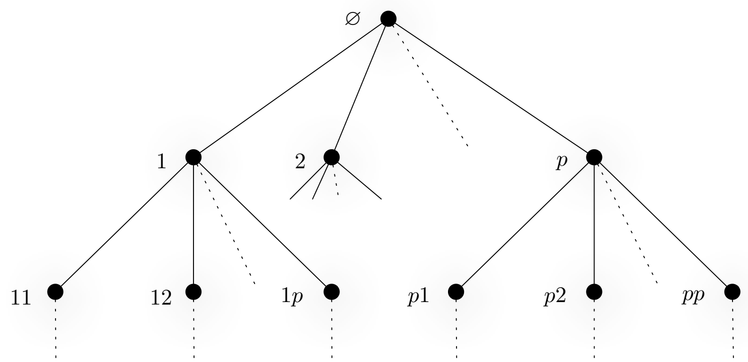

Figure 2.1: The p-adic regular rooted tree T over the alphabet $X = \{1, \ldots, p\}$

From the graph-theoretical point of view, every word over X represents a vertex in a rooted tree. The empty word $\emptyset$ represents the *root*; the p one-letter words $x_1, \ldots, x_p \in X$ represent the p children of the root etc. The ordered pairs (u, v), where u and v are vertices of the form $u = x_1 \ldots x_n$ and $v = x_1 \ldots x_{n+1}$, are precisely the (directed) edges of the tree. The distance of a vertex u from the root vertex is denoted by $|u|$ and is called the *length* of u. This is the number of edges in the unique shortest path from $\emptyset$ to u.

More generally, if u is a word over X, then the words ux for $x \in X$ of length $|u| + 1$, represent the p *children* of the vertex u (see Figure 2.1). The cardinality of the set of vertices which are both adjacent to the vertex u and of length $|u| + 1$ is called the *out-degree* of u.

A *rooted path* in the tree T is a sequence of adjacent vertices which starts at the root and is such that each vertex occurs at most once. If the tree T is infinite, the set of all infinite paths is called the *boundary* of T and is denoted by ∂T.

The graph structure of the tree T induces a distance function on the set of words by

$$d(u, v) = |u| + |v| - 2|u \wedge v|,$$

where $u \wedge v$ is the longest common prefix of u and v. In particular, the words of length n represent the vertices that are at distance n from the root. Such vertices constitute the *level* n of the tree.

The subtree of T containing only vertices from levels 0 to n is denoted by $T_{[n]}$. We write T_u for the full subtree of T rooted at the vertex u and consisting of all vertices v

with $u \leq v$. Since the tree T is regular, any two subtrees T_u and T_v, where u and v are words over X of the same length, are "canonically" isomorphic under the isomorphism that deletes the prefix u and replaces it by the prefix v. Based on this observation, we denote by $T_{|u|}$ any subtree rooted at a vertex in the same level as the vertex u. Thus we write T_n for the subtrees rooted at any vertex at level n. This is partly motivated by a more general setting described in Remark 2.1.1. In fact, it is clear that any subtree T_u, and hence any T_n, is canonically isomorphic to the tree T via the map $uv \mapsto v$.

Remark 2.1.1. There is a more general construction to that of a regular tree; see [11]. The tree T is constructed over a sequence of alphabets $X_1, X_2, X_3, \ldots$ with $|X_i| = m_i \geq 2$, where $\overline{m} = \{m_n\}_{n=1}^{\infty}$ is a sequence of natural numbers. Such a tree is called *spherically homogeneous* because vertices of the same length have the same out-degree. All notions introduced above carry over to this more general situation. The main difference is that the trees T_n are in general no longer isomorphic to T, but depend on n.

2.2 Tree automorphisms

As in the previous section, let T be a p-adic regular rooted tree on an alphabet X.

Definition 2.2.1. A *tree automorphism* of T is a permutation of the words over X that preserves the prefix relation.

In the language of graph theory an automorphism of T is just a graph automorphism that fixes the root. We denote the group of automorphisms of T by $\mathrm{Aut}(T)$. The orbits of the action of $\mathrm{Aut}(T)$ on the vertices of the tree T are precisely its levels. For any given vertex, a tree automorphism can be regarded as a *labelling* of the vertices by elements of the symmetric group which acts on the edges (or vertices) below that vertex.

Consider an automorphism f of T and a word u over X. We denote the image of u under the automorphism f by u^f. For a letter x in X we have $(ux)^f = u^f x'$ where x' is a uniquely determined letter in X. Clearly the induced map $x \mapsto x'$ is a permutation of X. We denote this permutation by $f(u)$ and we call it the *vertex permutation* of f at u. Denoting the image of x under $f(u)$ by $x^{f(u)}$, we get

$$(ux)^f = u^f x^{f(u)}, \tag{2.1}$$

which easily extends to

$$(x_1x_2\dots x_n)^f = x_1^{f(\emptyset)}x_2^{f(x_1)}\dots x_n^{f(x_1x_2\dots x_{n-1})}. \tag{2.2}$$

By using (2.1), it is clear that

$$fg(u) = f(u)\circ g(u^f) \quad \text{and} \quad f^{-1}(u) = (f(u^{f^{-1}}))^{-1}, \tag{2.3}$$

for all words u over X and $f, g \in \mathrm{Aut}(T)$.

Definition 2.2.2. Let f be an automorphism of T and u a word over X. The *section* of f at u is the unique automorphism f_u of $T_{|u|} \cong T$ defined by

$$v^{f_u} = w \quad \text{if} \quad (uv)^f = u^f w$$

for every word v over X.

Definition 2.2.3. Let G be a subgroup of $\mathrm{Aut}(T)$. The *set of sections* at the vertex u, denoted by $\mathrm{Sec}_u(G) = \{g_u \mid g \in G\}$, is called the *section* of G at u.

We now introduce some special classes of automorphisms which will be used in various constructions.

Definition 2.2.4. An automorphism $f \in \mathrm{Aut}(T)$ is called *rooted*, if $f(u) = 1$ for all $u \neq \emptyset$.

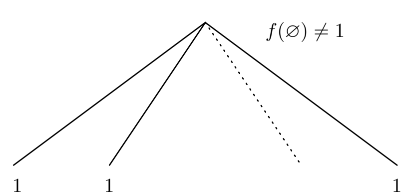

Figure 2.2: A rooted automorphism

Definition 2.2.5. Let f be an automorphism of T. The *labelling support*

$$\ell\mathrm{supp}(f) = \{u \mid f(u) \neq 1\}$$

of f, is the set of vertices at which f has non-trivial labelling.

Note that this notion differs from the usual concept of support. For instance, a rooted automorphism $f \neq 1$ has $\ell\mathrm{supp} = \{\emptyset\}$, while it does not move the root $\emptyset$ but other vertices of the tree T.

Definition 2.2.6. An automorphism $f \in \mathrm{Aut}(T)$ is called *finitary*, if the labelling support $\ell\mathrm{supp}(f) = \{u \mid f(u) \neq 1\}$ of its labelling is finite.

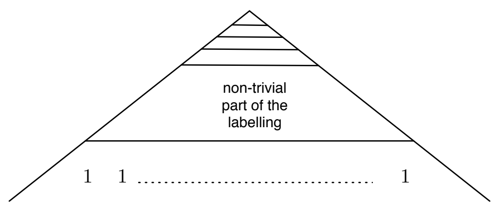

Figure 2.3: A finitary automorphism

Recall that if the tree T is infinite, the set of all infinite paths is called the boundary of T and is denoted by ∂T.

Definition 2.2.7. An automorphism $f \in \mathrm{Aut}(T)$ is called *directed*, with directing path $l \in \partial T$, if $f(u) = 1$ whenever u is not at distance 1 from l and f is not finitary.

Note that a directed automorphism is not necessarily non-trivial at every vertex of distance 1 from the directing path l.

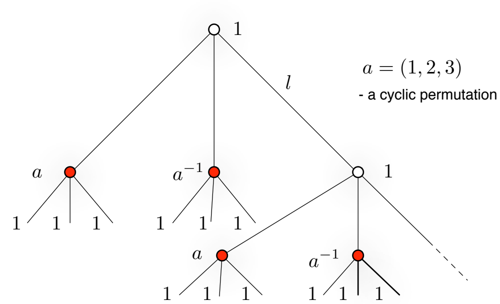

Figure 2.4: A directed automorphism "$f = (a, a^{-1}, f)$ " of the 3-adic regular rooted tree

2.3 Level and rigid stabilisers

Let T be a p-adic regular rooted tree and G a subgroup of $\mathrm{Aut}(T)$ acting transitively on every level of the tree. In this section we introduce some important subgroups of G which are going to be used in later chapters.

Definition 2.3.1. The subgroup $\mathrm{Stab}_G(u)$ consisting of all automorphisms in G that fix the vertex u, is called the *vertex stabiliser* of u in G.

Definition 2.3.2. The subgroup $\mathrm{Stab}_G(n) = \bigcap_{|v|=n} \mathrm{Stab}_G(v)$ consisting of all automorphisms in G that fix all vertices at level n, is called the n-th *level stabiliser* in G.

More generally, the subgroup $\mathrm{Stab}(n)$ of $\mathrm{Aut}(T)$ consisting of the automorphisms that fix the n-th level is called the *n-th level stabiliser*.

Note that the elements in $\mathrm{Stab}(n)$ fix all vertices of the finite tree $T_{[n]}$. Moreover, the subgroup $\mathrm{Stab}(n)$ is the kernel of the natural epimorphism

$$\pi_n\colon\ \mathrm{Aut}(T) \to \mathrm{Aut}(T_{[n]})$$

obtained by restriction. Hence $\mathrm{Stab}(n)$ is normal in $\mathrm{Aut}(T)$ and

$$\mathrm{Aut}(T)/\operatorname{Stab}(n) \cong \mathrm{Aut}(T_{[n]}).$$

Since $\mathrm{Aut}(T_{[n]})$ is a finite group, it is clear that $\mathrm{Stab}(n)$ is of finite index in $\mathrm{Aut}(T)$. In particular, since $\mathrm{Stab}_G(n) = \mathrm{Stab}(n) \cap G$ it follows that $\mathrm{Stab}_G(n)$ has finite index in G.

Recall from Section 2.1 that a positive word of length n over X is any formal product of the form $w = x_1x_2\cdots x_n$, where $x_i \in X$ for all $i \in \{1,\dots,n\}$. Note that any $g \in \mathrm{Stab}_G(n)$ can be identified in a natural way with the collection $(g_1,\dots,g_{N_n})$ of elements of $\mathrm{Aut}(T_n)$, where $N_n = p^n$ is the number of vertices at level n. Indeed, $g_i \in \mathrm{Aut}(T_n)$ corresponds to the restriction $g|_{T_u}$ of g to the subtree $T_u \cong T_n$ where the root is the i-th vertex u at level n.

Since T is a regular tree, $\mathrm{Aut}(T)$ is isomorphic to $\mathrm{Aut}(T_n)$ after the natural identification of subtrees. Therefore the decomposition $g = (g_1,\dots,g_{N_n})$ defines an embedding

$$\psi_n\colon\ \mathrm{Stab}_G(n) \to \mathrm{Aut}(T_{u_1}) \times \overset{N_n}{\cdots} \times \mathrm{Aut}(T_{u_{N_n}}) \cong \mathrm{Aut}(T) \times \overset{p^n}{\cdots} \times \mathrm{Aut}(T).$$

In the case where $n = 1$,

$$\psi_1\colon\ \mathrm{Stab}_G(1) \to \mathrm{Aut}(T_{u_1}) \times \overset{p}{\cdots} \times \mathrm{Aut}(T_{u_p}) \cong \mathrm{Aut}(T) \times \overset{p}{\cdots} \times \mathrm{Aut}(T).$$

We write U_u^G for the restriction of the vertex stabiliser $\mathrm{Stab}_G(u)$ to the subtree T_u rooted at the vertex u. Clearly U_u^G is a subgroup of $\mathrm{Aut}(T_u)$. Notice that since G acts transitively on all levels of the tree T, the vertex stabilisers at every level are conjugate in G.

We write $U_n^G \subseteq \mathrm{Aut}(T_n)$ for the common isomorphism type of the restrictions of the n-th level vertex stabilisers, and simply U_n when the group G is fixed. We call U_n^G the n-th *upper companion* group of G. Since T is a regular tree, U_n^G is a subgroup of $\mathrm{Aut}(T)$ under the canonical identification $T_n \cong T$, determined up to conjugation.

In general, the upper companion group U_n^G may not be a subgroup of G or may be a subgroup of infinite index. However, in certain classes of groups acting on regular rooted trees such as branch groups, the upper companion groups are isomorphic to finite index subgroups of G under the canonical identification of the original tree with its subtrees; see Section 2.4 for the definition of branch groups.

In analogy to principal congruence subgroups of arithmetic groups such as $\mathrm{SL}_d(\mathbb{Z})$, the level stabilisers can be considered as natural principal congruence subgroups for $\mathrm{Aut}(T)$; see [17, Ch. 6] for an overview on congruence subgroups in arithmetic groups. Moreover, if G is a subgroup of $\mathrm{Aut}(T)$ we refer to the quotient $G/\,\mathrm{Stab}_G(n)$ as the *n-th congruence quotient* of G.

Definition 2.3.3. The subgroup $G \leq \mathrm{Aut}(T)$ has the *congruence subgroup property* if every subgroup of finite index in G contains the group $\mathrm{Stab}_G(n)$ for some n.

Another important class of subgroups of groups acting on regular rooted trees are the *rigid vertex stabilisers* and the *rigid level stabilisers*.

Definition 2.3.4. The subgroup $\mathrm{Rstab}_G(u)$ consisting of all automorphisms in G that fix all vertices not having u as a prefix, is called the *rigid vertex stabiliser* of u in G.

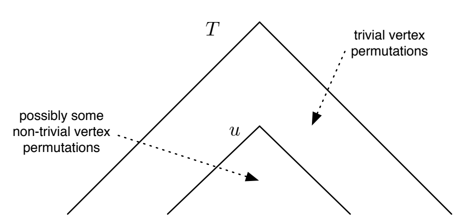

Figure 2.5: An automorphism in the rigid stabiliser of u

Definition 2.3.5. The group $\mathrm{Rstab}_G(n)$ is the product of the rigid vertex stabilisers

$$\mathrm{Rstab}_G(u_1), \dots, \mathrm{Rstab}_G(u_{N_n})$$

of the vertices $u_1, \dots, u_{N_n}$ at the n-th level, and is called the *rigid n-th level stabiliser* in G.

We write L_n^G for the common isomorphism type of the n-th level rigid vertex stabilisers and simply L_n when the group G is fixed. We call L_n^G the n-th *lower companion* group of G. Notice that the lower companion group L_u^G at a vertex u is a subgroup of the corresponding upper companion group. Because the group G acts transitively on every level of the tree, we have the inclusion

$$L_u^G \subseteq U_u^G \subseteq \mathrm{Sec}_u(G).$$

Clearly automorphisms in different rigid vertex stabilisers on the same level commute. Since G acts transitively on all levels, it follows that all vertex stabilisers $\mathrm{Stab}_G(u)$ corresponding to vertices on the same level are conjugate in G. Furthermore, since the lower companion group L_u^G at a vertex u is a subgroup of the upper companion group U_u^G, it follows that all rigid vertex stabilisers $\mathrm{Rstab}_G(u)$ corresponding to vertices on the same level are also conjugate in G. Thus taking the direct product of the rigid vertex stabilisers of vertices at the same level, we see that the rigid level stabilisers $\mathrm{Rstab}_G(n)$ are normal in G. In contrast to the level stabilisers, the rigid level stabilisers may have infinite index, and may even be trivial.

2.4 Branch groups

Among the families of groups acting on rooted trees, there is a class of groups of particular importance to group theorists. That is the class of branch groups.

We state the definition of branch groups slightly more generally for spherically homogeneous rooted trees.

Recall from Remark 2.1.1 in Section 2.1, that a spherically homogeneous rooted tree T is constructed over a sequence of alphabets $X_1, X_2, X_3, \dots$ with $|X_i| = m_i \geq 2$, where $\overline{m} = \{m_n\}_{n=1}^{\infty}$ is a sequence of natural numbers. Our main interest is in the case $X = X_1 = \dots$ and $p = m_1 = \dots$ an odd prime, leading to a regular p-adic rooted tree.

There are two approaches to the definition of branch groups; see [3]. The first one is purely algebraic, defining branch groups as groups whose lattice of subnormal subgroups is similar to the structure of a spherically homogeneous rooted tree.

Definition 2.4.1. A group G is said to be a *branch group*, if it has a descending sequence of normal subgroups H_n of finite index in G with trivial intersection $\bigcap_{n=1}^{\infty} H_n = 1$, such that:

(1) H_n admits a factorisation $H_n = L_n^{(1)} \times \cdots \times L_n^{(r)}$ as a product of finitely many copies $L_n^{(1)}, \ldots, L_n^{(r)}$ of a group L_n, where $r = r(n)$.

(2) The factorisation of H_n subdivides the factorisation of H_{n-1} for each $n \in \{2, 3, \ldots\}$.

(3) The group G acts transitively by conjugation on each set of factors as in (1).

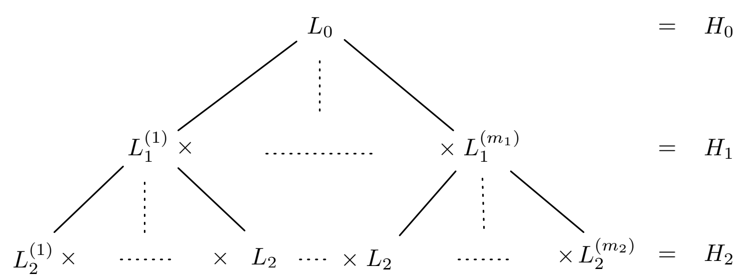

Figure 2.6: Branch structure of a branch group

The second approach is based on a geometric point of view according to which a branch group is a group acting transitively on a spherically homogeneous rooted tree T, such that it admits a structure of subnormal subgroups similar to the corresponding structure in the full automorphism group $\mathrm{Aut}(T)$ of the tree T. In this thesis we use the geometric definition of branch groups.

Definition 2.4.2. A group G is said to be a *branch group*, if there is a spherically homogeneous rooted tree $T = T_{\overline{m}}$, with branching sequence $\overline{m} = (m_1, m_2, \ldots)$, and an embedding $G \hookrightarrow \mathrm{Aut}(T)$ such that:

(1) The group G acts transitively on each level of the tree.

(2) For each level n there exists a subgroup L_n of the automorphism group $\mathrm{Aut}(T_n)$ of the full subtree T_n rooted at a level n vertex, such that the direct product

$$H_n = L_n^{(1)} \times \cdots \times L_n^{(N_n)} \leq \mathrm{Stab}_{\mathrm{Aut}(T)}(n), \quad \text{where} \quad L_n^{(i)} \cong L_n,$$

of $N_n = m_1 m_2 \cdots m_n$ copies of L_n is normal and of finite index in G.

Recall from Section 2.3 that the lower companion groups L_n^G denote the common isomorphism type of the rigid vertex stabilisers $\mathrm{Rstab}_G(u)$, where $u \in T$ runs through all vertices at level n. Thus condition (2) of Definition 2.4.2 means that all rigid level stabilisers $\mathrm{Rstab}_G(n)$ are of finite index in G.

Therefore the question whether a subgroup G of $\mathrm{Aut}(T)$ is branch, with respect to the given realisation as a group of tree autmorphisms, reduces to checking two properties. Firstly, that the group in question acts transitively on all levels of the tree. Secondly, that every rigid level stabiliser $\mathrm{Rstab}_G(n)$ is of finite index in G.

Recall that every lower companion group L_n^G is a subgroup of the corresponding upper companion group U_n^G. Thus we have the (geometrical) embedding

$$L_n^G \times \cdots \times L_n^G \cong \mathrm{Rstab}_G(n) \leq \mathrm{Stab}_G(n) \hookrightarrow U_n^G \times \cdots \times U_n^G,$$

where each product contains N_n factors, corresponding to the number of vertices at the n-th level of the tree T.

In addition, when the tree T is regular we have the following definition.

Definition 2.4.3. A subgroup G of $\mathrm{Aut}(T)$ is said to be *fractal*, if every upper companion group U_u^G coincides with G, where u runs through all vertices of the tree T.

2.5 Profinite groups

Profinite groups are objects of interest in a variety of mathematical areas. In abstract group theory, they provide the means for focussing attention on properties of finite homomorphic images; see [30]. For the number theorist, profinite groups are the groups which arise as Galois groups of algebraic field extensions; see [35, Ch. 3]. And for the analyst, they are the quotient groups of compact Hausdorff topological groups modulo the connected component of the identity; see [15] and [14].

There are many characterisations of profinite groups; see [16, Ch. I]. But for our purposes, we introduce the main notions that are most relevant to the current thesis.

Perhaps the most relevant characterisation of profinite groups in the context of branch groups, is that of profinite groups as inverse limits and profinite completions.

Definition 2.5.1. A directed set is a partially ordered set $I = (I, \preceq)$ such that for all $i, j \in I$ there exists $k \in I$ such that $k \succeq i$ and $k \succeq j$.

Definition 2.5.2. An inverse system $(G_i; \varphi_{ij})$ of groups (or other mathematical structures such as sets, rings, etc. . .) over I consists of a family of groups (or sets, . . .) $G_i, i \in I$, and homomorphisms $\varphi_{ij} : G_i \to G_j$ whenever $i \succeq j$, satisfying the natural compatibility conditions

$$\varphi_{ii} = \mathrm{id}_{G_i} \quad \text{and} \quad \varphi_{ij}\varphi_{jk} = \varphi_{ik} \quad \text{for all } i, j, k \in I \text{ with } \; i \succeq j \succeq k.$$

Definition 2.5.3. The inverse limit of the inverse system $(G_i; \varphi_{ij})$ is the group (or set,. . .)

$$\varprojlim G_i := \left\{ (g_i)_{i \in I} \in \prod\nolimits_{i \in I} \; G_i \mid g_i \varphi_{ij} = g_j \;\; \text{whenever} \;\; i \succeq j \right\}$$

together with the natural coordinate maps $\varphi_i : G \to G_i$ for $i \in I$.

$$\begin{tikzcd}
& G = \varprojlim G_i \arrow[d, dashed, "\varphi_i"] \arrow[rrd, dashed, "\varphi_3"] \arrow[rrrd, dashed, "\varphi_2"] \arrow[rrrrd, dashed, "\varphi_1"] & & & & \\
\cdots \arrow[r, "\varphi_{i+1,i}"'] & G_i \arrow[r, "\varphi_{i,i-1}"'] & \cdots \arrow[r, "\varphi_{43}"'] & G_3 \arrow[r, "\varphi_{32}"'] & G_2 \arrow[r, "\varphi_{21}"'] & G_1
\end{tikzcd}$$

Figure 2.7: The inverse limit G of an inverse system $(G_i; \varphi_{ij})$, $I = \mathbb{N}$

If the G_i are finite groups, we give each of them the discrete topology , and $\prod_{i \in I} G_i$ the product topology. Then $\varprojlim G_i$ with the induced topology becomes a totally disconnected, compact, Hausdorff topological group.

Another way to look at profinite groups is as profinite completions. Let Γ be any group. We write $H \leq_f \Gamma$ to indicate that H is a subgroup of finite index in Γ. Note that the finite quotients of Γ form a natural inverse system Γ/N, $N \lhd_f \Gamma$, with φ_{MN} given by the natural projection $\Gamma/M \to \Gamma/N$ whenever $M \subseteq N$. The inverse limit of this system is the *profinite completion* $\hat{\Gamma} := \varprojlim \Gamma/N$ of Γ. There is a natural map from the original group into its profinite completion, namely

$$\vartheta : \Gamma \to \hat{\Gamma}, \quad g \mapsto (gN)_{N \lhd_f \Gamma}.$$

If Γ is residually finite, that is the intersection of all its finite index subgroups is trivial, then ϑ is injective. Typically, $\Gamma\vartheta$ is strictly contained in $\hat{\Gamma}$ but it is always a dense subgroup, that is the closure of $\Gamma\vartheta$ is equal to $\hat{\Gamma}$.

Profinite groups are also topological groups.

Definition 2.5.4. A *profinite group* is a compact Hausdorff topological group whose open subgroups form a base for the neighbourhoods of the identity.

Thus a profinite group is a compact Hausdorff topological group G, such that every open neighbourhood of the identity contains an open subgroup. This means that the open subsets of a profinite group G are precisely those sets which can be written as unions of cosets gN of open normal subgroups $N \trianglelefteq_o G$.

Let G be a profinite group and $X \subseteq G$. Then X is said to *generate* G (topologically) if X generates a dense subgroup of G. Accordingly, G is *finitely generated* (as topological group) if it admits a finite (topological) generating set. We denote by $d(G)$ the minimal cardinality of a (topological) generating set for G. In order to check whether a given subset X generates a profinite group G, it suffices to show that X generates G modulo every open normal subgroup $N \trianglelefteq_o G$. Thus one has $d(G) = \sup\{d(G/N) \mid N \trianglelefteq_o G\}$.

Definition 2.5.5. An infinite group G is called *just infinite* if all its non-trivial normal subgroups have finite index; if G is profinite it means that all non-trivial closed normal subgroups have finite index. The group is called *hereditarily just infinite* if each of its subgroups of finite index is just infinite; if G is profinite it means that every open subgroup is just infinite.

The *Frattini subgroup* $\Phi(G)$ of a profinite group G is the intersection of all maximal proper open subgroups of G. Since every open subgroup is closed, it follows that $\Phi(G)$ is a closed subgroup of G. Furthermore, one can show that $X \subseteq G$ generates G if and only if X generates G modulo $\Phi(G)$.

A *pro-p group* is a topological group which is isomorphic to the inverse limit of finite p-groups. Every group Γ admits a *pro-p completion* $\hat{\Gamma}_p$, which is the pro-p group arising from the inverse system of finite quotients Γ/N where N runs through all normal subgroups of p-power index in Γ.

Let G be a pro-p group. Then every closed subgroup of G is a pro-p group and any quotient of G by a closed normal subgroup is a pro-p group. In particular, the index of any open subgroup of G is a power of p. The Frattini subgroup of G is equal to the subgroup $G^p[G,G]$, i.e. $\Phi(G) = G^p[G,G]$, where G^p is the closed subgroup generated by all p-powers of G and $[G,G]$ is the closed commutator subgroup of G; see [7, Proposition 1.13]. In particular, one has $d(G) = \dim_{\mathbb{F}_p} G/\Phi(G)$.

2.6 Branch groups as profinite groups

Let T be a p-adic regular rooted tree on the alphabet $X = \{1, \ldots, p\}$. Recall that $T_{[n]}$ is the subtree of T ending at level n. The fact that $\mathrm{Aut}(T)$ is a profinite group becomes clear if one observes that the sequence of groups $\mathrm{Aut}(T_{[n]})$ forms an inverse system. The homomorphisms

$$\mathrm{Aut}(T_{[n+1]}) \to \mathrm{Aut}(T_{[n]})$$

are defined by restricting the action of $\mathrm{Aut}(T_{[n+1]})$ to $T_{[n]}$, and the group $\mathrm{Aut}(T)$ is the inverse limit of the system

$$\mathrm{Aut}(T) = \varprojlim_{n\to\infty} \mathrm{Aut}(T_{[n]}).$$

Note that the topology of this group is defined by the open subgroups $\{\mathrm{Stab}_{\mathrm{Aut}(T)}(n)\}_{n=1}^{\infty}$.

A Sylow pro-p subgroup of $\mathrm{Aut}(T)$ is obtained as follows. Fix a cyclic permutation $a = (1\,2\,\cdots\,p)$ of the alphabet X, and regard a as an element in the symmetric group $\mathrm{Sym}(p)$.

Define $\mathrm{Aut}_*(T) \leq \mathrm{Aut}(T)$ to be the set of all elements whose labelling $\{g(u)\}$ takes values in $\langle a\rangle \cong C_p$, a cyclic group of order p, where u runs through all vertices of the tree. Thus for every vertex $u \in T$ we have $g(u) = a^{i(u)}$ for some $i(u)$ with $0 \leq i(u) \leq p-1$. The group $\mathrm{Aut}_*(T)$ is isomorphic to the *infinitely iterated wreath product*

$$\varprojlim_{n\to\infty} (C_p \wr C_p \wr \ldots \wr C_p)_{n \text{ factors}},$$

and is a Sylow pro-p subgroup of $\mathrm{Aut}(T)$. In the current thesis all groups will be constructed inside this Sylow pro-p subgroup of $\mathrm{Aut}(T)$. For the finite tree $T_{[n]}$, ending at level n, $\mathrm{Aut}(T_{[n]})$ is isomorphic to the finite iterated wreath product

$$(\mathrm{Sym}(p) \wr \mathrm{Sym}(p) \wr \ldots \wr \mathrm{Sym}(p))_{n \text{ factors}}.$$

Chapter 3

Multi-Edge Spinal Groups

In this chapter we introduce the class of multi-edge spinal groups. In Section 3.1 we give the general construction of the groups and establish some definitions for the rest of the thesis. In Section 3.2 we prove some general properties of such groups.

Again, we work with the p-adic regular rooted tree for an odd prime p. We choose the spine to be the rightmost infinite path starting at the root vertex of the tree.

3.1 Construction of multi-edge spinal groups

Let T be the p-adic regular rooted tree over the alphabet $X = \{1, \ldots, p\}$ for an odd prime p.

Definition 3.1.1. Let $L = (l_n)_{n\geq 0}$ be an infinite path in T starting at the root. If we consider, for every $n \geq 1$, immediate descendants $s_{n,k}$, for $k \in \{1, \ldots, p\}$, of l_{n-1} not lying in L, we say that the doubly indexed sequence $S = (s_{n,k})_{n\geq 1,k}$ is a *multi-edge spine* of T.

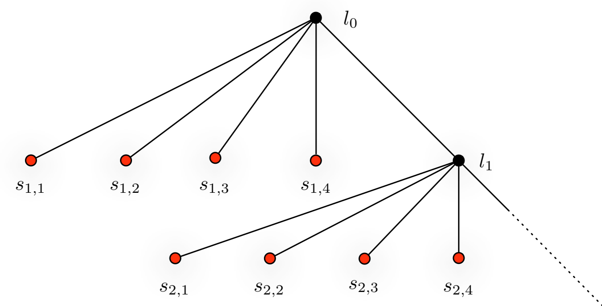

Figure 3.1: A multi-edge spine $(s_{n,k})_{n\geq 1,k}$ in the 5-adic rooted tree, associated to the right-most path $(l_n)_{n\geq 0}$, where $l_n = (x_1 \cdots x_n)$

We denote by a the rooted automorphism corresponding to $(1\,2\,\cdots\,p) \in \mathrm{Sym}(p)$ which cyclically permutes the vertices of the first level of the tree T. Clearly a is of order p. Recall from Section 2.3 the map

$$\psi_n\colon\ \mathrm{Stab}_G(n) \to \mathrm{Aut}(T_{u_1}) \times \overset{N_n}{\cdots} \times \mathrm{Aut}(T_{u_{N_n}}) \cong \mathrm{Aut}(T) \times \overset{p^n}{\cdots} \times \mathrm{Aut}(T),$$

where $N_n = p^n$. In particular, for $n = 1$

$$\psi_1\colon\ \mathrm{Stab}_G(1) \to \mathrm{Aut}(T_{u_1}) \times \overset{p}{\cdots} \times \mathrm{Aut}(T_{u_p}) \cong \mathrm{Aut}(T) \times \overset{p}{\cdots} \times \mathrm{Aut}(T).$$

Given a finite set $\mathbf{E}$ of $(\mathbb{Z}/p\mathbb{Z})$-linearly independent vectors

$$\begin{aligned}
\mathbf{e}_1 &= (e_{1,1}, e_{1,2}, \ldots, e_{1,p-1})\\
\mathbf{e}_2 &= (e_{2,1}, e_{2,2}, \ldots, e_{2,p-1})\\
\mathbf{e}_3 &= (e_{3,1}, e_{3,2}, \ldots, e_{3,p-1})\\
&\ \ \vdots\\
\mathbf{e}_r &= (e_{r,1}, e_{r,2}, \ldots, e_{r,p-1})
\end{aligned}$$

where $r \in \{1, 2, \ldots, p-1\}$ and each $\mathbf{e}_i = (e_{i,1}, \ldots, e_{i,p-1}) \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$ for $i \in \{1, \ldots, r\}$, we define recursively a finite set $B = \{b_1, \ldots, b_r\} \subseteq \mathrm{Stab}(1)$ of directed automorphisms via

$$\begin{aligned}
\psi_1(b_1) &= (a^{e_{1,1}}, a^{e_{1,2}}, \ldots, a^{e_{1,p-1}}, b_1)\\
\psi_1(b_2) &= (a^{e_{2,1}}, a^{e_{2,2}}, \ldots, a^{e_{2,p-1}}, b_2)\\
\psi_1(b_3) &= (a^{e_{3,1}}, a^{e_{3,2}}, \ldots, a^{e_{3,p-1}}, b_3)\\
&\vdots\\
\psi_1(b_r) &= (a^{e_{r,1}}, a^{e_{r,2}}, \ldots, a^{e_{r,p-1}}, b_r).
\end{aligned}$$

Definition 3.1.2. We say that the subgroup $G_{\mathbf{E}} = \langle \{a\} \cup B \rangle$ of $\mathrm{Aut}(T)$ is the *multi-edge spinal group* corresponding to the set of defining vectors $\mathbf{E}$.

Remark 3.1.3. We observe that $\mathcal{B} = \langle b_1, \ldots, b_r \rangle$ is an elementary abelian p-group of rank r. Indeed, the map

$$\langle e_1, \ldots, e_r \rangle \longrightarrow \langle b_1, \ldots, b_r \rangle$$

given by

$$\sum_{i=1}^{r} j_i e_i \longmapsto \prod_{i=1}^{r} b_i^{j_i}$$

is an isomorphism of groups, and $\langle e_1, \ldots, e_r \rangle \cong C_p^r$.

Remark 3.1.4. By choosing the defining vectors $\mathbf{e}_i \in \mathbf{E}$ linearly independent, we avoid that one of the directed automorphisms $b_1, \ldots, b_r$ of a multi-edge spinal group is redundant. To see this, assume the contrary. That is there exist constants $c_1, \ldots, c_r \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$, not all zero, such that

$$c_1\mathbf{e}_1 + \cdots + c_r\mathbf{e}_r \equiv \mathbf{0} \pmod p.$$

Without loss of generality assume that $c_r \equiv -1 \pmod p$. Then $b_r = b_1^{c_1} \cdots b_{r-1}^{c_{r-1}} \in \langle b_1, \ldots, b_{r-1} \rangle$. Therefore we can find a set $\tilde{\mathbf{E}} \subseteq \mathbf{E}$ of linearly independent defining vectors $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_{r-1}$, giving rise to directed automorphisms $\tilde{b}_1, \ldots, \tilde{b}_{r-1}$ such that the set $\{a\} \cup \{\tilde{b}_1, \ldots, \tilde{b}_{r-1}\}$ generates the same multi-edge spinal group. Observe that, the assertion above is an immediate consequence of part (3) of Lemma 4.1.9.

Definition 3.1.5. A defining vector $\mathbf{e}_i \in \mathbf{E}$ is said to be *symmetric*, if $e_{i,j} = e_{i,p-j}$ for all $j \in \{1, \ldots, \frac{p-1}{2}\}$. Otherwise, $\mathbf{e}_i$ is said to be *non-symmetric*.

Definition 3.1.6. A multi-edge spinal group $G_{\mathbf{E}}$ is said to be *symmetric* with respect to $\mathbf{E}$ if every defining vector $\mathbf{e}_i \in \mathbf{E}$ giving rise to a generating directed automorphism is symmetric. Otherwise, $G_{\mathbf{E}}$ is said to be *non-symmetric* with respect to $\mathbf{E}$.

Remark 3.1.7. By choosing only one defining vector

$$\mathbf{e} = (e_1, \ldots, e_{p-1}) \in (\mathbb{Z}/p\mathbb{Z})^{p-1},$$

for p an odd prime, and defining an automorphism b of $\mathrm{Aut}(T)$ via

$$\psi_1(b) = (a^{e_1}, \ldots, a^{e_{p-1}}, b),$$

we obtain the GGS-group $G_{\mathbf{e}} = \langle a, b \rangle$ corresponding to the defining vector $\mathbf{e}$. For instance, the Gupta-Sidki group arises by choosing p an odd prime and $\mathbf{e} = (1, -1, 0, \ldots, 0)$. Thus the GGS-groups form a subclass of all multi-edge spinal groups. As a reference for the GGS-groups, the reader can consult Section 2.3 of the monograph [3] by L. Bartholdi, R. Grigorchuk and Z. Šuniḱ, or the papers [33] by T. Vovkivsky and [22, 27] by E. Pervova.

The next theorem, adapted to the context of multi-edge spinal groups, gives the condition for a group in the class of multi-edge spinal groups to be an infinite p-group.

Theorem 3.1.8 (Grigorchuk [11], Vovkivsky [33])**.** *Let $G = \langle \{a\} \cup B \rangle$ be a multi-edge spinal group corresponding to a set of defining vectors $\mathbf{E}$. Then G is an infinite p-group if and only if for every $\mathbf{e}_i \in \mathbf{E}$*

$$\sum_{j=1}^{p-1} e_{i,j} \equiv 0 \pmod{p}.$$

We do not know a proof that the GGS-group $\mathcal{G}$ in (3.7) is not branch. From properties that were established in [36] we derive the following result.

3.2 General properties of multi-edge spinal groups

We continue to work with the p-adic regular rooted tree T for an odd prime p. We fix a multi-edge spinal group $G = \langle a, b_1, \ldots, b_r \rangle$ as defined in Section 3.1.

Recall from Definition 2.4.3 that a subgroup G of $\mathrm{Aut}(T)$ is fractal, if every upper companion group U_u^G coincides with G. This means that for every vertex $u \in T$, the restriction of its stabiliser to the subtree rooted at u, denoted by $\mathrm{Stab}_G(u)_u$, coincides with G.

Lemma 3.2.1. *Every multi-edge spinal group G is fractal.*

Proof. Clearly $\mathrm{Stab}_G(u)_u \subseteq G$ for every vertex $u \in T$. To show that $G \subseteq \mathrm{Stab}_G(u)_u$, we induct on the length n of the vertex u. Firstly, assume that $n = 0$. That is, u is the root vertex of the tree T and hence the result is trivial.

Let u be a vertex of length $n > 0$. Writing u as $u = vk$, where $k \in \{1, \dots, p\}$ and v a vertex of length $n-1$, we can conclude by the induction hypothesis that $\mathrm{Stab}_G(v)_v = G$. Therefore there exist elements $\tilde{a}, \tilde{b}_i \in \mathrm{Stab}_G(v)_v$ for all $i \in \{1, \dots, r\}$, acting as a and b_i on the subtree T_v rooted at the vertex v.

Fix $i \in \{1, \dots, r\}$. If $k = p$, then the directed automorphism $\tilde{b}_i \in \mathrm{Stab}_G(v)_v$ is acting as b_i on the subtree T_u. Otherwise, conjugating $\tilde{b}_i$ by a suitable power $\tilde{a}^j$ of $\tilde{a}$, we can get b_i at the k-th coordinate of the decomposition vector of $\tilde{b}_i^{\tilde{a}^j}$. Furthermore, choosing suitable $j \in \{1, \dots, p\}$ we can get a power of the rooted automorphism at the k-th coordinate in the decomposition vector of $\tilde{b}_i^{\tilde{a}^j}$. In addition, we can choose a suitable $l \in \{1, \dots, p-1\}$ such that $(\tilde{b}_i^{\tilde{a}^j})^l$ produces the rooted automorphism a at the k-th coordinate in the decomposition vector of $(\tilde{b}_i^{\tilde{a}^j})^l$. Thus $(\tilde{b}_i^{\tilde{a}^j})^l \in \mathrm{Stab}_G(u)_u$ is acting as the rooted automorphism a on the subtree T_u. Therefore $G = \langle a, b_1, \dots, b_r \rangle \subseteq \mathrm{Stab}_G(u)_u$. □

Proposition 3.2.2. *Every multi-edge spinal group G acts transitively on every level of the tree T.*

Proof. Let u, $v \in T$ be two vertices at level n. We induct on the length $|u| = |v| = n$. For $n = 1$, it is clear that $u^{a^k} = v$ for some $k \in \{1, \dots, p\}$, where a is the rooted automorphism permuting the p vertices of the first level.

Next suppose that $n > 1$ and the result is true for all vertices up to level $n-1$. There are two cases. Suppose that the vertices u and v begin with same letter $j \in \{1, \dots, p\}$. That is, $u = ju'$ and $v = jv'$. Then by the induction hypothesis there exists an element $f \in G$ such that $f(u') = v'$. By Lemma 3.2.1 $\mathrm{Stab}_G(w)_w = G$ for every $w \in T$. Therefore there exists an element $g \in \mathrm{Stab}_G(1)$ having the decomposition

$$\psi_1(g) = (*, \dots, *, f, *, \dots, *)$$

with the automorphism f located at the j-th coordinate in the decomposition vector of the automorphism g. Then $u^g = v$.

Finally, suppose that u and v begin with different letters. Acting by a suitable power of the rooted automorphism a on u, we can arrange that u^{a^k}, for $k \in \{1, \dots, p-1\}$, and v begin with the same letter. Thus we have reduced the argument to the previous case. □

In the next lemma we show that our group remains essentially unchanged when we reorder the p labellings of the branches in the p-adic regular rooted tree T.

Lemma 3.2.3. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a multi-edge spinal group. Then there exists an automorphism $f \in \mathrm{Aut}(T)$ of the form $f = f_0(f, \ldots, f) = (f, \ldots, f)f_0$, where f_0 is a rooted automorphism corresponding to a permutation $\pi \in \mathrm{Sym}(p)$, such that $G \cong G^f = \langle \tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_r \rangle$. The group G^f is a multi-edge spinal group generated by the rooted automorphism $\tilde{a} = a$ and directed automorphisms $\tilde{b}_1, \ldots, \tilde{b}_r$ of the form $\tilde{b}_i = (\tilde{a}^{\tilde{e}_{i,1}}, \ldots, \tilde{a}^{\tilde{e}_{i,p-1}}, \tilde{b}_i)$ for all $i \in \{1, \ldots, r\}$, and such that $\tilde{e}_{1,1} = 1$.*

Proof. Since the defining vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ have been chosen to be linearly independent over $\mathbb{Z}/p\mathbb{Z}$, each defining vector $\mathbf{e}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ satisfies

$$\mathbf{e}_i = (e_{i,1}, \ldots, e_{i,p-1}) \not\equiv \mathbf{0} \pmod{p}.$$

In particular, $\mathbf{e}_1 \not\equiv \mathbf{0} \pmod{p}$. Without loss of generality, assume that $e_{1,k} = k$ for some $k \in \{1, \ldots, p-1\}$ (otherwise we can replace b_1 by a power of b_1). There exists some $l \in \{1, \ldots, p-1\}$ such that $kl \equiv 1 \pmod{p}$.

Let $X = \{1, \ldots, p\}$ be the alphabet on p-letters over $\mathbb{Z}/p\mathbb{Z}$. We consider a permutation $\pi \in \mathrm{Sym}(p)$ defined by $x\pi = lx$, where $x \in \{1, \ldots, p\}$ is a vertex in the first level of the tree T. Observe that $x\pi^{-1} = kx$ for all $x \in \{1, \ldots, p\}$. We proceed to construct an automorphism $f \in \mathrm{Aut}(T)$ of the form $f = f_0(f, \ldots, f) = (f, \ldots, f)f_0$, where f_0 is a rooted automorphism corresponding to the permutation $\pi \in \mathrm{Sym}(p)$.

Set $\tilde{a} = (a^k)^f = (a^k)^{f_0}$, a rooted automorphism. Then

$$\begin{aligned} x\tilde{a} &= x f_0^{-1} a^k f_0 \\ &= (kx)a^k f_0 \\ &= (kx + k) f_0 \\ &= (kx + k)l \\ &= x + 1 \\ &= xa \end{aligned}$$

for every $x \in X$. Hence $\tilde{a} = a$. It follows that $a = (a^k)^f = (a^f)^k$ which implies $a^l = a^f$.

Now let $\tilde{b}_i = (b_i)^f$ for $i \in \{1, \dots, r\}$. Then

$$\begin{aligned}
\tilde{b}_i &= (b_i)^f \\
&= f^{-1} b_i f \\
&= (f^{-1}, \dots, f^{-1}) f_0^{-1} b_i f_0 (f, \dots, f) \\
&= (f^{-1}, \dots, f^{-1})(a^{e_{i,k}}, \dots, \underbrace{a^{e_{i,1}}}_{l^{\text{th}}}, \underbrace{a^{e_{i,k+1}}}_{(l+1)^{\text{st}}}, \dots, a^{e_{i,p-k}}, b_i)(f, \dots, f) \\
&= (a^{le_{i,k}}, \dots, a^{le_{i,p-k}}, b_i^f) \\
&= (\tilde{a}^{le_{i,k}}, \dots, \tilde{a}^{le_{i,p-k}}, \tilde{b}_i)
\end{aligned}$$

where the intermediate values represented by the dots denote unspecified powers of the rooted automorphism a. In particular, choosing $i = 1$ we conclude that $\psi_1(\tilde{b}_1) = (a^{lk}, \dots, \tilde{b}_1) = (a, \dots, \tilde{b}_1)$ which implies that $\tilde{e}_{1,1} = 1$. $\square$

Lemma 3.2.4. *Let $G_{\mathbf{E}} = \langle a, b_1, \dots, b_r \rangle$ be a multi-edge spinal group associated to an r-tuple $\mathbf{E}$ with $r \geq 2$. Then there exists an r-tuple of defining vectors $\tilde{\mathbf{E}}$ such that $G_{\tilde{\mathbf{E}}}$ is conjugate to $G_{\mathbf{E}}$ by an element $f \in \mathrm{Aut}(T)$ as in Lemma 3.2.3 and the following hold:*

(1) $\tilde{e}_{i,1} \equiv 1 \pmod p$ *for each* $i \in \{1, \dots, r\}$,

(2) *if* $r = 2$ *and* $p = 3$, *then* $\tilde{\mathbf{e}}_1 = (1, 0)$, $\tilde{\mathbf{e}}_2 = (1, 1)$,

(3) *if* $r = 2$ *and* $p > 3$, *then either*

   (a) *for each* $i \in \{1, 2\}$ *there exists* $k \in \{2, \dots, p-2\}$ *such that* $\tilde{e}_{i,k-1}\tilde{e}_{i,k+1} \not\equiv \tilde{e}_{i,k}^2 \pmod p$, *or*

   (b) $\tilde{\mathbf{e}}_1 = (1, 0, \dots, 0, 0)$, $\tilde{\mathbf{e}}_2 = (1, 0, \dots, 0, 1)$,

(4) *if* $r \geq 3$ *then for each* $i \in \{1, \dots, r\}$ *there exists* $k \in \{2, \dots, p-2\}$ *such that* $\tilde{e}_{i,k-1}\tilde{e}_{i,k+1} \not\equiv \tilde{e}_{i,k}^2 \pmod p$.

Proof. We split the proof into two cases: $r \geq 3$ and $r = 2$.

Case 1: $r \geq 3$. Observe that $p \geq 5$ and consider the $r \times (p-1)$-matrix

$$M(\mathbf{E}) = \begin{pmatrix} e_{1,1} & \cdots & e_{1,p-1} \\ e_{2,1} & \cdots & e_{2,p-1} \\ \vdots & \ddots & \vdots \\ e_{r,1} & \cdots & e_{r,p-1} \end{pmatrix}$$

encoding the defining vectors for the group $G_{\mathbf{E}}$. By Lemma 3.2.3, we may assume that $e_{1,1} \not\equiv 0 \pmod p$. Using elementary row operations, we transform $M(\mathbf{E})$ into reduced row-echelon form:

$$\begin{pmatrix} 1 & a_1 & \cdots & a_m & 0 & * & \cdots & * & 0 & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * \end{pmatrix},$$

where $m \geq 0$, $a_1, \ldots, a_m \in \mathbb{Z}/p\mathbb{Z}$ and the symbols $*$ denote other, unspecified elements. Adding the 1st row to every other row, we obtain

$$M(\tilde{\mathbf{E}}) = \begin{pmatrix} 1 & a_1 & \cdots & a_m & 0 & * & \cdots & * & 0 & * & \cdots & * \\ 1 & a_1 & \cdots & a_m & 1 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & a_1 & \cdots & a_m & 0 & * & \cdots & * & 1 & * & \cdots & * \end{pmatrix} \tag{3.1}$$

The row operations that we carried out yield a new set of generators for $\langle b_1, \ldots, b_r \rangle$, corresponding to an r-tuple $\tilde{\mathbf{E}}$ of defining vectors that are encoded in the rows of $M(\tilde{\mathbf{E}})$.

Let $i \in \{1, \ldots, r\}$ and consider the ith row of $M(\tilde{\mathbf{E}})$. We identify two patterns which guarantee that the ith row satisfies the condition in (4):

$$\text{(A)} \qquad (* \ \ldots \ * \ x\, y\, 0 \ * \ \ldots \ *),$$

$$\text{(B)} \qquad (* \ \ldots \ * \ 0\, y\, x \ * \ \ldots \ *),$$

where $x, y \in \mathbb{Z}/p\mathbb{Z}$ with $y \not\equiv 0$ and the symbols $*$ again denote unspecified elements. Observe that, if the patterns (A) and (B) do not appear in the ith row, then the row does not have any zero entries at all or must be of the form $(*\ 0\ \ldots\ 0\ *)$.

Suppose first that $2 \leq i \leq r-1$. In this case the ith row contains at least one zero entry and cannot be of the form $(*\ 0\ \ldots\ 0\ *)$. Hence the pattern (A) or (B) occurs.

Next suppose that $i = r$ and assume that patterns (A) or (B) do not appear. As $r \geq 3$ the rth row contains at least one zero entry and consequently has the form $(1\ 0\ \ldots\ 0\ 1)$. Changing generators, we may replace the rth row by the rth row minus the 2nd row plus the 1st row, yielding

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & -1 & * & \ldots & * & 1 \end{pmatrix} \tag{3.2}$$

with m zeros between the entries 1 and -1. If $m > 0$ then pattern (B) occurs in this new row. Suppose that $m = 0$. Then the row takes the form

$$(\,1 \quad -1 \quad * \quad \dots \quad * \quad 1\,). \tag{3.3}$$

For the condition in (4) to fail, we would need the row to be equal to

$$(1 \; -1 \; 1 \; -1 \, \dots \, 1 \; -1)$$

with the final entry being -1 as $p-1$ is even. This contradicts (3.3).

Finally, suppose that $i = 1$. Similarly as above, we assume that patterns (A) and (B) do not occur. Since it contains at least one zero entry, the 1st row is of the form

$$(\,1 \quad 0 \quad \dots \quad 0 \quad *\,) \tag{3.4}$$

and we change generators as follows. Generically, we replace the 1st row by the 1st row plus the 2nd row minus the 3rd row. Only if $r = 3$ and we already changed the rth row as described above, we replace the 1st row by 2 times the 1st row minus the 3rd row. In any case, this gives a new 1st row:

$$(\,1 \quad 0 \quad \dots \quad 0 \quad 1 \quad * \quad \dots \quad * \quad -1 \quad * \quad \dots \quad *\,)$$

with m zeros between the entries 1 and 1. If $m > 0$ then pattern (B) occurs. Suppose that $m = 0$ so that the new row takes the form

$$(\,1 \quad 1 \quad * \quad \dots \quad * \quad -1 \quad * \quad \dots \quad *\,). \tag{3.5}$$

For the condition in (4) to fail, the row would have to be of the form $(1 \; 1 \; \dots \; 1)$ contradicting (3.5).

Case 2: $r = 2$. The statement in (2) for $p = 3$ can clearly be achieved by a simple change of generators. Now we suppose that $p > 3$. By Lemma 3.2.3, we may assume that $e_{1,1} \not\equiv 0 \pmod{p}$. Using elementary row operations, we transform the $2 \times (p-1)$-matrix $M(\mathbf{E})$ encoding the defining vectors into reduced row-echelon form:

$$\begin{pmatrix} 1 & \mathbf{a} & 0 & \mathbf{b} \\ 0 & \mathbf{0} & 1 & \mathbf{c} \end{pmatrix},$$

where at most one of $\begin{pmatrix} \mathbf{a} \\ \mathbf{0} \end{pmatrix}$ or $\begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix}$ could be the empty matrix. Further row operations, corresponding to multiplication on the left by $\begin{pmatrix} 1 & y \\ 1 & z \end{pmatrix}$, where $y, z \in \mathbb{Z}/p\mathbb{Z}$ with $y \not\equiv z$ are to be specified below, yield

$$M(\tilde{\mathbf{E}}) = \begin{pmatrix} 1 & \mathbf{a} & y & \mathbf{b} + y\mathbf{c} \\ 1 & \mathbf{a} & z & \mathbf{b} + z\mathbf{c} \end{pmatrix} \tag{3.6}$$

encoding an r-tuple $\tilde{\mathbf{E}}$ of defining vectors for a new set of generators.

First suppose that $\mathbf{a} = \mathbf{0}$ is not empty and zero. If $\mathbf{b} = ()$ then

$$M(\tilde{\mathbf{E}}) = \begin{pmatrix} 1 & 0 & \dots & 0 & y \\ 1 & 0 & \dots & 0 & z \end{pmatrix}$$

leads to (3)(b). Otherwise, if $\mathbf{b} \neq ()$, we choose $y \equiv 1$ and $z \equiv -1 \pmod p$, yielding pattern (B) in both rows so that the condition in (3)(a) holds.

Next suppose that $\mathbf{a} = (a_1 \dots a_m) \neq \mathbf{0}$ is not empty and non-zero. Suppose further that the truncated rows $(1\ \mathbf{a}\ y)$, $(1\ \mathbf{a}\ z)$ do not yet satisfy the condition in (3)(a). Then pattern (B) does not occur in these and $\mathbf{a}$ cannot have any zero entries. Consequently, there exists $\lambda \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ such that $M(\tilde{\mathbf{E}})$ is of the form

$$M(\tilde{\mathbf{E}}) = \begin{pmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^m & y & * & \dots & * \\ 1 & \lambda & \lambda^2 & \dots & \lambda^m & z & * & \dots & * \end{pmatrix}.$$

As $p > 3$, we can choose $y, z \in \mathbb{Z}/p\mathbb{Z}$ with $y \not\equiv z$ and $y, z \not\equiv \lambda^{m+1} \pmod p$ so that the condition in (3)(a) is satisfied.

Finally suppose that $\mathbf{a} = ()$. Then

$$M(\tilde{\mathbf{E}}) = \begin{pmatrix} 1 & y & b_1 + yc_1 & * & \dots & * \\ 1 & z & b_1 + zc_1 & * & \dots & * \end{pmatrix}$$

for suitable $b_1, c_1 \in \mathbb{Z}/p\mathbb{Z}$. We can choose $y, z \in \mathbb{Z}/p\mathbb{Z}$ with $y \not\equiv z$ such that

$$y^2 \not\equiv b_1 + yc_1 \quad \text{and} \quad z^2 \not\equiv b_1 + zc_1 \pmod p,$$

because quadratic equations have at most two solutions and $p > 3$. Once more, the condition in (3)(a) is fulfilled. □

Definition 3.2.5. The *lower central series* of a group Γ is the series of characteristic subgroups of Γ

$$\Gamma = \gamma_1(\Gamma) \geqslant \gamma_2(\Gamma) \geqslant \gamma_3(\Gamma) \geqslant \dots,$$

where $\gamma_i(\Gamma) = [\gamma_{i-1}(\Gamma), \Gamma]$ for all $i > 1$. The sections $\gamma_i(\Gamma)/\gamma_{i+1}(\Gamma)$ are the *lower central factors* of Γ.

The next result mimicks [36, Lemma 3.3.1], which applies to GGS-groups. We remark that there are no new exceptions, in addition to the GGS-group

$$\mathcal{G} = \langle a, b \rangle \quad \text{with} \quad \psi_1(b) = (a, a, \ldots, a, b), \tag{3.7}$$

arising from a constant defining vector $(1, \ldots, 1)$.

Lemma 3.2.6. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a multi-edge spinal group that is not conjugate to $\mathcal{G}$ in $\mathrm{Aut}(T)$. Then*

$$\psi_1(\gamma_3(\mathrm{Stab}_G(1))) = \gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G).$$

In particular,

$$\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) \subseteq \psi_1(\gamma_3(G)).$$

Proof. Since $\psi_1(\mathrm{Stab}_G(1))$ is contained in $G \times \overset{p}{\cdots} \times G$ by Lemma 3.2.1, it suffices to prove the inclusion $\supseteq$. When $r = 1$, the result follows by [36, Lemma 3.3.1], so we may assume that $r \geq 2$. Without loss of generality, we perform a suitable automorphism $f \in \mathrm{Aut}(T)$ of the form $f_0(f, \ldots, f)$ on G as in Lemma 3.2.3 (here $\gamma_3(G)$ remains unchanged). That is

$$\psi_1(\gamma_3(\mathrm{Stab}_G(1)))^{f_0(f,\ldots,f)} = \psi_1(\gamma_3(\mathrm{Stab}_G(1)^f) = \psi_1(\mathrm{Stab}_{G^f}(1)).$$

Thus

$$(\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)) = \gamma_3(G^f) \times \overset{p}{\cdots} \times \gamma_3(G^f).$$

So we may assume that $\psi_1(b_1) = (a^{e_{1,1}}, \ldots, a^{e_{1,p-1}}, b_1)$ with $e_{1,1} \not\equiv 0 \pmod p$. For $(r, p) \neq (2, 3)$, we further apply Lemma 3.2.4, and so assume here that our set $\mathbf{E}$ satisfies the conditions (1) and (2) of Lemma 3.2.4.

By Proposition 3.2.2, G is acting transitively on all levels of the tree T, and hence it suffices to show

$$\gamma_3(G) \times 1 \times \cdots \times 1 \subseteq \psi_1(\gamma_3(\mathrm{Stab}_G(1))).$$

We divide the proof into two cases.

Case 1: Suppose $(r, p) \neq (2, 3)$.

If $e_{i,p-1} \equiv 0 \pmod p$ for some $i \in \{1, \ldots, r\}$, then

$$\psi_1(b_i) = (a, a^{e_{i,2}}, \ldots, a^{e_{i,p-2}}, 1, b_i),$$

(the intermediate values represented by the dots denote powers of the rooted automorphism) and consequently

$$\psi_1([b_i, b_i^a, b_i]) = ([a, b_i, a], 1, \ldots, 1),$$

$$\psi_1([b_i, b_i^a, b_i^a]) = ([a, b_i, b_i], 1, \ldots, 1),$$

and for $j \in \{1, \ldots, r\}$ with $j \neq i$,

$$\psi_1([b_i, b_i^a, b_j^a]) = ([a, b_i, b_j], 1, \ldots, 1),$$

$$\psi_1([b_i, b_j^a, b_i^a]) = ([a, b_j, b_i], 1, \ldots, 1),$$

where the intermediate values represented by the dots are all 1 in this case.

Suppose $e_{i,p-1} \not\equiv 0 \pmod p$ for some $i \in \{1, \ldots, r\}$. By Lemma 3.2.4, there exists $k \in \{2, \ldots, p-2\}$ such that $(e_{i,k-1}, e_{i,k})$ and $(e_{i,k}, e_{i,k+1})$ are not proportional, i.e. $e_{i,k-1} \cdot e_{i,k+1} \neq e_{i,k}^2$.

Let us set

$$g_{i,k} = (b_i^{a^{p-k+1}})^{e_{i,k}} \, (b_i^{a^{p-k}})^{-e_{i,k-1}}$$

for $k \in \{2, \ldots, p-2\}$, so that

$$\psi_1(g_{i,k}) = (a^{e_{i,k}^2 - e_{i,k-1}e_{i,k+1}}, \ldots, 1),$$

where the intermediate values represented by the dots are not necessarily 1 in this case.

In the exceptional case, as in part (3) of Lemma 3.2.4, in which $\tilde{e}_1 = (1, 0, \ldots, 0)$ and $\tilde{e}_2 = (1, 0, \ldots, 0, 1)$, giving $\tilde{b}_1 = (a, 1, \ldots, 1, \tilde{b}_1)$ and $\tilde{b}_2 = (a, 1, \ldots, a, \tilde{b}_2)$, we can replace $g_{2,k}$ by $\tilde{b}_2^{a^2} = (a, \tilde{b}_2, a, 1, \ldots, 1)$.

Since $(e_{i,k-1}, e_{i,k})$ and $(e_{i,k}, e_{i,k+1})$ are not proportional, we have

$$e_{i,k}^2 - e_{i,k-1}e_{i,k+1} \not\equiv 0 \pmod p.$$

Hence there is a power g_i of $g_{i,k}$ such that

$$\psi_1(g_i) = (a, \ldots, 1),$$

where the intermediate values represented by the dots are not necessarily 1 in this case. Additionally, since

$$\psi_1(b_i^a(b_i^{a^{p-1}})^{-e_{i,p-1}}) = (b_i a^{-e_{i,2}e_{i,p-1}}, \dots, 1),$$

with the help of g_i we can get an element $h_i \in \mathrm{Stab}_G(1)$ such that

$$\psi_1(h_i) = (b_i, \dots, 1),$$

where the intermediate values represented by the dots are not necessarily 1 in this case. Consequently,

$$\psi_1([b_i, b_i^a, g_i]) = ([a, b_i, a], 1, \dots, 1),$$

$$\psi_1([b_i, b_i^a, h_i]) = ([a, b_i, b_i], 1, \dots, 1),$$

and for $j \in \{1, \dots, r\}$ with $j \neq i$,

$$\psi_1([b_i, b_j^a, h_i]) = ([a, b_j, b_i], 1, \dots, 1),$$

$$\psi_1([b_i, b_i^a, h_j]) = ([a, b_i, b_j], 1, \dots, 1),$$

or

$$\psi_1([b_i, b_i^a, b_j^a]) = ([a, b_i, b_j], 1, \dots, 1),$$

if $e_{j,p-1} \equiv 0 \pmod p$, where the intermediate values represented by the dots are all 1 in this case.

If $e_{j,p-1} \not\equiv 0 \pmod p$ as in the previous case. For u a level one vertex, we note that $\mathrm{Stab}_G(u)_u = \mathrm{Stab}_G(1)_u$. Since

$$\gamma_3(G) = \langle [a, b_i, a], [a, b_i, b_j] \mid i, j \in \{1, \dots, r\} \rangle^G,$$

using Lemma 3.2.1, we have

$$\gamma_3(G) \times 1 \times \cdots \times 1 \subseteq \psi_1(\gamma_3(\mathrm{Stab}_G(1))).$$

Case 2: Suppose $(r, p) = (2, 3)$.

Without loss of generality, by Lemma 3.2.4, we may assume that $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (1, -1)$. Note that $\psi_1(b_2) = (a, a^{-1}, b_2)$.

Then

$$\psi_1(b_2 b_2^a) = (ab_2, 1, b_2 a^{-1})$$

and

$$\psi_1(b_2^a b_2^{a^2}) = (b_2 a^{-1}, ab_2, 1).$$

Hence

$$\psi_1([b_2, b_2^a, b_2^a b_2^{a^2}]) = ([a, b_2, b_2 a^{-1}], 1, 1)$$

and

$$\psi_1([b_2^{-a^2}, b_2^a, b_2 b_2^a]) = ([a, b_2, ab_2], 1, 1).$$

Now, since $G' = \langle [a, b_1], [a, b_2] \rangle^G$ and $\langle ab_2, b_2 a^{-1} \rangle = \langle b_2^2, b_2 a^{-1} \rangle = \langle a, b_2 \rangle$, we have

$$\gamma_3(G) = \langle [a, b_1, a], [a, b_1, b_1], [a, b_1, b_2], [a, b_2, b_1], [a, b_2, ab_2], [a, b_2, b_2 a^{-1}] \rangle^G.$$

Hence

$$\gamma_3(G) \times 1 \times \cdots \times 1 \subseteq \psi_1(\gamma_3(\mathrm{Stab}_G(1))).$$

□

Recall from Section 2.3 that the rigid vertex stabiliser, denoted by $\mathrm{Rstab}_G(u)$, of a vertex $u \in T$, is the subgroup consisting of all automorphisms in G that fix all vertices not having u as a prefix. Again, for a vertex $u \in T$, we write $\mathrm{Rstab}_G(u)_u$ for the restriction of this rigid vertex stabiliser to the subtree T_u rooted at the vertex u.

Proposition 3.2.7. *Let G be a multi-edge spinal group that is not* $\mathrm{Aut}(T)$*-conjugate to the GGS-group $\mathcal{G}$ in* (3.7)*. Then $\gamma_3(G) \subseteq \mathrm{Rstab}_G(u)_u$ for every vertex u of T, after the natural identification of subtrees.*

Proof. Let $u \in T$ be a vertex of length n. We induct on n. Firstly, assume that $n = 0$. That is $u = \emptyset$ is the root vertex of the tree T and hence $\mathrm{Rstab}_G(\emptyset)_\emptyset = G$. Therefore $\gamma_3(G) \subseteq \mathrm{Rstab}_G(\emptyset)_\emptyset$.

Now suppose that $n > 0$. Writing u as $u = vk$, where $k \in \{1, \ldots, p\}$ and v a vertex of length $n - 1$, we can conclude by the induction hypothesis that $\gamma_3(G) \subseteq \mathrm{Rstab}_G(v)_v$. By Lemma 3.2.6

$$\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) \subseteq \psi_1(\mathrm{Rstab}_G(v)_v).$$

In particular,

$$1 \times \cdots \times 1 \times \gamma_3(G) \times 1 \times \cdots \times 1 \subseteq \psi_1(\mathrm{Rstab}_G(v)_v)$$

where $\gamma_3(G)$ is located at position u in the subtree T_v rooted at the vertex v. Hence $\gamma_3(G) \subseteq \mathrm{Rstab}_G(u)_u$. $\square$

Proposition 3.2.8. *Let G be a multi-edge spinal group that is not* $\mathrm{Aut}(T)$*-conjugate to the GGS-group $\mathcal{G}$ in* (3.7)*. Then G is a branch group.*

Proof. We show that every rigid level stabiliser $\mathrm{Rstab}_G(n)$ is of finite index in G. Since G is a finitely generated group, every quotient of G is also finitely generated. The abelianisation $G/\gamma_2(G)$ is a finitely generated abelian group, and since each generator of G is of finite order, $\gamma_2(G)$ is of finite index in G. Using the surjective bi-additive map $G/\gamma_2(G) \times G/\gamma_2(G) \to \gamma_2(G)/\gamma_3(G)$, given by $(x\gamma_2(G), y\gamma_2(G)) \mapsto [x, y]\gamma_3(G)$, we further have that $\gamma_3(G)$ is of finite index in G. By Proposition 3.2.7, the image of $\mathrm{Rstab}_G(n)$ under the maps ψ_n contains the direct product of p^n copies of $\gamma_3(G)$. Since the image of any level stabiliser $\mathrm{Stab}_G(n)$ under the map ψ_n is contained in the direct product of p^n copies of G, we have $\psi_n(\mathrm{Rstab}_G(n))$ is of finite index in $\psi_n(\mathrm{Stab}_G(n))$. Therefore $\mathrm{Rstab}_G(n)$ is of finite index in $\mathrm{Stab}_G(n)$ and hence in G. $\square$

Recall from Definition 2.4.1 that a group K is branch with respect to a branch structure $(\{L_n\}, \{H_n\})$. Recall also that by L_n^K we denote the common isomorphism type of the rigid level stabilisers $\mathrm{Rstab}_K(n)$; see Section 2.3. When the group K is fixed we simply write L_n. The next theorem gives a criterion for a branch group to be just infinite.

Theorem 3.2.9 (see [11], Theorem 4)**.** *A branch group K is just infinite if and only if for each $n \geq 1$, the index of the commutator subgroup $[L_n, L_n]$ in L_n is finite. Moreover, if this condition holds, then every non-trivial normal subgroup N of K contains the subgroup $[H_n, H_n] = [L_n, L_n] \times \cdots \times [L_n, L_n]$ for some $n = n(N)$.*

For the next result, we require our group to be torsion.

Proposition 3.2.10. *Let $G = \langle a, b_1, \ldots, b_r\rangle$ be a torsion multi-edge spinal group. Then G is just infinite.*

Proof. By Proposition 3.2.8, G is a branch group. Hence $L_n^G = L_n$ is of finite index in G. Since G is a finitely generated torsion group, it follows that L_n is also finitely generated and torsion. Therefore its abelianisation $L_n/[L_n, L_n]$ is a finitely generated abelian torsion group. Therefore, by Theorem 3.2.9, G is just infinite. $\square$

We do not know a proof that the GGS-group $\mathcal{G}$ in (3.7) is not branch. From properties that were established in [36] we derive the following result.

Proposition 3.2.11 (see [2], Proposition 3.10)**.** *The GGS-group $\mathcal{G}$ in* (3.7) *is not just infinite.*

Proof. Write $G = \mathcal{G} = \langle a, b\rangle$ with $\psi_1(b) = (a, \dots, a, b)$, and put $K = \langle ba^{-1}\rangle^G$. From [36, Section 3.4] we have that

(1) $|G : K| = p$ and $K' = \langle [(ba^{-1})^a, ba^{-1}]\rangle^G \leq \mathrm{Stab}_G(1)$;

(2) $|G/K' \,\mathrm{Stab}_G(n)| = p^{n+1}$ for every $n \in \mathbb{N}$ with $n \geq 2$.

Hence K' is a non-trivial normal subgroup of infinite index in G. □

What about just infinite multi-edge spinal groups that are not torsion? For $p \geq 5$, it is shown in [11, Example 7.1] that the non-torsion group $G = \langle a, b\rangle$ with $\psi_1(b) = (a, 1, \dots, 1, b)$ is just infinite, and more generally in [11, Example 10.2] that $G = \langle a, b\rangle$ with $\psi_1(b) = (a^{e_1}, a^{e_2}, \dots, a^{e_{p-4}}, 1, 1, 1, b)$ where $e_1 \not\equiv 0$ is just infinite. For the latter example, when $\sum_{i=1}^{p-4} e_i \not\equiv 0 \pmod p$, then the group is non-torsion.

Let G be the multi-edge spinal group with defining vectors $\mathbf{e}_i$ of the form

$$(e_{i,1}, e_{i,2}, \dots, e_{i,p-2}, e_{i,p-1})$$

satisfying $e_{i,1} \not\equiv 0 \pmod p$ and $e_{i,p-3} \equiv e_{i,p-2} \equiv e_{i,p-1} \equiv 0 \pmod p$ for every $i \in \{1, \dots, r\}$. In similar spirit, it can be shown that G is just infinite, and furthermore when $\sum_{j=1}^{p-4} e_{i,j} \not\equiv 0 \pmod p$ for at least one $i \in \{1, \dots, r\}$, then G is non-torsion. It is not always the case that the last three elements of the defining vectors are to be zero. For example, the non-torsion multi-edge spinal group G with $e_{i,1} \equiv e_{i,p-2} \equiv e_{i,p-1} \equiv 0 \pmod p$ and $e_{i,2} \not\equiv 0 \pmod p$ is likewise just infinite.

Chapter 4

Theta Maps

Let T be the p-adic regular rooted tree over the alphabet $X = \{1, \ldots, p\}$ for an odd prime p. We fix a multi-edge spinal group $G_{\mathbf{E}} = \langle a, b_1, \ldots, b_r \rangle$ with respect to $\mathbf{E}$ as defined in Section 3.1.

We assume that the group $G_{\mathbf{E}}$ is torsion. According to Theorem 3.1.8, we require for every defining vector $\mathbf{e}_i \in \mathbf{E}$, $i \in \{1, \ldots, r\}$, that

$$\sum_{j=1}^{p-1} e_{i,j} \equiv 0 \pmod{p}.$$

In Section 4.1 we describe the abelianisation $G/[G, G]$ of the group G. In addition, we define a natural length function on elements of the commutator subgroup $[G, G]$. In Section 4.2 we introduce the theta maps

$$\Theta_1, \Theta_2 : [G, G] \to [G, G]$$

and we prove that the length of every element of the commutator subgroup of length at least 3 decreases under repeated applications of a combination of these maps.

4.1 Abelianisation of multi-edge spinal groups

We recall some elementary facts about free products of groups; for a detailed account see [28, Ch. 6].

Definition 4.1.1. Let $\{\Gamma_\lambda \mid \lambda \in \Lambda\}$ be a non-empty family of groups over an index set Λ. By a *free product* of the groups Γ_λ, $\lambda \in \Lambda$, we mean a group Γ and a family of homomorphisms $\iota_\lambda : \Gamma_\lambda \to \Gamma$, $\lambda \in \Lambda$, with the following universal property. Given a family of homomorphisms $\varphi_\lambda : \Gamma_\lambda \to W$, $\lambda \in \Lambda$, into some group W, there is a *unique* homomorphism $\varphi : \Gamma \to W$ such that $\iota_\lambda \varphi = \varphi_\lambda$, for all $\lambda \in \Lambda$. That is the following diagram commutes.

$$\begin{tikzcd}
\Gamma_\lambda \arrow[r, "\iota_\lambda"] \arrow[d, "\varphi_\lambda"'] & \Gamma \arrow[dl, dashed, "\varphi"] \\
W &
\end{tikzcd}$$

Figure 4.1: Universal property of free products

Proposition 4.1.2 (see [28], Propositions 6.2.1 and 6.2.2)**.** *For every non-empty family of groups $\{\Gamma_\lambda \mid \lambda \in \Lambda\}$ there corresponds a free product. If Γ and $\overline{\Gamma}$ are free products of a family of groups $\{\Gamma_\lambda \mid \lambda \in \Lambda\}$, then Γ is isomorphic to $\overline{\Gamma}$.*

Definition 4.1.3. Let $\Gamma = \mathrm{Fr}_{\lambda\in\Lambda}\,\Gamma_\lambda$ be the free product of a non-empty family of groups $\{\Gamma_\lambda \mid \lambda \in \Lambda\}$. A word in $\bigcup_{\lambda\in\Lambda}\Gamma_\lambda$ is called *reduced* (or in *normal form*) if none of its symbols is an identity and no two consecutive symbols belong to the same group Γ_λ.

Theorem 4.1.4 (see [18], Theorem 1.2)**.** *Every element $g \in \mathrm{Fr}_{\lambda\in\Lambda}\,\Gamma_\lambda$ can be uniquely represented as a reduced word of the form*

$$g = g_1 g_2 \cdots g_r \quad \text{for } r \geq 0$$

where $1 \neq g_i \in \Gamma_{\lambda_i}$, $\lambda_i \neq \lambda_{i+1}$, and r the length *of g as an element of the free product.*

Before we turn our attention to multi-edge spinal groups, it is worth mentioning that in 1972 S. Alešin [1] found a family of finitely generated infinite p-groups, arising as groups of automatic transformations. In [19], Y. Merzlyakov showed that the groups introduced in [1] are very closely related to the Grigorchuk and the Gupta-Sidki groups. In the Russian literature the groups introduced in [1] have become known as Alešin type of groups (or AT-groups). A closer look in the literature shows that the class of AT-groups is essentially the class of special groups that was described in [11, §8] by R. Grigorchuk. It is therefore clear that the class of multi-edge spinal groups is a subclass of AT-groups. As a reference for AT-groups, the reader can consult the papers [24, 25] by E. Pervova, or [29] by A. Rozhkov.

The following proposition due to A. Rozhkov plays a crucial role in proving Proposition 4.1.10, in which we describe the abelianisation of multi-edge spinal groups.

Proposition 4.1.5 (see [29], Proposition 1)**.** *Any relation in an Alešin type group , is a relation of some finite rank.*

Remark 4.1.6. Before we proceed, let's try to understand Proposition 4.1.5 in the context of multi-edge spinal groups.

Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a multi-edge spinal group acting on the p-adic regular rooted treet T for an odd prime p. Denote by $\mathcal{A} = \langle a \rangle$ the subgroup of G generated by the rooted automorphism, and by $\mathcal{B} = \langle b_1, \ldots, b_r \rangle$ the subgroup of G generated by the directed automorphisms. Let $R_{\mathcal{A}}$ be the set of relators in the subgroup $\mathcal{A}$, and $R_{\mathcal{B}}$ the set of relators in the subgroup $\mathcal{B}$.

Relations of rank 0 in the group G are elements of the normal subgroup generated by $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$, i.e. $a^p = 1, b_i^p = 1$ for all $i \in \{1, \ldots, r\}$, and $[b_i, b_j] = 1$ for $i, j \in \{1, \ldots, r\}$ with $i \neq j$. If the relations of rank $m-1$ are already defined, then a relation of rank m is any relation in the group G such that any u-section of it, where u is a first level vertex, is a relation of rank $m-1$ in the group $\mathrm{Stab}_G(u)_u$.

Loosely speaking, what A. Rozhkov means by finite rank, is that given a relation in the group, its sections, descending in the tree, reduce/restrict to just elements from the normal subgroup generated by $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$.

So by Proposition 4.1.5, all relators of G lie in a subgroup $K = \cup_{n \geq 0} K_n$, where K_0 are the relators of rank 0, K_1 are the relators of rank 1 (so you have to go down to the sections of just one level below, to get to the stage where the relators reduce/restrict to just relators of $\mathcal{A}$ and relators of $\mathcal{B}$, in each such section), etc.

Let $G = G_{\mathbf{E}} = \langle a, b_1, \ldots, b_r \rangle$ be a multi-edge spinal group acting on the regular p-adic rooted tree T, for an odd prime p. Here $\mathbf{E}$ is the r-tuple of defining vectors $\mathbf{e}_i = (e_{i,1}, \ldots, e_{i,p-1})$, for $i \in \{1, \ldots, r\}$.

In order to study G/G' we consider

$$H = \langle \hat{a}, \hat{b}_1, \ldots, \hat{b}_r \mid \hat{a}^p = \hat{b}_1^p = \ldots = \hat{b}_r^p = 1, \text{ and } [\hat{b}_i, \hat{b}_j] = 1 \text{ for } 1 \leq i, j \leq r \rangle, \quad (4.1)$$

the free product $\langle \hat{a} \rangle * \langle \hat{b}_1, \ldots, \hat{b}_r \rangle$ of a cyclic group $\langle \hat{a} \rangle \cong C_p$ and an elementary abelian group $\langle \hat{b}_1, \ldots, \hat{b}_r \rangle \cong C_p^r$. There is a unique epimorphism $\pi \colon H \to G$ such that $\hat{a} \mapsto a$ and $\hat{b}_i \mapsto b_i$ for $i \in \{1, \ldots, r\}$, inducing an epimorphism from $H/H' \cong \langle \hat{a} \rangle \times \langle \hat{b}_1, \ldots, \hat{b}_r \rangle \cong C_p^{r+1}$ onto G/G'. We want to show that the latter is an isomorphism; see Proposition 4.1.10 below.

Let $h \in H$. As discussed, each h can be uniquely represented in the form

$$h = \hat{a}^{s_1} \cdot (\hat{b}_1^{\beta_{1,1}} \cdots \hat{b}_r^{\beta_{r,1}}) \cdot \hat{a}^{s_2} \cdot \ldots \cdot \hat{a}^{s_m} \cdot (\hat{b}_1^{\beta_{1,m}} \ldots \hat{b}_r^{\beta_{r,m}}) \cdot \hat{a}^{s_{m+1}}, \tag{4.2}$$

where $m \in \mathbb{N} \cup \{0\}$ and $s_1, \ldots, s_{m+1}, \beta_{1,1}, \ldots, \beta_{r,m} \in \mathbb{Z}/p\mathbb{Z}$ with

$$s_i \not\equiv 0 \pmod p \qquad \text{for } i \in \{2, \ldots, m\},$$

and for each $j \in \{1, \ldots, m\}$,

$$\beta_{i,j} \not\equiv 0 \pmod p \qquad \text{for at least one } i \in \{1, \ldots, r\}.$$

We denote by $\partial(h) = m$ the *length* of h, with respect to the factor $\langle b_1, \ldots, b_r \rangle$. Clearly, for $h_1, h_2 \in H$ we have

$$\partial(h_1 h_2) \leq \partial(h_1) + \partial(h_2). \tag{4.3}$$

In addition, we define *exponent maps*

$$\begin{aligned} \varepsilon_{\hat{a}}(h) &= \sum\nolimits_{j=1}^{m+1} s_j \in \mathbb{Z}/p\mathbb{Z} \quad \text{and} \\ \varepsilon_{\hat{b}_i}(h) &= \sum\nolimits_{j=1}^{m} \beta_{i,j} \in \mathbb{Z}/p\mathbb{Z} \quad \text{for } i \in \{1, \ldots, r\} \end{aligned} \tag{4.4}$$

with respect to the generating set $\hat{a}, \hat{b}_1, \ldots, \hat{b}_r$.

The surjective homomorphism

$$H \to (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^r, \quad h \mapsto (\varepsilon_{\hat{a}}(h), \varepsilon_{\hat{b}_1}(h), \ldots, \varepsilon_{\hat{b}_r}(h)) \tag{4.5}$$

has kernel H' and provides an explicit model for the abelianisation H/H'. The group $L(H) = \langle \hat{b}_1, \ldots, \hat{b}_r \rangle^H$ is the kernel of the surjective homomorphism

$$H \to \mathbb{Z}/p\mathbb{Z}, \quad h \mapsto \varepsilon_{\hat{a}}(h).$$

Each element $h \in L(H)$ can be uniquely represented by a word of the form

$$h = (\hat{c}_1)^{\hat{a}^{t_1}} \cdots (\hat{c}_m)^{\hat{a}^{t_m}}, \tag{4.6}$$

where $m \in \mathbb{N} \cup \{0\}$ and $t_1, \ldots, t_m \in \mathbb{Z}/p\mathbb{Z}$ with $t_j \not\equiv t_{j+1} \pmod p$ for $j \in \{1, \ldots, m-1\}$, and for each $j \in \{1, \ldots, m\}$,

$$\hat{c}_j = \hat{b}_1^{\beta_{1,j}} \cdots \hat{b}_r^{\beta_{r,j}} \in \langle \hat{b}_1, \ldots, \hat{b}_r \rangle \setminus \{1\}. \tag{4.7}$$

Let α denote the cyclic permutation of the factors of $H \times \overset{p}{\ldots} \times H$ corresponding to the p-cycle $(1\,2\,\ldots\,p)$. We consider the homomorphism

$$\Phi \colon L(H) \to H \times \overset{p}{\ldots} \times H$$

defined by

$$\Phi(\hat{b}_i^{\hat{a}^k}) = (\hat{a}^{e_{i,1}}, \ldots, \hat{a}^{e_{i,p-1}}, \hat{b}_i)^{\alpha^k} \quad \text{for } i \in \{1, \ldots, r\},\ k \in \mathbb{Z}/p\mathbb{Z}.$$

We have the following commutative diagram:

$$\begin{tikzcd}
H \arrow[r, two heads, "\pi"] & G \\
L(H) \arrow[u, phantom, "\cup|"] \arrow[r, two heads, "\pi|_{L(H)}"] \arrow[d, "\Phi"'] & \mathrm{Stab}_G(1) \arrow[u, phantom, "\cup|"] \arrow[d, hook] \\
H \times \cdots \times H \arrow[r, "\pi \times \cdots \times \pi"] & G \times \cdots \times G
\end{tikzcd}$$

Figure 4.2: A commutative diagram

For completion, we give a self-contained interpretation of Proposition 4.1.5 in the context of multi-edge spinal groups.

Lemma 4.1.7. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a multi-edge spinal group acting on the p-adic regular rooted tree T, for an odd prime p. Let*

$$H = \langle \hat{a}, \hat{b}_1, \ldots, \hat{b}_r \mid \hat{a}^p = \hat{b}_1^p = \cdots = \hat{b}_r^p = 1,\ [\hat{b}_i, \hat{b}_j] = 1,\ i, j \in \{1, \ldots, r\} \text{ with } i \neq j \rangle$$

be the free product $\langle \hat{a} \rangle * \langle \hat{b}_1, \ldots, \hat{b}_r \rangle$ of the cyclic group $\langle \hat{a} \rangle \cong C_p$ and the elementary abelian group $\langle \hat{b}_1, \ldots, \hat{b}_r \rangle \cong C_p^r$ for the odd prime p. Then there is a unique epimorphism $\pi : H \twoheadrightarrow G$ such that $\hat{a} \mapsto a$ and $\hat{b}_i \mapsto b_i$ for all $i \in \{1, \ldots, r\}$. We also have

$$H \to H/[H, H] \cong C_p \times C_p^r.$$

Proof. It follows immediately from Definition 4.1.1 of free products of groups, and the fact that H is generated by $r+1$ elements each one of prime order p. □

Lemma 4.1.8. *Let H be as above, and $h \in L(H)$ with $\Phi(h) = (h_1, \dots, h_p)$. Then $\sum_{i=1}^{p} \partial(h_i) \leq \partial(h)$, and $\partial(h_i) \leq \lceil \frac{\partial(h)}{2} \rceil$ for each $i \in \{1, \dots, p\}$.*

Proof. Let $h \in L(H)$ and consider

$$\Phi(h) = (h_1, \dots, h_p). \tag{4.8}$$

Every $h \in L(H)$ of length $\partial(h) = m$ can be written in the form

$$h = \hat{c}_1^{\hat{a}^*} \cdot \hat{c}_2^{\hat{a}^*} \cdots \hat{c}_{m-1}^{\hat{a}^*} \cdot \hat{c}_m^{\hat{a}^*} \tag{4.9}$$

where the symbols $\hat{a}^*$ represent unspecified powers of $\hat{a}$.

Each factor $\hat{c}_i^{\hat{a}^*}$ in (4.9) contributes to the i-th coordinate of the vector (4.8), either a power $\hat{a}^*$ of $\hat{a}$ or an element of $\langle \hat{b}_1, \dots, \hat{b}_r \rangle$. Therefore there are at most $\left\lceil \frac{\partial(h)}{2} \right\rceil$ factors of $\hat{c}_i$ in each section h_i in (4.8). It is also clear that $\partial(h) \geq \sum_{i=1}^{p} \partial(h_i)$. □

Lemma 4.1.9. *Let $G = \langle a, b_1, \dots, b_r \rangle$ be a multi-edge spinal group acting on the p-adic regular rooted tree T, for an odd prime p. Let*

$$H = \langle \hat{a}, \hat{b}_1, \dots, \hat{b}_r \mid \hat{a}^p = \hat{b}_1^p = \cdots = \hat{b}_r^p = 1,\, [\hat{b}_i, \hat{b}_j] = 1,\, i, j \in \{1, \dots, r\} \text{ with } i \neq j \rangle$$

be the free product $\langle \hat{a} \rangle * \langle \hat{b}_1, \dots, \hat{b}_r \rangle$ of the cyclic group $\langle \hat{a} \rangle \cong C_p$ and the elementary abelian group $\langle \hat{b}_1, \dots, \hat{b}_r \rangle \cong C_p^r$ for the odd prime p.

Consider the subgroup $K = \bigcup_{n=1}^{\infty} K_n$ of H obtained from the sequence of subgroups

$$K_0 = \{1\} \quad \text{and} \quad K_n = \Phi^{-1}(K_{n-1} \times \overset{p}{\cdots} \times K_{n-1}), \dots \quad \text{for } n \geq 1.$$

For $L(H) = \langle \hat{b}_1, \dots, \hat{b}_r \rangle^H \trianglelefteq H$, we have that

(1) *$K_n \trianglelefteq L(H)$ and $K_n \trianglelefteq H$,*

(2) *$K_{n-1} \subseteq K_n$,*

(3) *$\mathrm{Ker}(\pi) = K$, where $\pi : H \twoheadrightarrow G$. In particular, $G \cong H/K$.*

Proof. (1) We induct on n. For $n = 0$, $K_0 = \{1\}$ and hence the result holds.

Suppose that $n \geq 1$. By the induction hypothesis $K_{n-1} \trianglelefteq H$ which implies that

$$K_{n-1} \times \overset{p}{\cdots} \times K_{n-1} \trianglelefteq H \times \overset{p}{\cdots} \times H.$$

Therefore

$$\begin{aligned} K_n &= \Phi^{-1}(K_{n-1} \times \overset{p}{\cdots} \times K_{n-1}) \\ &\trianglelefteq \Phi^{-1}(H \times \overset{p}{\cdots} \times H) \\ &= L(H). \end{aligned}$$

Since H is generated by $\{\hat{a}\} \cup L(H)$, it suffices to show that $K_n^{\hat{a}} \subseteq K_n$. The definition of the map Φ gives that for all $h \in L(H)$

$$\Phi(h) = (h_1, \dots, h_p)$$

if and only if

$$\Phi(h^{\hat{a}}) = (h_p, h_1, \dots, h_{p-1}).$$

Therefore

$$\begin{aligned} K_n^{\hat{a}} &= (\Phi^{-1}(K_{n-1} \times \overset{p}{\cdots} \times K_{n-1}))^{\hat{a}} \\ &= \Phi^{-1}(K_{n-1} \times \overset{p}{\cdots} \times K_{n-1})^{\hat{a}} \\ &= \Phi^{-1}(K_{n-1} \times \overset{p}{\cdots} \times K_{n-1}) \\ &= K_n. \end{aligned}$$

(2) Again we induct on n. For $n = 1$, $K_0 = \{1\} \subseteq K_1$ because $K_1 \leq L(H)$.

Suppose that $n \geq 2$. By the induction hypothesis we have that $K_{n-2} \subseteq K_{n-1}$, which implies that

$$K_{n-2} \times \overset{p}{\cdots} \times K_{n-2} \subseteq K_{n-1} \times \overset{p}{\cdots} \times K_{n-1}.$$

Therefore

$$K_{n-1} = \Phi^{-1}(K_{n-2} \times \overset{p}{\cdots} \times K_{n-2}) \subseteq \Phi^{-1}(K_{n-1} \times \overset{p}{\cdots} \times K_{n-1}) = K_n$$

as required.

(3) Recall that the epimorphism $\pi : H \twoheadrightarrow G$ is given by $\hat{a} \mapsto a$ and $\hat{b}_i \mapsto b_i$ for all $i \in \{1, \dots, r\}$. Hence π induces an isomorphism $H/L(H) \cong G/\operatorname{Stab}_G(1)$ of cyclic groups of order p. By the definition of K, it follows that $K \subseteq L(H)$ and hence we need

to show that for all $h \in L(H)$, $\pi(h) = 1$ if and only if $h \in K$.

Suppose that $h \in K$. Then $h \in K_n$ for some $n \geq 0$. We induct on n. If $n = 0$, then $h \in K_0$ and hence $h = 1$. Therefore $\pi(h) = 1$.

Suppose that $n \geq 1$. From the commutative diagram in Figure 4.2 it follows that

$$\Phi(h) = (h_1, \dots, h_p) \in K_{n-1} \times \overset{p}{\cdots} \times K_{n-1},$$

and

$$\begin{aligned}(\pi(h_1), \dots, \pi(h_p)) &\in \pi(K_{n-1}) \times \overset{p}{\cdots} \times \pi(K_{n-1}) \\ &= \{1\} \times \overset{p}{\cdots} \times \{1\} \quad \text{(by induction on } n) \\ &= \{1\}\end{aligned}$$

in $G \times \overset{p}{\cdots} \times G$. Therefore $\pi(h) = 1$.

Now suppose that $h \in L(H)$ with $\pi(h) = 1$. We induct on the length $\partial(h)$ of $h \in L(H)$. Suppose that $\partial(h) \in \{0, 1\}$. If $\partial(h) = 0$, then $h = 1$ and hence $h \in K$.

Suppose $\partial(h) = 1$, so that $h = \hat{a}^{-s} \cdot \hat{c} \cdot \hat{a}^s$ with $\hat{c}$ a non-trivial element of $\langle \hat{b}_1, \dots, \hat{b}_r \rangle$. Writing $c = \pi(\hat{c}) \in \langle b_1, \dots, b_r \rangle \setminus \{1\}$, we see that $\pi(h) = a^{-s} \cdot c \cdot a^s \neq 1$, contrary to our assumption, and hence the case $\partial(h) = 1$ does not arise.

Now suppose that $\partial(h) \geq 2$. Using again the commutative diagram in Figure 4.2, from $\pi(h) = 1$ we deduce that $\Phi(h) = (h_1, \dots, h_p)$ with $\pi(h_i) = 1$ for all $i \in \{1, \dots, p\}$. In particular, $h_i \in L(H)$ for all $i \in \{1, \dots, p\}$.

By Lemma 4.1.8, we have that $\partial(h_i) < \partial(h)$ for all $i \in \{1, \dots, p\}$. Hence, by induction, there exists $m \in \mathbb{N}_0$ such that

$$(h_1, \dots, h_p) \in K_m \times \overset{p}{\cdots} \times K_m.$$

Consequently,

$$h \in \Phi^{-1}(K_m \times \overset{p}{\cdots} \times K_m) = K_{m+1} \subseteq K.$$

□

In the following proposition we describe the abelianisation of multi-edge spinal groups.

Proposition 4.1.10. *Let $G = \langle a, b_1, \dots, b_r \rangle$ be a multi-edge spinal group, and H as in (4.1). Then the map $H \to (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})^r$ in (4.5) factors through G/G'. Consequently,*

$$G/G' \cong H/H' \cong C_p^{r+1}.$$

Proof. Below we prove that

$$\Phi^{-1}(H' \times \overset{p}{\dots} \times H') \leq H'. \tag{4.10}$$

Let $K = \bigcup_{n=0}^{\infty} K_n \leq L(H)$ be as in Proposition 4.1.9 so that the natural epimorphism $\pi \colon H \to G$ has $\ker(\pi) = K$, and $G \cong H/K$. From (4.10), we deduce by induction that $K_n \leq H'$ for all $n \in \mathbb{N} \cup \{0\}$, hence $K \leq H'$ and $G/G' \cong H/H'K = H/H'$.

It remains to justify (4.10). Consider an arbitrary element $h \in L(H)$ as in (4.6) and (4.7). We write $\Phi(h) = (h_1, \dots, h_p)$. For $i \in \{1, \dots, r\}$ and $k \in \{1, \dots, p\}$, let $\varepsilon_{\hat{b}_i,k}(h)$ be the sum of exponents $\beta_{i,j}$, $j \in \{1, \dots, m\}$ with $t_j = k$, so that $\varepsilon_{\hat{b}_i}(h_k) = \varepsilon_{\hat{b}_i,k}(h)$. It follows that for each $i \in \{1, \dots, r\}$,

$$\varepsilon_{\hat{b}_i}(h) = \sum\nolimits_{j=1}^{m} \beta_{i,j} = \sum\nolimits_{k=1}^{p} \varepsilon_{\hat{b}_i,k}(h) = \sum\nolimits_{k=1}^{p} \varepsilon_{\hat{b}_i}(h_k). \tag{4.11}$$

Now suppose that $h \notin H'$. From (4.5) and $\varepsilon_{\hat{a}}(H) = 0$ we deduce that $\varepsilon_{\hat{b}_i}(h) \not\equiv 0$ for at least one $i \in \{1, \dots, r\}$. Thus (4.11) implies that $\varepsilon_{\hat{b}_i}(h_k) \not\equiv 0$ for some $k \in \{1, \dots, p\}$ and $\Phi(h) \notin H' \times \overset{p}{\dots} \times H'$. Therefore (4.10) holds. □

We now define a length function on elements of multi-edge spinal groups.

Definition 4.1.11. Let $G = \langle a, b_1, \dots, b_r \rangle$ be a multi-edge spinal group, and $\pi \colon H \to G$ the natural epimorphism with H as in (4.1). The *length* of $g \in G$ is

$$\partial(g) = \min\{\partial(h) \mid h \in \pi^{-1}(g)\}.$$

Based on (4.3), one easily shows that for $g_1, g_2 \in G$,

$$\partial(g_1 g_2) \leq \partial(g_1) + \partial(g_2). \tag{4.12}$$

Moreover, using Proposition 4.1.10 we may define $\varepsilon_a(g), \varepsilon_{b_1}(g), \dots, \varepsilon_{b_r}(g) \in \mathbb{Z}/p\mathbb{Z}$ via any pre-image $h \in \pi^{-1}(g)$:

$$(\varepsilon_a(g), \varepsilon_{b_1}(g), \dots, \varepsilon_{b_r}(g)) = (\varepsilon_{\hat{a}}(h), \varepsilon_{\hat{b}_1}(h), \dots, \varepsilon_{\hat{b}_r}(h)). \tag{4.13}$$

Lemma 4.1.12. *Let G be a multi-edge spinal group as above, and $g \in \mathrm{Stab}_G(1)$ with $\psi_1(g) = (g_1, \dots, g_p)$. Then $\sum_{i=1}^{p} \partial(g_i) \leq \partial(g)$, and $\partial(g_i) \leq \lceil \frac{\partial(g)}{2} \rceil$ for each $i \in \{1, \dots, p\}$.*

In particular, if $\partial(g) > 1$ then $\partial(g_i) < \partial(g)$ for every $i \in \{1, \dots, p\}$.

Proof. It follows immediately from Lemma 4.1.8. □

4.2 Length reduction

We continue to consider a multi-edge spinal group $G_{\mathbf{E}} = G = \langle a, b_1, \dots, b_r \rangle$ with respect to a set of defining vectors $\mathbf{E}$; see Section 3.1. Again, we work with the p-adic regular rooted tree T for an odd prime p.

In this section we introduce the theta maps

$$\Theta_1, \Theta_2 : [G, G] \to [G, G]$$

and we prove that the length of every element of the commutator subgroup of length at least 3 decreases under repeated applications of a combination of these maps.

The maps Θ_1 and Θ_2 are defined in such a way to investigate maximal subgroups of multi-edge spinal groups. More precisely, in Chapter 5, we are interested in looking to upper companion groups further down in the tree T in certain coordinates.

Let $g \in \mathrm{Stab}_G(1)$ be an element in the first level stabiliser. Recall from Section 2.3 that every such element has a decomposition

$$\psi_1(g) = (g_1, \dots, g_p)$$

where each g_i for $i \in \{1, \dots, p\}$ is acting on the corresponding subtree rooted at a first level vertex $u_1, \dots, u_p$.

For $i \in \{1, \dots, p\}$ we define the map

$$\varphi_i(g) : \mathrm{Stab}_G(1) \longrightarrow \mathrm{Aut}(T_{u_i}), \quad \varphi_i(g) = g_i \in U_{u_i}^G \tag{4.14}$$

where $U_{u_i}^G$ is the upper companion group acting on the subtree rooted at the i-th vertex of the first level.

Let

$$\psi_1(b_i) = (a^{e_{i,1}}, \dots, a^{e_{i,p-1}}, b_i)$$

be a directed automorphism where the last non-trivial power of the rooted automorphism a is positioned at the n-th coordinate, with $n \in \{2, \ldots, p-1\}$. While the value of n does not feature, n is fixed throughout.

By Lemma 3.2.3, we may assume that

$$\psi_1(b_1) = (a^{e_{1,1}}, \ldots, a^{e_{1,p-1}}, b_1)$$

with the last non-trivial power of the rooted automorphism a positioned at the n-th coordinate, where $n \in \{2, \ldots, p-1\}$, and also that $e_{1,1} \equiv 1 \pmod p$. We construct the maps Θ_1 and Θ_2 as follows:

Conjugating b_1 by $(az)^{-1}$, for $z \in [G, G]$, we have that

$$\begin{aligned}
\psi_1(b_1^{(az)^{-1}}) &= (z \cdot (a, a^{e_{1,2}}, \ldots, a^{e_{1,p-1}}, b_1) \cdot z^{-1})^{a^{-1}} \\
&= ((z_1, \ldots, z_p) \cdot (a, a^{e_{1,2}}, \ldots, a^{e_{1,p-1}}, b_1) \cdot (z_1^{-1}, \ldots, z_p^{-1}))^{a^{-1}} \\
&= (a^{z_1^{-1}}, (a^{e_{1,2}})^{z_2^{-1}}, \ldots, (a^{e_{1,p-1}})^{z_{p-1}^{-1}}, b_1^{z_p^{-1}})^{a^{-1}} \\
&= ((a^{e_{1,2}})^{z_2^{-1}}, \ldots, (a^{e_{1,p-1}})^{z_{p-1}^{-1}}, b_1^{z_p^{-1}}, a^{z_1^{-1}})
\end{aligned}$$

Therefore

$$\varphi_p((b_1)^{(az)^{-1}}) = a^{z_1^{-1}} = a[a, z_1^{-1}].$$

We define

$$\Theta_1 : [G, G] \to [G, G]$$

by

$$\Theta_1(z) = [a, z_1^{-1}].$$

Similarly, as with Θ_1, by considering $b_1^{(az)^{p-n}}$ we get

$$\begin{aligned}
\psi_1(b_1^{(az)^{p-n}}) &= ((az)^{-1} \cdot (a, a^{e_{1,2}}, \ldots, a^{e_{1,p-1}}, b_1) \cdot az)^{(az)^{p-n-1}} \\
&= (z^{-1}a^{-1} \cdot (a, a^{e_{1,2}}, \ldots, a^{e_{1,p-1}}, b_1) \cdot az)^{(az)^{p-n-1}} \\
&= ((z_1^{-1}, \ldots, z_p^{-1}) \cdot (b_1, a, a^{e_{1,2}}, \ldots, a^{e_{1,p-1}}) \cdot (z_1, \ldots, z_p))^{(az)^{p-n-1}} \\
&= (b_1^{z_1}, a^{z_2}, (a^{e_{1,2}})^{z_3}, \ldots, (a^{e_{1,p-1}})^{z_p})^{(az)^{p-n-1}} \\
&\;\;\vdots \\
&= (*, *, \ldots, *, (a^{e_{1,n-1}})^{z_{n+1} \cdots z_p}),
\end{aligned}$$

where the symbols $*$ denote unspecified powers of the rooted automorphism a and b_1's.

Therefore

$$\varphi_p((b_1)^{(az)^{p-n}}) = a^{(z_{n+1}\cdots z_p)} = a[a, z_{n+1}\cdots z_p].$$

So we define

$$\Theta_2 : [G,G] \to [G,G]$$

by

$$\Theta_2(z) = [a, z_{n+1}\cdots z_p].$$

To deal with the case $n = 1$, we define $\boldsymbol{\mathcal{E}}$ to be the family of all multi-edge spinal groups $G = \langle a, b_1, \ldots, b_r\rangle$ that satisfy

$$\begin{cases} \psi_1(b_1) = (a, 1, \ldots, 1, b_1) & \\ e_{i,1} \equiv 1 \pmod p & \text{for every } i \in \{1,\ldots,r\} \\ e_{i,p-1} \not\equiv 0 \pmod p & \text{for at least one } i \in \{1,\ldots,r\}. \end{cases} \tag{4.15}$$

We remark that, by Theorem 3.1.8, there are no torsion groups in $\boldsymbol{\mathcal{E}}$.

The next Theorem is one of the main results in this thesis. In Appendix A, we "test" Theorem 4.2.1, using a three generated multi-edge spinal group.

Theorem 4.2.1. *Let $G = \langle a, b_1, \ldots, b_r\rangle$ be a multi-edge spinal group acting on the regular p-adic rooted tree T, for an odd prime p. Suppose G is not* $\mathrm{Aut}(T)$*-conjugate to a group in $\boldsymbol{\mathcal{E}}$. Then the length $\partial(z)$ of an element $z \in G'$ decreases under repeated applications of a suitable combination of the maps Θ_1 and Θ_2 down to length 0 or 2.*

Proof. Let $z \in G'$. We observe that $\partial(z) \neq 1$; see Proposition 4.1.10. Suppose that $\partial(z) = m \geq 3$. Then $z \in G' \subseteq \mathrm{Stab}_G(1)$ has a decomposition

$$\psi_1(z) = (z_1, \ldots, z_p).$$

From Lemma 4.1.12 and (4.12) we obtain $\partial(z_j) \leq \lceil \frac{m}{2} \rceil$ for $j \in \{1,\ldots,p\}$ and

$$\partial(z_1) + \partial(z_{n+1}\cdots z_p) \leq m.$$

If $\partial(z_1) < \frac{m}{2}$ then $\partial(\Theta_1(z)) < m$, and likewise if $\partial(z_{n+1}\cdots z_p) < \frac{m}{2}$ then $\partial(\Theta_2(z)) < m$. Hence we may suppose that $m = 2\mu$ is even and

$$\partial(z_1) = \partial(z_{n+1}\cdots z_p) = \mu.$$

We write $z_{n+1} \cdots z_p$ as

$$a^{s_1} \cdot c_1 \cdot a^{s_2} \cdot \ldots \cdot a^{s_\mu} \cdot c_\mu \cdot a^{s_{\mu+1}},$$

where $s_1, \ldots, s_{\mu+1} \in \mathbb{Z}/p\mathbb{Z}$ with $s_i \not\equiv 0 \pmod p$ for $i \in \{2, \ldots, \mu\}$ and $c_1, \ldots, c_\mu \in \langle b_1, \ldots, b_r \rangle \setminus \{1\}$, and distinguish two cases. To increase the readability of exponents we use at times also the notation $s(i) = s_i$.

Case 1: $s_{\mu+1} \equiv 0 \pmod p$. Expressing

$$\Theta_2(z) = [a, z_{n+1} \cdots z_p] = [a, a^{s_1} c_1 a^{s_2} \cdots a^{s_\mu} c_\mu]$$

as a product of conjugates of the $c_i^{\pm 1}$ by powers of a and relabelling the $c_i^{\pm 1}$ as $\bar{c}_j$ for $j \in \{1, \ldots, m\}$, we get

$$\Theta_2(z) = \bar{c}_1^{a} \cdot \bar{c}_2^{a^{1+s(\mu)}} \cdot \ldots \cdot \bar{c}_\mu^{a^{1+s(\mu)+\ldots+s(2)}} \cdot \bar{c}_{\mu+1}^{a^{s(\mu)+\ldots+s(2)}} \cdot \ldots \cdot \bar{c}_{m-1}^{a^{s(\mu)}} \cdot \bar{c}_m. \tag{4.16}$$

Consider now

$$\psi_1(\Theta_2(z)) = ((\Theta_2(z))_1, \ldots, (\Theta_2(z))_p).$$

If $\partial((\Theta_2(z))_1) < \mu$ then $\Theta_1(\Theta_2(z))$ has length less than m. Hence we suppose

$$\partial((\Theta_2(z))_1) = \mu.$$

Using the symbol $*$ for unspecified exponents, we deduce from (4.16) that the first components $(\bar{c}_j^{a^*})_1$ for odd $j \in \{1, \ldots, m-1\}$ must be non-trivial elements of $\langle b_1, \ldots, b_r \rangle$, and the $(\bar{c}_j^{a^*})_1$ for even $j \in \{2, \ldots, m-2\}$ must be non-trivial powers of a. In particular, looking at the $(m-1)$th term we require $s_\mu \equiv 1 \pmod p$. This implies that the second factor in (4.16) is $\bar{c}_2^{a^2}$.

In the special case $n = 1$, $e_{i,p-1} \equiv 0 \pmod p$ for every $i \in \{1, \ldots, r\}$ and so we immediately get a contradiction, because $(\bar{c}_2^{a^2})_1$ contributes a trivial factor 1 to $(\Theta_2(z))_1$ instead of a non-trivial power of a.

In the generic case $n \geq 2$ we claim $\partial((\Theta_2(z))_{n+1} \cdots (\Theta_2(z))_p) < \mu$, leading to

$$\partial(\Theta_2(\Theta_2(z))) < m.$$

Indeed, only factors $\bar{c}_j^{a^*}$ in (4.16) for even $j \in \{2, \ldots, m\}$ can contribute non-trivial elements of $\langle b_1, \ldots, b_r \rangle$ to the product $(\Theta_2(z))_{n+1} \cdots (\Theta_2(z))_p$. But since $n \geq 2$, the second factor $\bar{c}_2^{a^2}$ in (4.16) contributes only a power of a.

Case 2: $s_{\mu+1} \not\equiv 0 \pmod{p}$. Similarly as in Case 1, we write

$$\Theta_2(z) = \bar{c}_1^{a^{1+s(\mu+1)}} \cdot \bar{c}_2^{a^{1+s(\mu+1)+s(\mu)}} \cdot \ldots \cdot \bar{c}_{m-1}^{a^{s(\mu+1)+s(\mu)}} \cdot \bar{c}_m^{a^{s(\mu+1)}}, \tag{4.17}$$

where the $c_i^{\pm 1}$ are relabeled as $\bar{c}_j$ for $j \in \{1, \ldots, m\}$. As before, it suffices to show that $\partial((\Theta_2(z))_1) < \mu$ or $\partial((\Theta_2(z))_{n+1} \cdots (\Theta_2(z))_p) < \mu$.

Suppose $\partial((\Theta_2(z))_1) = \mu$. Then either:

(i) $(\bar{c}_j^{a^*})_1$ for odd $j \in \{1, \ldots, m-1\}$ is a non-trivial element of $\langle b_1, \ldots, b_r \rangle$, and $(\bar{c}_j^{a^*})_1$ for even $j \in \{2, \ldots, m-2\}$ is a non-trivial power of a; or

(ii) $(\bar{c}_j^{a^*})_1$ for even $j \in \{2, \ldots, m\}$ is a non-trivial element of $\langle b_1, \ldots, b_r \rangle$, and $(\bar{c}_j^{a^*})_1$ for odd $j \in \{3, \ldots, m-1\}$ is a non-trivial power of a.

In case (i), we deduce from the $(m-1)$th term in (4.17) that

$$s_{\mu+1} + s_\mu \equiv 1 \pmod{p},$$

and the second term in (4.17) is equal to $\bar{c}_2^{a^2}$. We may argue as in Case 1 that $\partial((\Theta_2(z))_{n+1} \cdots (\Theta_2(z))_p) < \mu$ so that $\partial(\Theta_2(\Theta_2(z))) < m$.

In case (ii), we deduce from the mth term in (4.17) that

$$s_{\mu+1} \equiv 1 \pmod{p},$$

and the first term in (4.16) is $\bar{c}_1^{a^2}$. In the generic situation $n \geq 2$ we argue similarly as in Case 1 that $\partial((\Theta_2(z))_{n+1} \cdots (\Theta_2(z))_p) < \mu$ so that $\partial(\Theta_2(\Theta_2(z))) < m$. It remains to deal with the special situation $n = 1$, which makes use of the fact that the defining vectors satisfy $e_{i,p-1} \equiv 0$ for every $i \in \{1, \ldots, r\}$. For $m \geq 6$ the argument follows as before. For $m = 4$, proceeding similarly, we obtain $\Theta_2(z) = \bar{c}_1^{a^2} \bar{c}_2^{a} \bar{c}_3 \bar{c}_4^{a}$, so $(\Theta_2(z))_1 = ba^w c$ for some $b, c \in \langle b_1, \ldots, b_r \rangle$ and $w \in \mathbb{Z}/p\mathbb{Z}$. Thus subject to relabelling,

$$\Theta_1(\Theta_2(z)) = \bar{c}_1^{a} \bar{c}_2^{a^{1+w}} \bar{c}_3^{a^w} \bar{c}_4. \tag{4.18}$$

As before, for $\partial(\Theta_1(\Theta_2(z))) = 2$, we need $(\bar{c}_3^{a^*})_1$ to be a non-trivial element of $\langle b_1, \ldots, b_r \rangle$ and $(\bar{c}_2^{a^*})_1$ to be a non-trivial power of a. Looking at the third term of (4.18), we require $w = 1$. However, then $\bar{c}_2^{a^2}$ contributes a trivial factor 1 to $\Theta_1(\Theta_2(z))_1$ instead of a non-trivial power of a. So we see that the length decreases, as required. $\square$

Chapter 5

Maximal Subgroups

In this chapter we extend the methods developed in [26] to the class of torsion multi-edge spinal groups which are known to be just infinite. In particular, we prove that no torsion multi-edge spinal group contains proper dense subgroups with respect to the profinite topology. As a corollary, we obtain that such a group has maximal subgroups only of finite index. Moreover, we show that for G a torsion multi-edge spinal group, all its maximal subgroups are normal of index p, where p is the odd prime such that G acts on the p-adic regular rooted tree.

5.1 Dense subgroups

Definition 5.1.1. Let G be a group and M an arbitrary subgroup. For every normal subgroup N of finite index in G, denote by ε_N the natural projection $G \to G/N$. Then M is *dense* in G with respect to the profinite topology, if for every normal subgroup N of finite index in G

$$\varepsilon_N(M) = \varepsilon_N(G).$$

Lemma 5.1.2. *Every dense subgroup M of a multi-edge spinal group G is necessarily infinite.*

Proof. Let M be a finite dense subgroup of a multi-edge spinal group G. Then there exists $n_0 \in \mathbb{N}$ such that $M \cap \mathrm{Stab}_G(n) = 1$ for $n \geq n_0$. Then since M is dense, we have

$$|G : \mathrm{Stab}_G(n)| = |M : \mathrm{Stab}_M(n)| = |M|, \quad \text{for every } n \geq n_0,$$

which is impossible, since $|G : \mathrm{Stab}_G(n)|$ goes to infinity as $n \to \infty$. $\square$

Lemma 5.1.3 (see [26], Lemma 3.1). *Let G be a group. Then every proper dense subgroup M of G with respect to the profinite topology has infinite index.*

Proposition 5.1.4 (see [26], Proposition 3.2). *Let T be a spherically homogeneous rooted tree and let $G \leq \mathrm{Aut}(T)$ be a just infinite group acting transitively on each level of T. Let M be a dense subgroup of G with respect to the profinite topology. Then*

(1) *the subgroup M acts transitively on each level of the tree T,*

(2) *for every vertex $u \in T$, $\mathrm{Stab}_M(u)_u$ is dense in $\mathrm{Stab}_G(u)_u$.*

Proposition 5.1.5 (see [26], Proposition 4.1). *Every maximal subgroup M of infinite index in a group G is dense with respect to the profinite topology.*

Assumption 5.1.6. *For the rest of this chapter we fix a just infinite multi-edge spinal group $G = \langle a, b_1, \ldots, b_r \rangle$ acting on the p-adic regular rooted tree T for an odd prime p. Let u be any vertex of the p-adic regular rooted tree T. We write G_u for $\mathrm{Stab}_G(u)_u$, i.e. the restriction of this vertex stabiliser in G, acting on the subtree rooted at the vertex u; see Section 2.3. Similarly, for a subgroup M of G, we write M_u for $\mathrm{Stab}_M(u)_u$.*

Recall from Section 4.2, that for $i \in \{1, \ldots, p\}$ the map

$$\varphi_i : \mathrm{Stab}_G(1) \longrightarrow \mathrm{Aut}(T_{u_i})$$

is defined by

$$\varphi_i(g) = g_i \in U_{u_i}^G$$

where $\psi_1(g) = (g_1, \ldots, g_p)$ and $U_{u_i}^G$ is the upper companion group acting on the subtree rooted at the i-th vertex u_i of the first level.

The next result extends [26, Lemma 3.3], which addresses just infinite GGS-groups. Here we give a different and shorter proof for just infinite multi-edge spinal groups.

Theorem 5.1.7. *Let M be a proper dense subgroup of G with respect to the profinite topology. Then M_u is a proper subgroup of G_u for all $u \in T$.*

Proof. Assume on the contrary, that there exists a vertex u of T such that $M_u = G_u$. Let u be a vertex of minimal length n with the specified property, and suppose $u = wx$ where $|w| = n-1$. By Proposition 5.1.4 and induction, M_w is a proper dense subgroup of G_w. Since G_w is isomorphic to G, we have $|u| = 1$, say $u = u_1$ among the vertices $u_1, \ldots, u_p$ at level 1.

Let $R = \mathrm{Rstab}_M(u)_u$. By our assumption, we have $R \trianglelefteq M_u = G_u$. Since $G_u \cong G$ is just infinite, either R has finite index in G_u, or R is trivial.

Suppose first that R has finite index in G_u. Then

$$\begin{aligned}
|G : M| &\leq |G : \mathrm{Rstab}_M(1)| = |G : \mathrm{Stab}_G(1)|\,|\mathrm{Stab}_G(1) : \mathrm{Rstab}_M(1)| \\
&\leq |G : \mathrm{Stab}_G(1)| \left| \prod\nolimits_{i=1}^{p} G_{u_i} : \prod\nolimits_{i=1}^{p} \mathrm{Rstab}_M(u_i)_{u_i} \right| \\
&\leq |G : \mathrm{Stab}_G(1)|\,|G_u : R|^p
\end{aligned}$$

is finite. But, being a proper dense subgroup, M has infinite index in G.

Hence R is trivial, and so $\mathrm{Rstab}_M(1)$ is trivial. By Lemma 5.1.2 M is infinite, and thus

$$|G/\,\mathrm{Rstab}_G(1)| \geq |M/\,\mathrm{Rstab}_M(1)| = |M|$$

is infinite.

By Proposition 3.2.11, G is not $\mathrm{Aut}(T)$-conjugate to the GGS-group $\mathcal{G}$ in (3.7). Hence Proposition 3.2.8 shows that G is a branch group. Thus $\mathrm{Rstab}_G(1)$ has finite index in G, a contradiction. □

We remark that by M_u^g we mean that we first take the restriction of the subgroup M to the subtree rooted at the vertex $u \in T$ and then we conjugate by an element $g \in G$.

Theorem 5.1.8. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a just infinite multi-edge spinal group and M a dense torsion subgroup of G, with respect to the profinite topology. For each $i \in \{1, \ldots, r\}$, there is a vertex u of T such that, under the natural identification of T_u and T, the following holds: there exist $g \in G$ and $b \in \langle b_1, \ldots, b_r \rangle$ with $\varepsilon_{b_i}(b) \neq 0$ such that $(M^g)_u$ is a dense subgroup of $G_u \cong G$ and $b \in (M^g)_u$. Furthermore there exists $k \in \mathbb{Z}/p\mathbb{Z}$ such that $(M^g)_u = (M_u)^{a^k}$.*

Proof. Clearly, it suffices to prove the claim for $i = 1$. Since $|G : G'|$ is finite, G' is open in the profinite topology. Thus we find $x \in M \cap b_1 G'$. In particular, $x \in \mathrm{Stab}_G(1)$ with $\varepsilon_{b_1}(x) \not\equiv 0 \pmod{p}$. We argue by induction on $\partial(x) \geq 1$.

First suppose that $\partial(x) = 1$. Then x has the form $x = b^{a^k}$, where $b \in \langle b_1, \ldots, b_r \rangle$ with $\varepsilon_{b_1}(b) \not\equiv 0 \pmod{p}$ and $k \in \{0, 1, \ldots, p-1\}$. Thus choosing the vertex u to be the root of the tree T and $g = a^{-k}$, we have $b \in M_u^g$.

Now suppose that $\partial(x) \geq 2$. Recall from (4.4) and (4.13) the definition of $\varepsilon_{b_1}(x)$, and from (4.14) the definition of the maps $\varphi_j\colon \mathrm{Stab}_G(1) \to G_{u_j}$, where $u_1, \ldots, u_p$ denote the first level vertices of T. For any vertex u of T, the subtree T_u has a natural identification with T and $G_u \cong G$. We freely use the symbols $a, b_1, \ldots, b_r$ to denote also automorphisms of T_u under this identification. We claim that

$$\varepsilon_{b_1}(\varphi_1(x)) + \ldots + \varepsilon_{b_1}(\varphi_p(x)) \equiv \varepsilon_{b_1}(x) \not\equiv 0 \pmod{p}. \tag{5.1}$$

To see this, write x as a product of conjugates $b_i^{a^*}$ of the directed generators b_i, $i \in \{1, \ldots, r\}$, by powers a^*, where the symbol $*$ represents unspecified exponents. Then $\varepsilon_{b_1}(x)$ is the number of factors of the form $b_1^{a^*}$. Each of these factors contributes a directed automorphism b_1 in a unique coordinate, and none of the other factors $b_2^{a^*}, \ldots, b_r^{a^*}$ contributes a b_1 in any of the coordinates. Hence (5.1) holds.

By (5.1), there exists $j \in \{1, \ldots, p\}$ such that $\varepsilon_{b_1}(\varphi_j(x)) \not\equiv 0 \pmod{p}$. Moreover, Lemma 4.1.12 shows that

$$\partial(\varphi_j(x)) \leq \lceil \partial(x)/2 \rceil \leq (\partial(x) + 1)/2 < \partial(x). \tag{5.2}$$

Suppose that $\tilde{x} = \varphi_j(x) \in M_{u_j}$ belongs to $\mathrm{Stab}_{G_{u(j)}}(1)$, where we write $u(j) = u_j$ for readability. By Proposition 5.1.4, the subgroup M_{u_j} is dense in $G_{u_j} \cong G$. Since $\varepsilon_{b_1}(\tilde{x}) \not\equiv 0 \pmod{p}$ and $\partial(\tilde{x}) < \partial(x)$, the result follows by induction.

Now suppose that $\varphi_j(x) \notin \mathrm{Stab}_{G_{u(j)}}(1)$. For $\ell \in \{1, \ldots, p\}$ we claim that

$$\varepsilon_{b_1}(\varphi_\ell(\varphi_j(x)^p)) \equiv \varepsilon_{b_1}(\varphi_j(x)) \not\equiv 0 \pmod{p}. \tag{5.3}$$

To see this, observe that $\varphi_j(x)$ is of the form

$$\varphi_j(x) = a^k h,$$

for $k \not\equiv 0 \pmod{p}$ and $h \in \mathrm{Stab}_{G_{u(j)}}(1)$ with $\psi_1(h) = (h_1, \ldots, h_p)$, say. Hence raising $\varphi_j(x)$ to the prime power p, we get

$$\varphi_j(x)^p = (a^k h)^p = h^{a^{(p-1)k}} h^{a^{(p-2)k}} \cdots h^{a^k} h,$$

and thus for $\ell \in \{1, \ldots, p\}$,

$$\varphi_\ell(\varphi_j(x)^p) \equiv h_1 h_2 \cdots h_p \pmod{G'_{u_{j\ell}}}.$$

Here $u_{j\ell}$ denotes the ℓth descendant of u_j. Arguing similarly as for (5.1), this yields

$$\varepsilon_{b_1}(\varphi_\ell(\varphi_j(x)^p)) \equiv \varepsilon_{b_1}(h_1) + \ldots + \varepsilon_{b_1}(h_p) \equiv \varepsilon_{b_1}(h) \equiv \varepsilon_{b_1}(\varphi_j(x)) \pmod p$$

and (5.3) holds.

Furthermore, we claim that

$$\partial(\varphi_\ell(\varphi_j(x)^p)) \le \partial(\varphi_j(x)) < \partial(x). \tag{5.4}$$

The second inequality comes from (5.2). To see that the first inequality holds, we note that

$$\varphi_\ell(\varphi_j(x)^p) = \varphi_\ell(h^{a^{(p-1)k}}) \cdots \varphi_\ell(h^{a^k})\varphi_\ell(h),$$

and $\partial(\varphi_j(x)) = \partial(h)$. We write h as a product of $\partial(h)$ conjugates $c_j^{a^*}$ of directed automorphisms $c_j \in \langle b_1, \ldots, b_r\rangle$, where the symbol $*$ represents unspecified exponents. Each factor $c_j^{a^*}$ contributes a directed automorphism c_j in a unique coordinate and powers of a in all other coordinates. Focusing on the ℓth coordinate, we can write $\varphi_\ell(\varphi_j(x)^p)$ as a product of powers of a and the $\partial(h)$ directed automorphisms $c_j \in \langle b_1, \ldots, b_r\rangle$. Hence (5.4) holds.

If $\tilde{x} = \varphi_\ell(\varphi_j(x)^p) \in M_{u_{j\ell}}$ belongs to $\mathrm{Stab}_{G_{u(j\ell)}}(1)$, we argue as follows. By Proposition 5.1.4, the subgroup $M_{u_{j\ell}}$ is dense in $G_{u_{j\ell}} \cong G$. Since $\varepsilon_{b_1}(\tilde{x}) \not\equiv 0 \pmod p$ and $\partial(\tilde{x}) < \partial(x)$, the result follows by induction.

In general, we apply the operation $y \mapsto \varphi_\ell(y^p)$ more than once. Since M is a torsion group, $x \in \mathrm{Stab}_M(1)$ and $\varphi_j(x)$ have finite order. Clearly, if $y \in G$ has finite order then $\varphi_\ell(y^p)$ has order strictly smaller than y. Thus after finitely many iterations, we inevitably reach an element

$$\tilde{x} = \varphi_\ell(\varphi_\ell(\cdots \varphi_\ell(\varphi_\ell(\varphi_j(x)^p)^p)^p \cdots)^p) \in M_{u_{j\ell\cdots\ell}}$$

which in addition to the inherited properties $\varepsilon_{b_1}(\tilde{x}) \not\equiv 0 \pmod p$ and $\partial(\tilde{x}) < \partial(x)$ satisfies $\tilde{x} \in \mathrm{Stab}_{G_{u(j\ell\cdots\ell)}}(1)$. As before, the proof concludes by induction. □

Recall the definition of the family $\boldsymbol{\mathcal{E}}$ of groups by means of (4.15).

Proposition 5.1.9. *Let $G = \langle a, b_1, \ldots, b_r\rangle$ be a multi-edge spinal group. Suppose G is not* $\mathrm{Aut}(T)$*-conjugate to a group in $\boldsymbol{\mathcal{E}}$. Let M be a dense subgroup of G, with respect to the profinite topology, and suppose that $b_1 \in M$. Then there exist a vertex u of T and $g \in G$ such that $a, b_1 \in (M^g)_u$. Furthermore there exists $b \in \langle b_1, \ldots, b_r\rangle$ such that $(M^g)_u = (M_u)^b$.*

Proof. Observe that G' is open in G. Since M is dense in G, there is $z \in G'$ such that $az \in M$. Write $\psi_1(z) = (z_1, \ldots, z_p)$.

Let u_p denote the pth vertex at level 1. The coordinate map φ_p allows us to restrict $\mathrm{Stab}_M(1)$ to M_{u_p}. Clearly, $b_1 \in M$ implies $b_1 = \varphi_p(b_1) \in M_{u_p}$. Based on Lemma 3.2.3, we assume that the defining vector $\mathbf{e}_1$ for b_1 has first coordinate $e_{1,1} = 1$. Consider the theta maps Θ_1, Θ_2 defined in Section 4.2, with reference to b_1. By their definition, $a\,\Theta_1(z)$ and $a\,\Theta_2(z)$ belong to M_{u_p}. Moreover, repeated application of φ_p corresponds to repeated applications of Θ_1 and Θ_2. By Proposition 5.1.4 and Theorem 4.2.1, we may assume that $\partial(z) \in \{0, 2\}$. If $\partial(z) = 0$ we are done.

Thus we may assume that $\partial(z) = 2$ and we write $z = b^{-a^m} c^{a^k}$ for $b, c \in \langle b_1, \ldots, b_r\rangle \setminus \{1\}$ and $m, k \in \mathbb{Z}/p\mathbb{Z}$ with $m \neq k$.

Case 1: $m, k \neq 1$. Here $z_1 = a^w$ for some $w \in \mathbb{Z}/p\mathbb{Z}$. Thus $\Theta_1(z) = [a, z_1^{-1}] = [a, a^{-w}] = 1$, and $a \in M_{u_p}$.

Case 2: $m = 1$, $k \neq 1$. Here

$$\psi_1(b^{-a}) = (b^{-1}, *, \ldots, *) \quad \text{and} \quad \psi_1(c^{a^k}) = (a^w, *, \ldots, *),$$

where $w \in \mathbb{Z}/p\mathbb{Z}$ and the symbols $*$ denote unspecified entries. Hence $z_1 = b^{-1}a^w$ so that $\Theta_1(z) = [a, z_1^{-1}] = [a, b]$. This gives $a\,\Theta_1(z) = b^{-1}ab$. Remembering that b_1 and b commute, we obtain $a, b_1 \in M_{u_p}^{b^{-1}}$.

Case 3: $m \neq 1$, $k = 1$. Here

$$\psi_1(b^{-a^m}) = (a^w, *, \ldots, *) \quad \text{and} \quad \psi_1(c^a) = (c, *, \ldots, *),$$

where $w \in \mathbb{Z}/p\mathbb{Z}$ and the symbols $*$ denote unspecified entries. Hence $z_1 = a^w c$ so that $\Theta_1(z) = [a, z_1^{-1}] = c^{a^{1-w}} c^{-a^{-w}}$. If $w \not\equiv -1 \pmod p$, we are back in Case 1 or Case 2.

Suppose that $w \equiv -1 \pmod p$. Then $\Theta_1(z) = c^{a^2} c^{-a}$, where

$$c^{a^2} = (*, c, *, \ldots, *) \quad \text{and} \quad c^{-a} = (c^{-1}, *, \ldots, *)$$

and the symbols $*$ denote unspecified power of a. We recall from the definition of Θ_2 that in the generic case $n \geq 2$ this gives $\Theta_2(\Theta_1(z)) = 1$, hence $a, b_1 \in \tilde{M}_{u_{pp}}$, where u_{pp} is the p^2th vertex at level 2. In the special case $n = 1$ we have $\Theta_1(\Theta_1(z)) = [a, c]$. In this case we proceed similarly as in Case 2. $\square$

Proposition 5.1.10. *Let G be a just infinite multi-edge spinal group. Suppose G is not $\mathrm{Aut}(T)$-conjugate to a group in $\boldsymbol{\mathcal{E}}$. Let M be a dense torsion subgroup of G, with respect to the profinite topology. Then there exists a vertex u of T such that $M_u = G_u \cong G$.*

Proof. By Theorem 5.1.8, there exist $g \in G$, a vector u_1 of T and $k \in \mathbb{Z}/p\mathbb{Z}$ such that $x_1 \in (M^g)_{u_1} = (M_{u_1})^{a^k}$ with $x_1 \in \langle b_1, \ldots, b_r\rangle$ and $\varepsilon_{b_1}(x_1) \neq 0$. We modify our generating set of directed automorphisms, by taking $\tilde{b}_1 = x_1$ instead of b_1. Denote $(M^g)_{u_1}$ by M_1. By Proposition 5.1.9, we have $a, \tilde{b}_1 \in ((M_1)^{\hat{h}})_v$ for $\hat{h} \in G_{u_1}$ and a vertex v of T. We denote $((M_1)^{\hat{h}})_v$ by M_2.

Applying Theorem 5.1.8 again, we see that there exist $l \in \mathbb{Z}/p\mathbb{Z}$ and a vertex u_2 of T such that $x_2 \in ((M_2)_{u_2})^{a^l}$ with $x_2 \in \langle \tilde{b}_1, b_2, \ldots, b_r\rangle$ and $\varepsilon_{b_2}(x_2) \neq 0$. Note that ε_{b_2} is now defined with respect to the new generating set $\{\tilde{b}_1, b_2, \ldots, b_r\}$ of directed automorphisms. Since $a, \tilde{b}_1 \in M_2$, it follows that:

(i) $a, \tilde{b}_1 \in (M_2)_w$ for all vertices w in T (recall Lemma 3.2.1 and Proposition 3.2.2 with respect to the multi-edge spinal group $\langle a, \tilde{b}_1\rangle$);

(ii) $((M_2)_w)^{a^l} = (M_2)_w$ as $a \in (M_2)_w$.

Hence $a, \tilde{b}_1, x_2 \in ((M_2)_{u_2})^{a^l} = (M_2)_{u_2}$. Again we replace b_2 by $\tilde{b}_2 = x_2$. So $a, \tilde{b}_1, \tilde{b}_2 \in (M_2)_{u_2}$.

Continuing in this manner, we obtain $a, \tilde{b}_1, \ldots, \tilde{b}_r \in (M_2)_{u_2 \ldots u_r}$, where $\{\tilde{b}_1, \ldots, \tilde{b}_r\}$ is a generating set of directed automorphisms.

Now $M_2 = (((M^g)_{u_1})^{\hat{h}})_v$. There exists an element $h \in G$ such that $(G^h)_{u_1} = (G_{u_1})^{\hat{h}}$. Therefore $M_2 = (M^{gh})_{u_1 v}$. Setting $u = u_1 v u_2 \ldots u_r$, we have $(M^{gh})_u = G_u \cong G$. Lastly, there exists $f \in G_u$ such that $(M^{gh})_u = (M_u)^f$. Then $M_u = (G_u)^{f^{-1}} = G_u \cong G$. $\square$

Theorem 5.1.11. *Let $G = \langle a, b_1, \ldots, b_r\rangle$ be a just infinite multi-edge spinal group. Then G does not contain any proper dense torsion subgroups, with respect to the profinite topology.*

Proof. Suppose on the contrary that M is a proper dense subgroup of G with respect to the profinite topology. By Theorem 5.1.7, for every vertex $u \in T$ we have M_u is properly contained in G_u. However, by Proposition 5.1.10, the subgroup M_u is all of $G \cong G_u$. This gives us our required contradiction. $\square$

As a corollary, we obtain the following.

Corollary 5.1.12. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a torsion multi-edge spinal group. Then G does not contain any maximal subgroups of infinite index.*

Proof. Suppose, for a contradiction, that M is a maximal subgroup of infinite index in G. By definition, M is dense in G with respect to the profinite topology. By Theorem 5.1.11, the group G does not contain dense proper subgroups with respect to the profinite topology, a contradiction and hence the result follows. $\square$

5.2 Normal subgroups

Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a torsion just infinite multi-edge spinal group acting on the p-adic regular rooted tree T, for an odd prime p. In this section we show that for such G, all its maximal subgroups are normal of index p.

Proposition 5.2.1 (see [28], Proposition 5.2.4)**.** *Let Γ be a finite group. Then the following are equivalent:*

(1) *The group Γ is nilpotent.*

(2) *The maximal subgroups of Γ are normal of prime index.*

(3) *Every subgroup of Γ is subnormal.*

(4) *The group Γ satisfies the normaliser condition.*

The *normaliser condition* in Proposition 5.2.1, means that every proper subgroup is properly contained in its normaliser.

Corollary 5.2.2. *Let $G = \langle a, b_1, \ldots, b_r \rangle$ be a just infinite multi-edge spinal group. Suppose G is not* $\mathrm{Aut}(T)$*-conjugate to a group in $\boldsymbol{\mathcal{E}}$. Then every maximal subgroup of G is normal of index p, where p is the odd prime such that G acts on the p-adic regulat rooted tree.*

Proof. Let M be any maximal subgroup of finite index in G. By Theorem 3.2.10, the group G is just infinite. That is every non-trivial normal subgroup N of G is of finite index in G. Hence M being maximal and of finite index in G, it contains a normal subgroup N of finite index in G. By Theorem 3.1.8 G is also a p-group. It follows that the quotient G/N is a finite p-group. It is clear that the image of M/N in G/N is a maximal subgroup. By part (2) of Proposition 5.2.1, it follows that M/N is normal of index p in G/N. Therefore M is of index p in G. By part (4) of Proposition 5.2.1, M/N is properly contained in its normaliser. Since M/N is a finite group, its normaliser is strictly bigger than M/N. Since M/N is of index p in G/N, it follows that its normaliser is G/N. Therefore M is normal in G. $\square$

Appendix A

A three generated multi-edge spinal group

The aim of this Appendix is to demonstrate a hands on computation using the theta maps Θ_1 and Θ_2 introduced in Section 4.2.

More precisely, we "test" Theorem 4.2.1 using the multi-edge spinal group $G = \langle a, b_1, b_2 \rangle$, generated by the rooted automorphism a and two directed automorphisms b_1 and b_2 of the form

$$\psi_1(b_1) = (a, a^{-1}, 1, \ldots, 1, b_1)$$
$$\psi_1(b_2) = (1, a, a^{-1}, 1, \ldots, 1, b_2).$$

Again, we work with the p-adic regular rooted tree T for an odd prime p.

Let $z \in [G, G]$ be an arbitrary element in the derived group of G. Since the derived group $[G, G]$ of G is a subgroup of $\mathrm{Stab}_G(1)$, every $z \in [G, G]$ has a decomposition

$$\psi_1(z) = (z_1, \ldots, z_p) \tag{A.1}$$

where every z_i for $i \in \{1, \ldots, p\}$ is acting on the subtree rooted at the i-th first level vertex; see Section 2.3.

Let $z \in [G, G]$ be an element of length $\partial(z) = r$, and p an odd prime. We define by $D(z)$ the matrix with row entries $z_{i,j}$ for $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, r\}$. We call $D(z)$ the *decomposition matrix* of z.

For example, if $z \in [G, G]$ is an arbitrary element of length $\partial(z) = r$, its decomposition matrix is the $(p \times r)$-matrix of the form:

$$D(z) = \begin{pmatrix} z_{1,1} & z_{1,2} & \cdots & \cdots & z_{1,r-1} & z_{1,r} \\ z_{2,1} & z_{2,2} & \cdots & \cdots & z_{2,r-1} & z_{2,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{p-1,1} & z_{p-1,2} & \cdots & \cdots & z_{p-1,r-1} & z_{p-1,r} \\ z_{p,1} & z_{p,2} & \cdots & \cdots & z_{p,r-1} & z_{p,r} \end{pmatrix}$$

Consider the maps

$$\Theta_1, \Theta_2 : [G,G] \to [G,G]$$

defined by

$$\begin{aligned} \Theta_1 &:= z \mapsto [a, z_1^{-1}] \\ \Theta_2 &:= z \mapsto [a, z_3 \cdots z_p]. \end{aligned}$$

Proposition A.0.3. *Let $G = \langle a, b_1, b_2 \rangle$ be the multi-edge spinal group defined above. The group G is acting on the p-adic regular rooted tree T for an odd prime p. Let z be an element of the derived group $[G,G]$ of length at least 3. Then the length $\partial(z)$ decreases under repeated applications of a combination of the maps Θ_1 and Θ_2.*

Proof. As in Theorem 4.2.1, for every $z = (z_1, \dots, z_p) \in [G,G]$ of length $\partial(z) = m$, the length $\partial(z_i)$ for $i \in \{1, \dots, p\}$ of each section is less than or equal to $m/2$. If $\partial(z_1)$ or $\partial(z_3 z_4 \cdots z_p)$ is strictly less than $m/2$, then $\partial([a, z_1^{-1}]) < m$ and $\partial([a, z_3 z_4 \cdots z_p]) < m$ respectively.

Suppose that $\partial(z_1) = \partial(z_3 z_4 \cdots z_p) = \frac{m}{2}$. Clearly m must be an even number.

For such an element $z \in [G,G]$ we have that $z_{n+1} \cdots z_p$ is of the form:

$$z_{n+1} \cdots z_p = a^{w(1)} b^* a^{w(2)} b^* a^{w(3)} b^* \cdots a^{w(m/2)} b^* \tag{A.2}$$

or

$$z_{n+1} \cdots z_p = a^{w(1)} b^* a^{w(2)} b^* a^{w(3)} b^* \cdots a^{w(m/2)} b^* a^{w((m/2)+1)}, \tag{A.3}$$

where $w(i) \not\equiv 0 \pmod p$ for $i \in \{2, \dots, \frac{m}{2} + 1\}$ and the symbols b^* represent non-trivial elements of $\langle b_1, b_2 \rangle$.

We need to examine four cases. We write b_{i_j} for non-trivial elements of $\langle b_1, b_2 \rangle$ which appear in the decomposition matrix $D(z)$ of $z \in G'$. Also, the exponents of the rooted automorphism are to be taken modulo p, for an odd prime p.

Case 1: We consider an element $z \in [G,G]$ of length $\partial(z) = 4$. In this case the decomposition matrix $D(z)$ is of the form:

$$D(z) = \begin{pmatrix} b_{i_1} & a^{d_1} & b_{i_3} & a^{f_3} \\ a^{c_1} & a^{d_2} & a^{e_1} & 1 \\ a^{c_2} & a^{d_3} & a^{e_2} & 1 \\ a^{c_3} & 1 & a^{e_3} & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & b_{i_4} \\ 1 & 1 & 1 & a^{f_1} \\ 1 & b_{i_2} & 1 & a^{f_2} \end{pmatrix}$$

It is clear that $\partial(z_1) = 2$. Note that in order to have $\partial(z_3 z_4 \cdots z_p) = 2$, the directed automorphism b_{i_4} cannot be in the first two and the last two rows. Thus we have in total $p-4$ matrices arising in this case. We need to consider two subcases.

Case 1.1: In this case $z_3 \cdots z_p$ is of the form:

$$z_3 z_4 \cdots z_p = a^{w_1} b^* a^{w_2} b^* a^{w_3}$$

where $w_i \not\equiv 0 \pmod{p}$ for $i \in \{1,2,3\}$ and the symbols b^* represent elements in $\langle b_1, b_2 \rangle$.

For example, the decomposition matrix is of the form:

$$D(z) = \begin{pmatrix} b_{i_1} & a^{d_1} & b_{i_3} & a^{f_3} \\ a^{c_1} & a^{d_2} & a^{e_1} & 1 \\ a^{c_2} & a^{d_3} & a^{e_2} & 1 \\ a^{c_3} & 1 & a^{e_3} & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & b_{i_4} \\ 1 & 1 & 1 & a^{w_2} \\ 1 & b_{i_2} & 1 & a^{w_3} \end{pmatrix}$$

Thus

$$z_3 z_4 \cdots z_p = a^{c_2} a^{d_3} a^{e_2} a^{c_3} a^{e_3} b_{i_4} a^{w_2} b_{i_2} a^{w_3}.$$

Setting

$$w_1 = c_2 + d_3 + e_2 + c_3 + e_3,$$

we get

$$z_3z_4\cdots z_p = a^{w_1}b^*a^{w_2}b^*a^{w_3}.$$

Applying the map Θ_2 on the element $z \in [G,G]$ we get

$$\begin{aligned}
\Theta_2(z) = [a, z_3z_4\cdots z_p] &= a^{-1}a^{-w_3}b_{i_2}^{-1}a^{-w_2}b_{i_4}^{-1}a^{-w_1}aa^{w_1}b_{i_4}a^{w_2}b_{i_2}a^{w_3}\\
&= a^{-(1+w_3)}b_{i_2}^{-1}a^{-w_2}b_{i_4}^{-1}ab_{i_4}a^{w_2}b_{i_2}a^{w_3}\\
&= a^{-(1+w_3)}b_{i_2}^{-1}a^{-w_2}b_{i_4}^{-1}ab_{i_4}a^{(w_2+w_3)}b_{i_2}^{a^{w_3}}\\
&= a^{-(1+w_3)}b_{i_2}^{-1}a^{-w_2}b_{i_4}^{-1}a^{(1+w_2+w_3)}b_{i_4}^{a^{(w_2+w_3)}}b_{i_2}^{a^{w_3}}\\
&= a^{-(1+w_3)}b_{i_2}^{-1}a^{-w_2}a^{(1+w_2+w_3)}(b_{i_4}^{-1})^{a^{(1+w_2+w_3)}}b_{i_4}^{a^{(w_2+w_3)}}b_{i_2}^{a^{w_3}}\\
&= a^{-(1+w_3)}b_{i_1}^{-1}a^{(1+w_3)}(b_{i_4}^{-1})^{a^{(1+w_2+w_3)}}b_{i_4}^{a^{(w_2+w_3)}}b_{i_2}^{a^{w_3}}\\
&= (b_{i_2}^{-1})^{a^{(1+w_3)}}(b_{i_4}^{-1})^{a^{(1+w_2+w_3)}}b_{i_4}^{a^{(w_2+w_3)}}b_{i_2}^{a^{w_3}}.
\end{aligned}$$

It is not hard to see that for $(b_{i_2}^{-1})^{a^{(1+w_3)}}$ to contribute a $b_{i_1}^{-1}$ in the first section of the decomposition we need $w_3 \equiv 0 \pmod p$. Hence $b_{i_2}^{a^{w_3}}$ contributes a rooted automorphism in the first section of the decomposition. Since $w_3 \equiv 0 \pmod p$, for $b_{i_4}^{a^{(w_2+w_3)}}$ to contribute b_{i_4} we need $w_2 = 1$. Therefore $(b_{i_4}^{-1})^{a^{(1+w_2+w_3)}}$ contributes the identity element in the first section of the decomposition. As a result

$$z_{11}([a, z_3z_4\cdots z_p]) = (b_{i_2}^{-1}b_{i_4}a, *, \ldots, *).$$

Therefore, applying the map Θ_1 on $z_{11}([a, z_3z_4\cdots z_p])$ we see that the length of its image is equal to two.

On the other hand, for $b_{i_2}^{a^{w_3}}$ to contribute a directed automorphism in the first section of the decomposition we need $w_3 = 1$. For $(b_{i_4}^{-1})^{a^{(1+w_2+w_3)}}$ to contribute a directed automorphism we need $w_2 = -1$. Then $(b_{i_4})^{a^{(w_2+w_3)}}$ gives a rooted automorphism and we see that the length is equal to two. But in this case

$$\begin{aligned}
(b_{i_2}^{-1})^{a^{(1+w_3)}} &= (1, b_{i_2}^{-1}, a^*, a^*, a^*, 1, \ldots, 1)\\
(b_{i_4}^{-1})^{a^{(1+w_2+w_3)}} &= (b_{i_4}^{-1}, a^*, a^*, a^*, 1, \ldots, 1)\\
(b_{i_4})^{a^{(w_2+w_3)}} &= (a^*, a^*, a^*, 1\ldots, 1, b_{i_4})\\
b_{i_2}^{a^{w_3}} &= (b_{i_2}, a^*, a^*, a^*, 1, \ldots, 1).
\end{aligned}$$

Therefore the length of $z_3z_4\cdots z_p$ in the product of the directed automorphisms above is equal to one. Hence applying the map Θ_2 again we see that the length of its image is equal to two.

Case 1.2: In this case $z_3 \cdots z_p$ is of the form:

$$z_3 z_4 \cdots z_p = a^{w_1} b^* a^{w_2} b^*$$

where $w_i \not\equiv 0 \pmod{p}$ for $i \in \{1, 2\}$ and the symbols b^* represent elements in $\langle b_1, b_2 \rangle$. For example, the decomposition matrix is of the form:

$$D(z) = \begin{pmatrix} b_{i_1} & a^{d_1} & b_{i_3} & 1 \\ a^{c_1} & a^{d_2} & a^{e_1} & 1 \\ a^{c_2} & a^{d_3} & a^{e_2} & 1 \\ a^{c_3} & 1 & a^{e_3} & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & b_{i_4} \\ 1 & 1 & 1 & a^{f_1} \\ 1 & 1 & 1 & a^{f_2} \\ 1 & 1 & 1 & a^{f_3} \\ 1 & b_{i_2} & 1 & 1 \end{pmatrix}$$

Thus

$$z_3 z_4 \cdots z_p = a^{c_2} a^{d_3} a^{e_2} a^{c_3} a^{e_3} b_{i_4} a^{f_1} a^{f_2} a^{f_3} b_{i_2}.$$

Setting

$$w_1 = c_2 + d_3 + e_2 + c_3 + e_3,$$

and

$$w_2 = f_1 + f_2 + f_3$$

we get

$$z_3 z_4 \cdots z_p = a^{w_1} b_{i_4} a^{w_2} b_{i_2}.$$

Applying the map Θ_2 on z we get

$$\begin{aligned}\Theta_2(z) = [a, z_3z_4\cdots z_p] &= a^{-1}b_{i_2}^{-1}a^{-w_2}b_{i_4}^{-1}a^{-w_1}aa^{w_1}b_{i_4}a^{w_2}b_{i_2}\\ &= a^{-1}b_{i_2}^{-1}a^{-w_2}b_{i_4}^{-1}ab_{i_4}a^{w_2}b_{i_2}\\ &= a^{-1}b_{i_2}^{-1}a^{-w_2}b_{i_4}^{-1}a^{(1+w_2)}b_{i_4}^{a^{w_2}}b_{i_2}\\ &= a^{-1}b_{i_2}^{-1}a^{(-w_2+1+w_2)}(b_{i_4}^{-1})^{a^{(1+w_2)}}b_{i_4}^{a^{w_2}}b_{i_2}\\ &= (b_{i_2}^{-1})^a(b_{i_4}^{-1})^{a^{(1+w_2)}}b_{i_4}^{a^{w_2}}b_{i_2}.\end{aligned}$$

It is not hard to see that $(b_{i_2}^{-1})^a$ and b_{i_2} contribute a directed and a rooted automorphism in the first section of the decomposition respectively. Also for $b_{i_4}^{a^{w_2}}$ to contribute a directed automorphism we need $w_2 = 1$. Then $(b_{i_4}^{-1})^{a^{(1+w_2)}}$ contributes the identity and hence applying the map Θ_1 on $z_{11}([a, z_3z_4\cdots z_p])$ we see that the length of its image is equal to two. The remaining cases are identical.

Case 2: We consider elements $z \in [G, G]$ of length $\partial(z) = 6$. As before, we need to consider two subcases.

Case 2.1: In this case $z_3\cdots z_p$ is of the form:

$$z_3z_4\cdots z_p = a^{w_1}b^*a^{w_2}b^*a^{w_3}b^*a^{w_4}$$

where $w_i \not\equiv 0 \pmod p$ for $i \in \{1, 2, 3, 4\}$ and the symbols b^* represent elements in $\langle b_1, b_2\rangle$.

For example, the decomposition matrix is of the form:

$$D(z) = \begin{pmatrix} b_{i_1} & a^{d_1} & b_{i_3} & a^{f_2} & b_{i_5} & 1\\ a^{c_1} & a^{d_2} & a^{e_1} & a^{f_3} & a^{h_1} & 1\\ a^{c_2} & a^{d_3} & a^{e_2} & 1 & a^{h_2} & 1\\ a^{c_3} & 1 & a^{e_3} & 1 & a^{h_3} & 1\\ 1 & 1 & 1 & 1 & 1 & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & 1 & 1 & b_{i_6}\\ 1 & 1 & 1 & 1 & 1 & a^{w_2}\\ 1 & 1 & 1 & b_{i_4} & 1 & a^{w_3}\\ 1 & b_{i_2} & 1 & a^{f_1} & 1 & a^{k_3} \end{pmatrix}$$

Thus

$$z_3z_4\cdots z_p = a^{c_2}a^{d_3}a^{e_2}a^{h_2}a^{c_3}a^{e_3}a^{h_3}b_{i_6}a^{w_2}b_{i_4}a^{w_3}b_{i_2}a^{f_1}a^{k_3}.$$

Setting

$$w_1 = c_2 + d_3 + e_2 + h_2 + c_3 + e_3 + h_3$$

and

$$w_4 = f_1 + k_3$$

we get

$$z_3 z_4 \cdots z_p = a^{w_1} b^* a^{w_2} b^* a^{w_3} b^* a^{w_4}.$$

Applying the map Θ_2 on the commutator z we get

$$\begin{aligned}\Theta_2(z) &= [a, z_3 z_4 \cdots z_p] \\ &= (b_{i_2}^{-1})^{a^{(1+w_4)}} (b_{i_4}^{-1})^{a^{(1+w_3+w_4)}} (b_{i_6}^{-1})^{a^{(1+w_2+w_3+w_2)}} \\ &\quad (b_{i_6})^{a^{(w_2+w_2+w_4)}} (b_{i_4})^{a^{(w_3+w_4)}} (b_{i_2})^{a^{w_4}}.\end{aligned}$$

For $(b_{i_2})^{a^{w_4}}$ to contribute a directed automorphism in the first section of the decomposition we need $w_4 = 1$. For $(b_{i_6})^{a^{(w_2+w_3+w_4)}}$ to contribute a directed automorphism we need $w_2 + w_3 \not\equiv 0 \pmod p$. But then $(b_{i_6}^{-1})^{a^{(1+w_2+w_3+w_4)}}$ contributes the identity in the first section of the decomposition. Therefore applying the map Θ_1 we see that the length of its image is at most four.

On the other hand, for $(b_{i_2}^{-1})^{a^{(1+w_4)}}$ to contribute a directed automorphism in the first section of the decomposition we need $w_4 \not\equiv 0 \pmod p$. But then for $(b_{i_4})^{a^{(k_2+w_4)}}$ to contribute a directed automorphism we need $w_3 = 1$. Hence $(b_{i_4}^{-1})^{a^{(1+w_3+w_4)}}$ gives the identity in the first section of the decomposition and therefore applying the map Θ_1 we see that the length of its image is at most four.

Case 2.2: In this case $z_3 \cdots z_p$ is of the form:

$$z_3 z_4 \cdots z_p = a^{w_1} b^* a^{w_2} b^* a^{w_3} b^*$$

where $w_i \not\equiv 0 \pmod p$ for $i \in \{1, 2, 3\}$ and the symbols b^* represent elements in $\langle b_1, b_2 \rangle$.

For example, the decomposition matrix is of the form:

$$D(z) = \begin{pmatrix} b_{i_1} & a^{d_2} & b_{i_3} & a^{f_1} & b_{i_5} & 1 \\ a^{c_1} & a^{d_3} & a^{e_1} & a^{f_2} & a^{h_1} & 1 \\ a^{c_2} & 1 & a^{e_2} & a^{f_3} & a^{h_2} & 1 \\ a^{c_3} & 1 & a^{e_3} & 1 & a^{h_3} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & b_{i_6} \\ 1 & 1 & 1 & 1 & 1 & a^{k_1} \\ 1 & 1 & 1 & 1 & 1 & a^{k_2} \\ 1 & b_{i_2} & 1 & 1 & 1 & a^{k_3} \\ 1 & a^{d_1} & 1 & b_{i_4} & 1 & 1 \end{pmatrix}$$

Thus

$$z_3 z_4 \cdots z_p = a^{w_1} b_{i_6} a^{w_2} b_{i_2} a^{w_3} b_{i_4}.$$

Applying the map Θ_2 on z we get

$$\Theta_2(z) = [a, z_3 z_4 \cdots z_p] = (b_{i_4}^{-1})^a (b_{i_2}^{-1})^{a^{(1+w_3)}} (b_{i_6}^{-1})^{a^{(1+w_2+w_3)}} (b_{i_6})^{a^{(w_2+w_3)}} (b_{i_2})^{a^{w_3}} b_{i_4}.$$

Now, $(b_{i_4}^{-1})^a$ gives a directed automorphism is the first section of the decomposition. For $(b_{i_2})^{a^{w_3}}$ to give a directed automorphism we need $w_3 = 1$. But then $(b_{i_2}^{-1})^{a^{(1+w_3)}}$ contributes the identity and hence the length of the first section of the decomposition is at most two. Therefore applying the map Θ_1 we see that the length of its image is at most four.

On the other hand, b_{i_4} gives a rooted automorphism in the first section of the decomposition and hence the length of the first section of the decomposition is at most two. Therefore applying the map Θ_1 we see that the length of its image is at most four. The remaining cases are identical.

Case 3: We consider elements $z \in [G, G]$ of length $\partial(z) = 8$. In this case it is not hard to see that in order to have $\partial(z_1) = \partial(z_3 z_4 \cdots z_p) = 4$, the decomposition matrix is of the form:

$$D(z) = \begin{pmatrix} b_{i_1} & a^{d_3} & b_{i_3} & a^{f_1} & b_{i_5} & a^{k_1} & b_{i_7} & 1 \\ a^{c_1} & 1 & a^{e_1} & a^{f_3} & a^{h_1} & a^{k_1} & a^{l_1} & 1 \\ a^{c_2} & 1 & a^{e_2} & 1 & a^{h_2} & a^{k_3} & a^{l_2} & 1 \\ a^{c_3} & 1 & a^{e_3} & 1 & a^{h_3} & 1 & a^{l_3} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & b_{i_8} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & a^{n_1} \\ 1 & b_{i_2} & 1 & & 1 & 1 & 1 & a^{n_2} \\ 1 & a^{d_1} & 1 & b_{i_4} & 1 & 1 & 1 & a^{n_3} \\ 1 & a^{d_2} & 1 & a^{f_1} & 1 & b_{i_6} & 1 & 1 \end{pmatrix}$$

Thus

$$z_3 z_4 \cdots z_p = a^{w_1} b_{i_8} a^{w_2} b_{i_2} a^{w_3} b_{i_4} a^{w_4} b_{i_6},$$

where

$$\begin{aligned} w_1 &= c_2 + e_2 + h_2 + k_3 + l_2 + c_3 + e_3 + h_3 + l_3 \\ w_2 &= n_1 \\ w_3 &= n_2 + d_1 \\ w_4 &= n_3 + d_2 + f_1. \end{aligned}$$

Applying the map Θ_2 on the commutator z we get

$$\begin{aligned} \Theta_2(z) &= [a, z_3 z_4 \cdots z_p] \\ &= (b_{i_6}^{-1})^a \, (b_{i_4}^{-1})^{a^{1+w_4}} \, (b_{i_2}^{-1})^{a^{1+w_3+w_4}} \, (b_{i_8}^{-1})^{a^{1+w_2+w_3+w_4}} \\ &\quad (b_{i_8})^{a^{w_2+w_3+w_4}} \, (b_{i_2})^{a^{w_3+w_4}} \, (b_{i_4})^{a^{w_4}} \, b_{i_6}. \end{aligned}$$

Now, $(b_{i_6}^{-1})^a$ contributes a directed automorphism in the first section of the decomposition. For $(b_{i_4})^{a^{w_4}}$ to contribute a directed automorphism we need $w_4 = 1$. But then $(b_{i_4}^{-1})^{a^{(1+w_4)}}$ gives the identity and hence the length of the first section of the decomposition is at most three. Therefore applying the map Θ_1 we see that the length its image is at most six.

On the other hand, b_{i_6} contributes a rooted automorphism in the first section of the decomposition. Hence its length is at most three and the result yields.

Case 4: We consider commutator words of length greater or equal to ten. Firstly note that in order to have $\partial(z_1) \geq 5$, the directed automorphisms in columns $2, 4, 6, \ldots, m-2$

can only be positioned in the last three rows. Thus the decomposition matrix is of the form:

$$D(z) = \begin{pmatrix} b_{i_1} & a^* & b_{i_3} & a^* & b_{i_5} & a^* & b_{i_7} & a^* & \cdots & a^* \\ a^* & 1 & a^* & 1 & a^* & 1 & a^* & a^* & \cdots & a^* \\ a^* & 1 & a^* & 1 & a^* & 1 & a^* & a^* & \cdots & a^* \\ a^* & 1 & a^* & 1 & a^* & 1 & a^* & a^* & \cdots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & b_{i_2} & 1 & 1 & 1 & b_{i_6} & 1 & 1 & \cdots & 1 \\ 1 & a^* & 1 & b_{i_4} & 1 & a^* & 1 & b_{i_8} & \cdots & 1 \\ 1 & a^* & 1 & a^* & 1 & a^* & 1 & a^* & \cdots & b_{i_m} \end{pmatrix}$$

Note that there is at most one directed automorphism either in position (p, m) or in position $(p-4, m)$. Also there is at most one directed automorphism in row $p-2$ and at most $\frac{m-5}{2}$ in row $p-1$. In total

$$\partial(z_3 z_4 \cdots z_p) \leq 1 + 1 + \frac{m-5}{2} = \frac{m-1}{2}.$$

Therefore applying the map Θ_2 we see that the length of its image is at most

$$\frac{m-1}{2} \times 2 = m - 1 \leq m = \partial(z).$$

The remaining cases for elements $z \in [G, G]$ of length $\partial(z) \geq 12$ are identical and the result yields. □

Appendix B

A MAGMA program

In [26], E. Pervova showed that the GGS-groups have maximal subgroups only of finite index. A key ingredient in the proof for the GGS-groups relies on a single theta map

$$\Theta_1 : [G, G] \mapsto [G, G]$$

defined by

$$z \mapsto [a, z_1^{-1}].$$

Recall from Section 4.2, that in the current thesis we have introduced a second theta map

$$\Theta_2 : [G, G] \mapsto [G, G]$$

defined by

$$z \mapsto [a, z_{n+1} \cdots z_p].$$

During the early stages of our research, we wrote a program in MAGMA [6] in our efforts to get a better insight of the theta map used in [26].

It turned out that introducing the second map Θ_2 allowed us to extend our result in the class of torsion just infinite multi-edge spinal groups. It is worth mentioning, that even in the case of the GGS-groups, by using the second map Θ_2 one can simplify to a great extent the methods used in [26].

We present a MAGMA [6] program which allows to investigate the map Θ_1 for the Gupta-Sidki group for $p = 3$. Recall from Remark 3.1.7, that the Gupta-Sidki group $G = \langle a, b \rangle$ is generated by the rooted automorphism a and the directed automorphism $\psi_1(b) = (a, a^{-1}, 1, \ldots, 1, b)$.

The text after // denotes some comments for making the code more readable.

```
// ———————————————— basic parameters

// Magma program to investigate the map
//
//   Theta_1 : G' --> G', z = (z_1,z_2,z_3) -> [a,z_1^{-1}]
//
// for the Gupta-Sidki group G for p=3
// with generators
//  a: rooted automorphism
//  b: directed automorphism (a,a^{-1},b)
//
// The program searches for fixed points under Theta_1 among elements
// of bounded length in G.
//
// September 2013

clear;

// ————————————————————————————————    basic parameters

p := 3;
r := 10; // maximal length of elements that are tested

// Candidates g in G are written as
//
//  a^{e_1} b^{d_1} a^{e_2} ... a^{e_r} b^{d_r}
//
// with e_i, d_j in {0,1,2}.
// The exponents are recorded as vectors
//
//  (e_1,e_2,...,e_r,d_1,d_2,...,d_r)
//
// in a vector space U \cong W \oplus W

// auxiliary vector spaces
//  U \cong W \oplus W
```

```
V := VectorSpace(FiniteField(p),r);
W := sub<V | [V.i - V.(i+1) : i in [1..r-1]]>;

// ---------------------------------------------------   functions

 function reduced(a) // reduces elements of W x W to standard form
   e := a[1]; // e = (e_1,...,e_r) exponents of a's
   d := a[2]; // d = (d_1,...,d_r) exponents of b's
   repeat
     finished := true;
// tests for 0s in the vector e and shifts them to the right
     for i in [2..r] do
       if e[i] eq 0 and d[i] ne 0 then
         d[i-1] := d[i-1] + d[i];
         for j in [i..r-1] do
           e[j] := e[j+1];
           d[j] := d[j+1];
         end for;
         e[r] := 0;
         d[r] := 0;
         finished := false;
         break i;
       end if;
     end for;
// tests for 0s in the vector d and shifts them to the right
     for i in [1..r-1] do
       if e[i+1] ne 0 and d[i] eq 0 then
         e[i] := e[i] + e[i+1];
         d[i] := d[i+1];
         for j in [i+1..r-1] do
           e[j] := e[j+1];
           d[j] := d[j+1];
         end for;
         e[r] := 0;
         d[r] := 0;
         finished := false;
         break i;
```

```
      end if;
    end for;
  until finished;

  return [e,d];
end function;

// ---

function theta(a) // theta map from W x W to W x W
// vectors e = (e_1,...,e_r), d = (d_1,...,d_r)
// and e_seq : e as a sequence [e_1,...,e_r,0]
  e := a[1];
  e_seq := ElementToSequence(e) cat [FiniteField(p)!0];
  d := a[2];
// index set for non-zero d_i's
  I_d := { i : i in [1..r] | d[i] ne 0};

// work out first coordinate z_1 of element z corresponding to e, d
  m := [];                    // sequence to encode factors of z_1
  sum_e := FiniteField(p)!0;
  for j in [i+1 : i in I_d] do
    sum_e := sum_e + e_seq[j];
  end for;
  for i in I_d do
    if sum_e eq 0 then        // shift by 0 -> (a,a^-1,b)^{d_i}
      entry := [1,d[i]];      // endcodes factor a^{d_i}
    else
      if sum_e eq 1 then      // shift by 1 -> (b,a,a^-1)^{d_i}
        entry := [2,d[i]];    // encodes factor b^{d_i}
      else                    // shift by 2 -> (a^-1,b,a)^{d_i}
        entry := [1,-d[i]];   // encodes factor a^{d_i}
      end if;
    end if;
    Append(~m,entry);
    sum_e := sum_e - e_seq[i+1];
  end for;
```

```
// form commutator [a,z_1^-1] of a with z_1^-1
  m_rev := Reverse(m);
// sequence c encodes commutator
  c_seq := [[1,-1]] cat                                    // a^-1
       m cat                                               // z_1
       [[1,1]] cat                                         // a
       [ [entry[1],-entry[2]] : entry in m_rev ];          // z_1^-1
// build exponents e',d' as elements of W x W corresponding to commutator
  c := [W!0,W!0];
  i := [1,0]; // start by determining e'_1
  type := 1; // search 1st for a's (a has type 1, b has type 2)
  for entry in c_seq do
    if entry[1] eq type then // collecting the same generator as previously
      new_vec := c[type];
      new_vec[i[type]] := new_vec[i[type]] + entry[2];
      if type eq 1 then
        c := [new_vec,c[2]];  // update e'_i
      else
        c := [c[1],new_vec];  // update d'_j
      end if;
    else  // start collecting the other generator now
      type := Integers()!entry[1];  // update type (a: 1, b: 2)
      i[type] := i[type]+1;  // move to the next position
      new_vec := c[type];
      new_vec[i[type]] := new_vec[i[type]] + entry[2];
      if type eq 1 then
        c := [new_vec,c[2]];  // update e'_i
      else
        c := [c[1],new_vec];  // update d'_j
      end if;
    end if;
  end for;

  c := reduced(c);  // reduce to normal form
  return c;
end function;
```

```
function UtoWxW(u) // transforms u into (w1,w2)
  w1 := W![ElementToSequence(u)[i] : i in [1..r]];
  w2 := W![ElementToSequence(u)[i] : i in [r+1..2*r]];

  return [w1,w2];
end function;

//—————————————————————————————— MAIN

// successively build supply of 'reduced' pairs of vectors
// and search for fixed points

print "";
print "Building set of candidates.  Please wait...";

WxW_red := [];

U1 := [W!([0,0] cat [0 : i in [3..r]]),
      W!([1,2] cat [0 : i in [3..r]]),
      W!([2,1] cat [0 : i in [3..r]])];
U2 := [W!([1,2] cat [0 : i in [3..r]]),
      W!([2,1] cat [0 : i in [3..r]])];

for k in [2..r-1] do
 U1_old := U1;
 U1 := [];
 for u_old in U1_old do
   for x in FiniteField(p) do
     if u_old[k] ne -x then
       u := u_old + x*W.k;
       Append(~U1,u);
       for u2 in U2 do
         Append(~WxW_red,[u,u2]);
       end for;
     end if;
   end for;
```

```
 end for;
 U2 := [];
 for u in U1 do
   if u[1] ne 0 and u[k+1] ne 0 then
     Append(~U2,u);
   end if;
 end for;
end for;

Number := #WxW_red;
print "There are ",Number," candidates to test.";

print "";

// start search among candidate set

print "Now searching in the Gupta-Sidki-group G (p=3) for";
print "fixed points of the map Theta_1 : G' --> G' defined by";
print "  z = (z_1,z_2,z_3) maps to [a,z_1^{-1}].";
print "";

count := 1;
Step := Number div 10;

for u in WxW_red do
  if count mod Step eq 0 then
    print Number - count," candidates remain...";
  end if;
  if theta(u) eq u then
    print "Found one:";
    print u;
  end if;
  count := count + 1;
end for;
```

The following example shows that one cannot expect to reduce the length in all cases, without using Θ_2; see Section 4.2.

```
/* OUTPUT

> load "MAGMA-Sep2013.txt";
Loading "MAGMA-Sep2013.txt"

Building set of candidates.  Please wait...
There are  174420  candidates to test.

Now searching in the Gupta-Sidki-group G (p=3) for
fixed points of the map Theta_1 : G' --> G' defined by
  z = (z_1,z_2,z_3) maps to [a,z_1^{-1}].

Found one:
[
    (2 1 0 0 0 0 0 0 0 0),
    (1 2 0 0 0 0 0 0 0 0)
]
Found one:
[
    (2 1 0 0 0 0 0 0 0 0),
    (2 1 0 0 0 0 0 0 0 0)
]
Found one:
[
    (1 1 2 1 1 2 1 0 0 0),
    (1 1 2 1 2 2 0 0 0 0)
]
156978  candidates remain...
139536  candidates remain...
Found one:
[
    (2 2 1 2 1 1 2 1 0 0),
    (1 1 1 2 1 2 2 2 0 0)
]
```

```
Found one:
[
    (2 2 1 2 1 1 2 1 0 0),
    (2 1 1 2 1 2 2 1 0 0)
]
122094  candidates remain...
104652  candidates remain...
87210  candidates remain...
69768  candidates remain...
52326  candidates remain...
34884  candidates remain...
17442  candidates remain...
0  candidates remain...
>

*/
```

Problem. Study the dynamical system $([G, G], \Theta_1)$. What are the fixed points (or periodic points)? Are there infinitely many? How do the answers depend on the multi-edge spinal group G?

Bibliography

[1] S. V. Alešin. Finite automata and the Burnside problem for periodic groups. *Mat. Zametki*, 11:319–328, 1972.

[2] T. Alexoudas, B. Klopsch, and A. Thillaisundaram. Maximal subgroups of multi-edge spinal groups. *preprint*, 2013.

[3] L. Bartholdi, R. I. Grigorchuk, and Z. Šunić. Branch groups. In *Handbook of algebra, Vol. 3*, pages 989–1112. North-Holland, 2003.

[4] L. Bartholdi and Z. Šunić. On the word and period growth of some groups of tree automorphisms. *Comm. Algebra*, 29(11):4923–4964, 2001.

[5] I. V. Bondarenko. Finite generation of iterated wreath products. *Arch. Math. (Basel)*, 95(4):301–308, 2010.

[6] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997.

[7] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic pro-p-groups*, volume 157 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1991.

[8] G. A. Fernández-Alcober and A. Zugadi-Reizabal. Spinal groups: semidirect product decompositions and Hausdorff dimension. *J. Group Theory*, 14(4):491–519, 2011.

[9] E. S. Golod. On nil-algebras and finitely approximable p-groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 28:273–276, 1964.

[10] R. I. Grigorchuk. On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.*, 14(1):53–54, 1980.

[11] R. I. Grigorchuk. Just infinite branch groups. In *New horizons in pro-p groups*, volume 184 of *Progr. Math.*, pages 121–179. Birkhäuser Boston, 2000.

[12] R. I. Grigorchuk and J. S. Wilson. A structural property concerning abstract commensurability of subgroups. *J. London Math. Soc. (2)*, 68(3):671–682, 2003.

[13] N. Gupta and S. Sidki. On the Burnside problem for periodic groups. *Math. Z.*, 182(3):385–388, 1983.

[14] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations.* Die Grundlehren der mathematischen Wissenschaften, Bd. 115. Academic Press Inc., Publishers, New York, 1963.

[15] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Band 152. Springer-Verlag, New York, 1970.

[16] B. Klopsch, N. Nikolov, and C. Voll. *Lectures on profinite topics in group theory*, volume 77 of *London Mathematical Society Student Texts*. Cambridge University Press, 2011.

[17] A. Lubotzky and D. Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, 2003.

[18] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[19] Y. I. Merzlyakov. Infinite finitely generated periodic groups. *Dokl. Akad. Nauk SSSR*, 268(4):803–805, 1983.

[20] J. Milnor. Growth of finitely generated solvable groups. *J. Differential Geometry*, 2:447–449, 1968.

[21] D. S. Passman. The semiprimitivity of group algebras. In *Methods in ring theory (Levico Terme, 1997)*, volume 198 of *Lecture Notes in Pure and Appl. Math.*, pages 199–211. Dekker, New York, 1998.

[22] E. L. Pervova. Profinite topologies in just infinite branch groups. *preprint of the Max Planck Institute for Mathematics, Bonn, Germany*, (154).

[23] E. L. Pervova. Everywhere dense subgroups of a group of tree automorphisms. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):356–367, 2000.

[24] E. L. Pervova. AT-groups over a sequence of abelian groups. *Vestnik Chelyab. Univ. Ser. 3 Mat. Mekh. Inform.*, (1(6)):8–12, 2002.

[25] E. L. Pervova. The congruence property of AT-groups. *Algebra Logika*, 41(5):553–567, 634, 2002.

[26] E. L. Pervova. Maximal subgroups of some non locally finite p-groups. *Internat. J. Algebra Comput.*, 15(5-6):1129–1150, 2005.

[27] E. L. Pervova. Profinite completions of some groups acting on trees. *J. Algebra*, 310(2):858–879, 2007.

[28] D. J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

[29] A. V. Rozhkov. On the theory of groups of Aleshin type. *Mat. Zametki*, 40(5):572–589, 1986.

[30] D. Segal. On the finite images of infinite groups. In *Groups: topological, combinatorial and arithmetic aspects*, volume 311 of *London Math. Soc. Lecture Note Ser.*, pages 542–563. Cambridge Univ. Press, 2004.

[31] S. Sidki. A primitive ring associated to a Burnside 3-group. *J. London Math. Soc. (2)*, 55(1):55–64, 1997.

[32] O. Siegenthaler and A. Zugadi-Reizabal. The equations satisfied by GGS-groups and the abelian group structure of the Gupta-Sidki group. *European J. Combin.*, 33(7):1672–1690, 2012.

[33] T. Vovkivsky. Infinite torsion groups arising as generalizations of the second Grigorchuk group. In *Algebra (Moscow, 1998)*, pages 357–377. de Gruyter, 2000.

[34] J. S. Wilson. Groups with every proper quotient finite. *Proc. Cambridge Philos. Soc.*, 69:373–391, 1971.

[35] J. S. Wilson. *Profinite groups*, volume 19 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1998.

[36] A. Zugadi-Reizabal. *Hausdorff dimension in groups acting on the p-adic tree*. PhD thesis, University of the Basque Country, 2011.